ASYMPTOTICS OF CHEBYSHEV POLYNOMIALS, II. DCT SUBSETS OF $\mathbb{R}$

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Abstract. We prove Szegő–Widom asymptotics for the Chebyshev polynomials of a compact subset of $\mathbb{R}$ which is regular for potential theory and obeys the Parreau–Widom and DCT conditions.

1. Introduction

Let $\mathfrak{e} \subset \mathbb{R}$ be a compact subset with logarithmic capacity $C(\mathfrak{e}) > 0$. Define

$$\|f\|_{\mathfrak{e}} = \sup_{x \in \mathfrak{e}} |f(x)| \tag{1.1}$$

The Chebyshev polynomial, $T_n(z)$, is the monic polynomial with

$$t_n \equiv \|T_n\|_{\mathfrak{e}} = \inf\{\|P\|_{\mathfrak{e}} \mid \deg P = n, P \text{ monic}\} \tag{1.2}$$

It is a consequence of the alternation theorem (a result of Borel [3] and Markov [13] using ideas that go back to Chebyshev; see [4] for a

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statement and proof) that T_n is unique and that

$$\mathfrak{e}_n \equiv T_n^{-1}([-t_n, t_n]) = \{z \in \mathbb{C} \mid -t_n \le T_n(z) \le t_n\} \tag{1.3}$$

is a subset of $\mathbb{R}$. Clearly, by definition of t_n,

$$\mathfrak{e} \subset \mathfrak{e}_n \tag{1.4}$$

Recall that the Green's function, $G_\mathfrak{e}(z)$, is the unique function on $\mathbb{C}$ which is positive and harmonic on $\mathbb{C} \setminus \mathfrak{e}$, upper semicontinuous on $\mathbb{C}$, so that $G_\mathfrak{e}(z) = \log(|z|) + \mathrm{O}(1)$ near $z = \infty$ and so that $G_\mathfrak{e}(x) = 0$ for quasi-every $x \in \mathfrak{e}$. A set, $\mathfrak{e}$, is called regular (for potential theory) if $G_\mathfrak{e}(x) = 0$ for all $x \in \mathfrak{e}$ (which implies that $G_\mathfrak{e}$ is continuous on $\mathbb{C}$). We'll assume that $\mathfrak{e}$ is regular. One has that near infinity

$$G_\mathfrak{e}(z) = \log(|z|) - \log(C(\mathfrak{e})) + \mathrm{O}(1/|z|) \tag{1.5}$$

Moreover, if $d\rho_\mathfrak{e}$ is the potential theoretic equilibrium measure for $\mathfrak{e}$, then

$$G_\mathfrak{e}(z) = -\log(C(\mathfrak{e})) + \int \log(|z-x|)\,d\rho_\mathfrak{e}(x) \tag{1.6}$$

For more on potential theory, see [19, Section 3.6].

It is not hard to see (see [4]) that the Green's function, G_n, for $\mathfrak{e}_n$ is

$$G_n(z) = \frac{1}{n} \log\left(\left|\frac{T_n(z)}{t_n} + i\sqrt{1 - \left(\frac{T_n(z)}{t_n}\right)^2}\right|\right) \tag{1.7}$$

which implies that

$$t_n = 2(C(\mathfrak{e}_n))^n \tag{1.8}$$

In particular, since $C(\mathfrak{e}) \le C(\mathfrak{e}_n)$, we get Schiefermayr's bound [16]

$$t_n \ge 2(C(\mathfrak{e}))^n \tag{1.9}$$

In [4], we introduced the term *Totik–Widom bound* (after [22, 24]) if for some constant D, one has that

$$t_n \le D(C(\mathfrak{e}))^n \tag{1.10}$$

A compact set $\mathfrak{e} \subset \mathbb{C}$ is said to obey a Parreau–Widom (PW) condition (after [15, 25]) if and only if

$$PW(\mathfrak{e}) \equiv \sum_{z_j \in \mathcal{C}} G_\mathfrak{e}(z_j) < \infty \tag{1.11}$$

where $\mathcal{C}$ is the set of points, z_j, where $\nabla G_\mathfrak{e}(z_j) = 0$. For regular subsets of $\mathbb{R}$, all these critical points are real and there is exactly one such point in each bounded open component, K_j, of $\mathbb{R} \setminus \mathfrak{e}$ and $G_\mathfrak{e}(z_j) = \max_{x \in K_j} G_\mathfrak{e}(x)$.

In [4], we proved that if $\mathfrak{e} \subset \mathbb{R}$ is a regular PW set, then one has an explicit Totik–Widom bound

$$t_n \leq 2 \exp{(PW(\mathfrak{e}))}(C(\mathfrak{e}))^n \tag{1.12}$$

Our methods there say nothing about the complex case. In this regard, we mention the recent interesting paper of Andrievskii [2] who has proven Totik–Widom bounds for a class of sets that, for example, includes the Koch snowflake.

One of our results in this paper (see Theorem 1.4 and Section 2) will be a kind of weak converse – under an additional condition on $\mathfrak{e}$ which should hold generically, if $\mathfrak{e} \subset \mathbb{C}$ is compact, regular and obeys a Totik–Widom bound, then $\mathfrak{e}$ is a PW set.

For a general positive capacity, regular, compact set $\mathfrak{e} \subset \mathbb{C}$, we define Ω to be its complement in the Riemann sphere, i.e.,

$$\Omega = (\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e} \tag{1.13}$$

which we suppose is connected (this always holds if $\mathfrak{e} \subset \mathbb{R}$). We let $\widetilde{\Omega}$ be its universal cover and $\pi : \widetilde{\Omega} \to \Omega$ the covering map. It is a consequence of the uniformization theorem (see [18, Section 8.7]) that $\widetilde{\Omega}$ is conformally equivalent to the disk, $\mathbb{D}$, a fact we will use. We denote by $\mathbf{x} : \mathbb{D} \to \Omega$ the unique covering map normalized by $\mathbf{x}(0) = \infty$ and near $z = 0$, $\mathbf{x}(z) = Dz^{-1} + \mathrm{O}(1)$ with $D > 0$.

There is an important multivalued analytic function, $B_\mathfrak{e}(z)$, on Ω determined by

$$|B_\mathfrak{e}(z)| = e^{-G_\mathfrak{e}(z)} \tag{1.14}$$

and that near ∞,

$$B_\mathfrak{e}(z) = C(\mathfrak{e})z^{-1} + \mathrm{O}(z^{-2}) \tag{1.15}$$

One way of constructing it is to use the fact that $-G_\mathfrak{e}$ has a harmonic conjugate locally so that locally on $\mathbb{C} \setminus \mathfrak{e}$, it is the real part of an analytic function whose exponential is $B_\mathfrak{e}(z)$. It is easy to see that this allows $B_\mathfrak{e}$ to be continued along any curve in $\widetilde{\Omega}$ so by the monodromy theorem ([18, Section 11.2]), $B_\mathfrak{e}(z)$ has an analytic continuation to $\widetilde{\Omega}$ which defines a multivalued analytic function on Ω.

By analyticity, (1.14) holds for all branches of $B_\mathfrak{e}(z)$. In particular, going around a closed curve, γ, can only change $B_\mathfrak{e}$ by a phase factor which implies there is a character, $\chi_\mathfrak{e}$, of the fundamental group, $\pi_1(\Omega)$, so that going around γ changes $B_\mathfrak{e}$ by $\chi_\mathfrak{e}([\gamma])$. It is not hard to see ([4, Theorem 2.7]) that

$$\chi_\mathfrak{e}(\gamma) = \exp\left(-2\pi i \int_\mathfrak{e} N(\gamma, x) d\rho_\mathfrak{e}(x)\right) \tag{1.16}$$

where $N(\gamma, x)$ is the winding number for the curve γ about x. Thus $B_{\mathfrak{e}}$ is a character automorphic function.

An alternate construction is to consider elementary Blaschke factors $b(z,w)(= (\bar{w}/|w|)[(w-z)/(1-\bar{w}z)]$ if $w \neq 0)$ for $z, w \in \mathbb{D}$. Then, lifted to $\mathbb{D}$,

$$B_{\mathfrak{e}}(z) = \prod_{\{w_j \,|\, \mathbf{x}(w_j)=\infty\}} b(z, w_j) \tag{1.17}$$

We will call $B_{\mathfrak{e}}$ the *canonical Blaschke product for* $\mathfrak{e}$ and $\chi_{\mathfrak{e}}$, *the canonical character.*

Similarly, we can define for each $w \in \Omega$, $B_{\mathfrak{e}}(z, w)$ either by using (1.17) with $\{w_j \,|\, \mathbf{x}(w_j) = \infty\}$ replaced by $\{w_j \,|\, \mathbf{x}(w_j) = w\}$ or by using the Green's function $G_{\mathfrak{e}}(z, w)$ with pole at w and demanding that $|B_{\mathfrak{e}}(z, w)| = \exp(-G_{\mathfrak{e}}(z, w))$ and fixing the phase by demanding that $B_{\mathfrak{e}}(\infty, w) > 0$.

One can consider character automorphic functions for general characters, $\chi \in \pi_1(\Omega)^*$, the full character group. In this regard the following theorem of Widom [25] (see also Hasumi [11, Theorem 5.2B]) is important:

Theorem 1.1. *(Widom) Suppose that* $\mathfrak{e}$ *is a compact set regular for potential theory. Then* $\mathfrak{e}$ *is a PW set if and only if for every character,* $\chi \in \pi_1(\Omega)^*$, *there is a non-zero analytic* χ*-automorphic function on* $\widetilde{\Omega}$ *which is bounded.*

Single-valued analytic functions on $\widetilde{\Omega}$ correspond to multi-valued functions on Ω and we will often refer to them as if they are ordinary functions. In essence we view Ω with the convex hull of $\mathfrak{e}$ removed as a subset of $\widetilde{\Omega}$.

For a PW set, $\mathfrak{e}$, and any character, χ, we let $H^\infty(\Omega, \chi)$ be the set of bounded analytic χ-automorphic functions on $\widetilde{\Omega}$ and denote by $\|\cdot\|_\infty$ the corresponding norm. We use $H^2(\Omega, \chi)$ or $\mathcal{H}_\chi$ for the set of analytic χ-automorphic functions, f, for which $|f|^2$ has a harmonic majorant in Ω. Evidently, $H^\infty(\Omega, \chi) \subset H^2(\Omega, \chi)$. It is easy to see that $H^2(\Omega, \chi)$ is precisely those χ-automorphic functions, f, on Ω whose lifts to $\mathbb{D}$ under $\mathbf{x}$ are in $H^2(\mathbb{D})$.

When $\mathfrak{e}$ is a PW set, there exist $h \in H^\infty(\Omega, \chi)$ with $h(\infty) \neq 0$, for if $f \in H^\infty(\Omega, \chi)$ with $f(z) = Cz^{-n} + \mathrm{O}(z^{-n-1})$; $C \neq 0$, then $h(z) = z^n f(z)$ is also in $H^\infty(\Omega, \chi)$ and $h(\infty) = C$.

For any χ, the Widom trial functions for χ is the set, $\{h \in H^\infty(\Omega, \chi) \,|\, h(\infty) = 1\}$. The *Widom minimizer*, $F_\chi(z)$, is a bounded χ-character automorphic function with $F_\chi(\infty) = 1$ so that

$$\|F_\chi\|_\infty = \inf\{\|h\|_\infty \,|\, h \in H^\infty(\Omega, \chi);\ h(\infty) = 1\} \tag{1.18}$$

Knowing that there are Widom trial functions, it is easy to prove using Montel's Theorem ([18, Section 6.2]) that minimizers exist. In Section 2, we'll prove that minimizers are unique (this is not a new result although our proof is simpler than previous ones).

We will also consider a dual problem. The dual Widom trial functions are $\{g \in H^\infty(\Omega,\chi) \mid \|g\|_\infty = 1\}$. The *dual Widom maximizer* is that function Q_χ in the dual Widom trial functions with

$$Q_\chi(\infty) = \sup\{g(\infty) \mid g \in H^\infty(\Omega,\chi),\ \|g\|_\infty = 1,\ g(\infty) > 0\} \tag{1.19}$$

If g is a dual Widom trial function with $g(\infty) \neq 0$, then $g/g(\infty)$ is a Widom trial function. Conversely, if h is a Widom trial function, then $h/\|h\|_\infty$ is a dual Widom trial function. This shows that for the two problems, either both or neither have unique solutions and

$$Q_\chi = F_\chi/\|F_\chi\|_\infty, \quad F_\chi = Q_\chi/Q_\chi(\infty), \quad Q_\chi(\infty) = 1/\|F_\chi\|_\infty \tag{1.20}$$

Suppose now that $\mathfrak{e} \subset \mathbb{C}$ is compact, connected and simply connected. Then Ω is simply connected and $B_\mathfrak{e}$ is analytic (rather than multivalued analytic) and is, in fact, the Riemann map of Ω to $\mathbb{D}$ (uniquely specified by $B_\mathfrak{e}(\infty) = 0$ and that near ∞, $B_\mathfrak{e}(z) = Cz^{-1} + \mathrm{O}(z^{-2})$ with $C > 0$). In 1919, assuming that $\partial\Omega$ is an analytic Jordan curve, Faber [7] proved that in this case

$$\frac{T_n(z)B_\mathfrak{e}(z)^n}{C(\mathfrak{e})^n} \to 1 \tag{1.21}$$

uniformly on $\overline{\Omega}$.

In 1969, Widom [24] considered $\mathfrak{e} \subset \mathbb{C}$ which is a finite union of C^{1+} Jordan curves and arcs. He noted that (1.21) couldn't hold when there was more than one arc or curve since, in that case, $B_\mathfrak{e}(z)^n$ is now a character automorphic function with character $\chi_\mathfrak{e}^n$. If $F_n \equiv F_{\chi_\mathfrak{e}^n}$, Widom suggested what we call the *Widom surmise*, that

$$\frac{T_n(z)B_\mathfrak{e}(z)^n}{C(\mathfrak{e})^n} - F_n(z) \to 0 \tag{1.22}$$

uniformly on compact subsets of $\widetilde{\Omega}$. He proved this when $\mathfrak{e}$ consisted only of (closed) Jordan curves and in [4], we proved it for $\mathfrak{e}$ a finite gap set in $\mathbb{R}$.

We say that T_n has *strong Szegő–Widom asymptotics* if (see [20, Section 6.6] for a discussion of almost periodic functions)

(a) (1.22) holds uniformly on compact subsets of $\widetilde{\Omega}$
(b) $n \mapsto \|F_n\|_\infty$ is an almost periodic function
(c) $n \mapsto F_n(z)$ is an almost periodic function uniformly on compact subsets of $\widetilde{\Omega}$.

We note that the above results of Widom [24] and [4] prove (b) and (c) also.

A final element we need before stating our main theorem is the notion of the Direct Cauchy Theorem (DCT) property. There are many equivalent definitions of DCT – see Hasumi [11] or Volberg–Yuditskii [23]. Rather than stating a formal definition, we first of all quote a theorem that could be used as one definition of DCT:

Theorem 1.2 (Hayashi [12], Hasumi [11])**.** *A PW set* $\mathfrak{e}$ *obeys a DCT if and only if the function* $\chi \mapsto Q_\chi(\infty)$ *of the dual Widom maximizer problem is a continuous function on* $\pi_1(\Omega)^*$*.*

We'll also quote as needed some other results that rely on the DCT condition. We note that any homogeneous subset of $\mathbb{R}$ (in the sense of Carleson [21]) obeys DCT [21]. On the other hand, Hasumi [11] has found rather simple explicit examples (with thin components) of subsets of $\mathbb{R}$ which obey PW but not DCT. Volberg–Yuditskii [23] have even found examples all of whose reflectionless measures are absolutely continuous.

We can now state the main result of this paper:

Theorem 1.3. *Let* $\mathfrak{e} \subset \mathbb{R}$ *be a compact set which is regular for potential theory and that obeys the PW and DCT conditions. Then its Chebyshev polynomials have strong Szegő–Widom asymptotics. Moreover,*

$$\lim_{n\to\infty} \frac{t_n}{C(\mathfrak{e})^n \|F_n\|_\infty} = 2 \tag{1.23}$$

Remarks. 1. Given the limit (1.22), the 2 in (1.23) may seem surprising. Widom noted the 2 in the easy special case $\mathfrak{e} = [-1, 1]$ and proved (1.23) for general finite gap subsets of $\mathbb{R}$. This fact was used in our proof of (1.22) for the finite gap case in [4]. Here we'll prove (1.22) first and then prove (1.23).

2. Our proof uses a partially variant strategy to the one in [4] and we believe is simpler even in the finite gap case (especially if you include the need there for some results of Widom that we don't need to prove a priori).

For our other main results, we need a new definition. We say a set $\mathfrak{e} \subset \mathbb{R}$ has a *canonical generator* if $\{\chi_\mathfrak{e}^n\}_{n=-\infty}^\infty$ is dense in the character group $\pi_1(\Omega)^*$. This holds if and only if for each decomposition $\mathfrak{e} = \mathfrak{e}_1 \cup \dots \cup \mathfrak{e}_\ell$ into closed disjoint sets and rational numbers $\{q_j\}_{j=1}^{\ell-1}$, we have that

$$\sum_{j=1}^{\ell-1} q_j \rho_\mathfrak{e}(\mathfrak{e}_j) \neq 0 \tag{1.24}$$

Remarks. 1. The class of regular PW sets can be parametrized by comb domains of the form

$$\Pi = \{x+iy \,|\, 0<x<1, y>0\} \setminus \cup_k \{\omega_k + iy \,|\, 0 < y \le h_k\} \tag{1.25}$$

with $\omega_k \in (0,1)$, $\omega_k \neq \omega_j$ for $k \neq j$ and $h_k > 0$, $\sum_k h_k < \infty$. Specifically, if $\mathfrak{e}$ is scaled to the interval $[0,1]$, then

$$\theta(z) = \frac{-\log B_{\mathfrak{e}}(z)}{\pi i} \tag{1.26}$$

is a conformal mapping of $\mathbb{C}_+$ onto such a domain (see [6] for more details). In that parametrization, the property of a canonical generator is generic. For one can show that $\omega_k = \rho_{\mathfrak{e}}(\{x \in \mathfrak{e} \,|\, x \le a_k\})$ and the collection of comb domains with rationally independent ω_k's clearly form a dense G_δ set.

2. It seems likely that the condition of a canonical generator holds in various other generic senses as well. For example, given a fixed nowhere dense, infinite gap set, we can pick a positive integer labeling of the gaps and, for any $\lambda \in \prod_1^\infty [1/2, 2]$, consider the set obtained by scaling the jth gap by λ_j. We suspect the set of λ's for which this set has a canonical generator, is a dense G_δ. In the finite gap case, that this is true follows from results of Totik [22].

Theorem 1.4. *Let $\mathfrak{e} \subset \mathbb{C}$ be a compact set regular for potential theory with a canonical generator. If $\mathfrak{e}$ has a Totik–Widom bound, then $\mathfrak{e}$ is a PW set.*

Remarks. 1. While we need to assume canonical generator, this result suggests that Totik–Widom fails if the set is not PW.

2. We emphasize that this result holds for $\mathfrak{e} \subset \mathbb{C}$ and not just $\mathfrak{e} \subset \mathbb{R}$.

Theorem 1.5. *Let $\mathfrak{e} \subset \mathbb{C}$ be a compact set regular for potential theory with a canonical generator. Suppose that $\mathfrak{e}$ is a PW set and that $n \mapsto \|F_n\|_\infty$ is a bounded almost periodic function on $\mathbb{Z}$. Then $\mathfrak{e}$ is a DCT set.*

Remarks. 1. Again, we emphasize that this holds for all $\mathfrak{e} \subset \mathbb{C}$ not just $\mathfrak{e} \subset \mathbb{R}$.

2. So, one small part of Szegő–Widom asymptotics, namely asymptotic almost periodicity of $\|T_n\|_{\mathfrak{e}}/C(\mathfrak{e})^n$ and the limit result (1.23), implies that $\mathfrak{e}$ is a DCT set (at least if $\mathfrak{e}$ has a canonical generator).

We will note results from [4] as needed but mention some that are needed to overview the contents of the paper. Let $B_n \equiv B_{\mathfrak{e}_n}$. Then [4] proved that

$$\frac{2T_n(z)}{t_n} = B_n(z)^n + B_n(z)^{-n} \tag{1.27}$$

Thus, instead of looking at

$$L_n(z) \equiv \frac{T_n(z) B_{\mathfrak{e}}(z)^n}{C(\mathfrak{e})^n} \tag{1.28}$$

we'll look at

$$M_n(z) = B_{\mathfrak{e}}(z)^n / B_n(z)^n \tag{1.29}$$

which obeys

$$|M_n(z)| = \exp(-n h_n(z)), \qquad h_n(z) \equiv G_{\mathfrak{e}}(z) - G_{\mathfrak{e}_n}(z) \tag{1.30}$$

By (1.27)

$$L_n(z) = (1 + B_n(z)^{2n}) H_n(z), \quad H_n(z) = \frac{C(\mathfrak{e}_n)^n}{C(\mathfrak{e})^n} \frac{B_{\mathfrak{e}}(z)^n}{B_n(z)^n} = \frac{M_n(z)}{M_n(\infty)} \tag{1.31}$$

The first equation in (1.31) explains the 2 in (1.23). By a simple argument,

$$\sup_{n, z \in K} |B_n(z)| < 1 \text{ for any compact set } K \subset \widetilde{\Omega} \tag{1.32}$$

so that $B_n(z)^{2n}$ goes to zero, but for $\sup_{z \in \Omega} |1 + B_n(z)^{2n}|$, we get 2 since there are points $x \in \mathfrak{e}_n$ with $B_n(x + i0) = 1$.

By the first equation in (1.31) and (1.32), (1.22) is equivalent to

$$H_n(z) - F_n(z) \to 0 \tag{1.33}$$

By the second equation in (1.31), it seems likely that it suffices to control limits of M_n and that is what we'll do. By the maximum principle for harmonic functions and (1.30), $|M_n(z)| \leq 1$. We will prove that $\lim_{n \to \infty} \|M_n\|_\infty = 1$ and that limit points of M_n with $n_j \to \infty$ so that $\chi_{\mathfrak{e}}^{n_j} \to \chi_0$ for some $\chi_0 \in \pi_1(\Omega)^*$ are dual Widom maximizers which will let us prove (1.33).

Here is an overview of the rest of this paper. In section 2, following ideas of Fisher [8], we prove uniqueness of solutions of the Widom minimization problem (this is not a new result – only a new proof – see the discussion there) and prove Theorem 1.4. In Section 3, we discuss continuity of F_χ in χ and prove Theorem 1.5. In Section 4, we prove that limit points of the M_n are Blaschke products of suitable $B(z, x_j)$ and in Section 5 that these products are dual Widom maximizers. This result has been obtained by Volberg–Yuditskii [23] but we found an alternate proof using ideas of Eichinger–Yuditskii [5]. Finally, in Section 6, we put things together and prove Theorem 1.3

2. Uniqueness of the Dual Widom Maximizer

In this section, we provide a proof of uniqueness of solutions of the dual Widom maximizer problem and so uniqueness of solutions of the Widom minimizer problem. If $\mathfrak{e}$ obeys a PW condition, $H^\infty(\Omega,\chi)$ is non-empty (by Theorem 1.1) and so contains h with $h(\infty) > 0$. By Montel's theorem ([18, Section 6.2]), $\{h \in H^\infty(\Omega,\chi) \mid \|h\|_\infty \leq 1, h(\infty) \geq 0\}$ is compact in the topology of uniform convergence on compact subsets of $\widetilde{\Omega}$. Thus, there exists a maximizer. We need to prove that this is unique.

Recall that the Ahlfors problem for a compact set $\mathfrak{e} \subset \mathbb{C}$ is to look for bounded analytic functions, f, on $\Omega = (\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$ with $\sup_{z\in\Omega} |f(z)| \leq 1$ and $f(\infty) = 0$ that maximize $f'(\infty)$ (defined by $f(z) = f(\infty) + f'(\infty)z^{-1} + \mathrm{O}(z^{-2})$ near $z = \infty$). This maximum is called the analytic capacity (because if "analytic" is replaced by "harmonic", the maximum is the potential theoretic capacity). There is an enormous literature on the Ahlfors problem, in particular two sets of lecture notes [9, 14] and a textbook presentation in [18, Section 8.8].

This is clearly analogous to the dual Widom maximizer problem so proofs of uniqueness for the Ahlfors problem should have analogs for our problem. In his original paper, Ahlfors [1] considered an n-connected domain Ω (i.e., $\mathfrak{e} \subset \mathbb{C}$ has n connected components) and proved that any maximizer, g, has limiting values for almost every point in $\partial\Omega$ (maybe only one sided if $\mathfrak{e}$ has a one dimensional component) with $|g(w)| = 1$ for $w \in \partial\Omega$. This can be used to prove uniqueness. In [24], Widom proved that uniqueness for the dual maximizer by proving any maximizer had absolute value one on $\partial\Omega$. The same idea occurs for general Parreau–Widom sets in Volberg–Yuditskii [23] who had the first proof of the result in this section.

A simple, elegant approach to uniqueness of the Ahlfors problem is due to Fisher [8]. We will modify his approach to accommodate change of character and the fact that the vanishing at ∞ is different.

Theorem 2.1. *Let $\mathfrak{e} \subset \mathbb{C}$ be a PW set regular for potential theory. Then for any character $\chi \in \pi_1(\Omega)^*$, the dual Widom maximizer (and so also the Widom minimizer) exists and is unique.*

Remarks. 1. As noted above this has already been proven by Volberg–Yuditskii [23] but starting from first principles, our proof is simpler.

2. Uniqueness implies that the maximizer in the dual problem is an extreme point in $H^\infty(\Omega,\chi)_1$, the closed unit ball in $H^\infty(\Omega,\chi)$. For if $Q_\chi = \frac{1}{2}(q_1 + q_2)$ with $q_j \in H^\infty(\Omega,\chi)_1$, then by the maximum property,

$q_j(\infty) = Q_\chi(\infty)$. So the q_j are also maximizers, and hence equal to Q_χ.

Proof. Without loss, we can suppose $\chi \not\equiv 1$ since if $\chi \equiv 1$, the unique dual maximizer is $f \equiv 1$. In particular, since $\chi \not\equiv 1$, we have that $f(\infty) < 1$ by the maximum principle. Let f_1 and f_2 be two maximizers and define

$$f = \tfrac{1}{2}(f_1 + f_2), \qquad k = \tfrac{1}{2}(f_1 - f_2) \tag{2.1}$$

Pick $q \in H^\infty(\Omega, \overline{\chi})$ with $q(\infty) \neq 0$ and $\|q\|_\infty = 1$ which exists by the PW condition and Theorem 1.1.

Since $\|f_j\|_\infty = 1$, we have that $\|f \pm k\|_\infty = 1$ so

$$|f|^2 + |k|^2 = \tfrac{1}{2}\left(|f+k|^2 + |f-k|^2\right) \leq 1 \tag{2.2}$$

Define

$$g = qk^2/2 \tag{2.3}$$

so $g \in H^\infty(\Omega, \chi)$. By (2.2),

$$|g| \leq \frac{1-|f|^2}{2} = (1-|f|)\left(\frac{1+|f|}{2}\right) \leq 1 - |f|$$

so

$$|g| + |f| \leq 1 \tag{2.4}$$

Since $f_1(\infty) = f_2(\infty)$ is the maximum value, $g(\infty) = 0$, so if $g \not\equiv 0$, then, near ∞, we can write

$$g(z) = \sum_{k=\ell}^{\infty} a_k z^{-k}, \qquad a_\ell \neq 0 \tag{2.5}$$

for some $\ell \geq 1$.

We'll consider as a trial function

$$h_\epsilon(z) = f(z) + \epsilon \bar{a}_\ell z^\ell g(z) \tag{2.6}$$

where ϵ will be picked below. Since $f(\infty) \in (0,1)$, we can pick $\epsilon_0 > 0$ so that

$$f(\infty) + \epsilon_0 |a_\ell|^2 < 1 \tag{2.7}$$

Therefore, we can find $R > 0$ so that

$$|z| > R \Rightarrow |f(z)| + \epsilon_0 |a_\ell| |z^\ell g(z)| < 1 \tag{2.8}$$

Pick $\epsilon_1 > 0$ so that

$$\epsilon_1 < \epsilon_0, \qquad \epsilon_1 |a_\ell| R^\ell < 1 \tag{2.9}$$

We claim that $\|h_{\epsilon_1}\| \leq 1$, for by (2.8) if $|z| > R$, then $|h_{\epsilon_1}(z)| \leq 1$, and, if $|z| \leq R$, then by (2.9)

$$|h_{\epsilon_1}(z)| \leq |f(z)| + \epsilon_1 |a_\ell| R^\ell |g(z)| < |f(z)| + |g(z)| \leq 1$$

by (2.4). Thus h_{ϵ_1} is a trial function for the dual Widom problem.

On the other hand,

$$h_{\epsilon_1}(\infty) = f(\infty) + \epsilon_1 |a_\ell|^2 > f(\infty) \tag{2.10}$$

violating maximality. We conclude that $g \equiv 0$, so $k \equiv 0$, and $f_1 = f_2$. □

Proof of Theorem 1.4. Suppose we have a Totik–Widom bound

$$t_n \le D(C(\mathfrak{e}))^n \tag{2.11}$$

Given $\chi_\infty \in \pi_1(\Omega)^*$, pick $n_j \to \infty$ so that $\chi_{\mathfrak{e}}^{n_j}$, the character of $B_{\mathfrak{e}}^{n_j}$, converges to χ_∞ (which we can do by the assumption of canonical generator). Let

$$f_j(z) = \frac{T_{n_j}(z) B_{\mathfrak{e}}(z)^{n_j}}{C(\mathfrak{e})^{n_j}} \tag{2.12}$$

By the maximum principle,

$$\|f_j\|_\infty \le \sup_{z \to \mathfrak{e}} |f_j(z)| \le t_{n_j} C(\mathfrak{e})^{-n_j} \le D$$

so by Montel's theorem, we can find $j_k \to \infty$, so that f_{j_k} converges to f_∞ uniformly on compacts. Since T_{n_j} is monic and $B_{\mathfrak{e}}(z) = C(\mathfrak{e})/z + \mathrm{O}(z^{-2})$, we have $f_j(\infty) = 1$ and, therefore, f_∞ is non-zero. Clearly, $f_\infty \in H^\infty(\Omega, \chi_\infty)$. By Theorem 1.1, $\mathfrak{e}$ obeys a PW condition. □

3. Continuity of the Widom Minimizer

In this section, we study continuity properties (in χ) of $Q_\chi(z)$, $F_\chi(z)$ and $\|F_\chi\|_\infty$. We'll show there is continuity if and only if the DCT holds. Applying this to $n \to F_{\chi_{\mathfrak{e}}^n}$, we'll see that DCT implies almost periodicity.

Theorem 3.1. *Let $\mathfrak{e} \subset \mathbb{C}$ be a compact, PW and DCT set that is regular for potential theory. Then $\chi \mapsto Q_\chi$ and $\chi \mapsto F_\chi$ are continuous in the topology of uniform convergence on compact subsets of $\widetilde{\Omega}$. Moreover, $\chi \mapsto \|F_\chi\|_\infty$ is continuous. Conversely, if $\chi \mapsto \|F_\chi\|_\infty$ is continuous for $\mathfrak{e}$ a regular PW set, then $\mathfrak{e}$ is a DCT set.*

Proof. By Theorem 1.2, if $\mathfrak{e}$ is a DCT set, then $Q_\chi(\infty)$ is continuous. If $\chi_n \to \chi$ for some sequence so that Q_{χ_n} converges to a function g uniformly on compact subsets of $\widetilde{\Omega}$, then by continuity, $g(\infty) = Q_\chi(\infty)$ and $\|g\|_\infty \le 1$. It follows by uniqueness of the minimizer that $g = Q_\chi$. By Montel's Theorem, $\chi \mapsto Q_\chi$ is continuous. Since $F_\chi(z) = Q_\chi(z)/Q_\chi(\infty)$ and $\|F_\chi\|_\infty = 1/Q_\chi(\infty)$, we conclude continuity of F_χ and $\|F_\chi\|_\infty$.

The converse follows from Theorem 1.2 and $Q_\chi(\infty) = 1/\|F_\chi\|_\infty$ □

Theorem 3.2. *Let $\mathfrak{e} \subset \mathbb{C}$ be a compact, PW and DCT set that is regular for potential theory. Then $n \mapsto F_{\chi_\mathfrak{e}^n}(z)$ and $n \mapsto Q_{\chi_\mathfrak{e}^n}(z)$ are almost periodic uniformly for z in compact subsets of $\widetilde{\Omega}$. Moreover, $n \mapsto \|F_{\chi_\mathfrak{e}^n}\|_\infty$ is a bounded almost periodic function.*

Proof. Almost periodicity of a function, f, on $\mathbb{Z}$ can be defined in terms of the family $f_m \equiv f(\cdot - m)$ lying in a compact family of functions. Since $\pi_1(\Omega)^*$ is compact, $\{F_\chi\}_{\chi\in\pi_1(\Omega)^*}$ and $\{Q_\chi\}_{\chi\in\pi_1(\Omega)^*}$ are the required compact families. Since $Q_\chi(\infty)$ is a continuous function, it takes its minimum value which is always non-zero. Thus $Q_\chi(\infty)$ is bounded away from zero and thus, $\|F_\chi\|_\infty = 1/Q_\chi(\infty)$ is bounded. □

We now turn to the proof of Theorem 1.5. The first two of four lemmas require neither almost periodicity nor canonical generator. We'll focus on the dual maximizer, Q_χ, given by (1.20).

Lemma 3.3. *Let $\mathfrak{e}$ be a regular PW set. Then $\chi \mapsto Q_\chi(\infty)$, the map from $\pi_1(\Omega)^*$ to $(0,1]$, is upper semicontinuous, i.e.,*

$$\chi_j \to \chi \;\Rightarrow\; \limsup_{j\to\infty} Q_{\chi_j}(\infty) \le Q_\chi(\infty) \tag{3.1}$$

Proof. By Montel's theorem, we can always pick a subsequence so that $Q_{\chi_{j_n}}(\infty) \to \limsup_{j\to\infty} Q_{\chi_j}(\infty)$ and so that $Q_{\chi_{j_n}}$ has a pointwise limit, g, on the universal cover which has $\|g\|_\infty \le 1$ and for which the convergence is uniform on compact subsets of the universal cover. Since $\chi_{j_n} \to \chi$, g is a trial function for the dual Widom problem with character χ. Since Q_χ is a maximizer, $g(\infty) \le Q_\chi(\infty)$, i.e., (3.1) holds. □

Lemma 3.4. *Let $\mathfrak{e}$ be a regular PW set. If $\chi \mapsto Q_\chi(\infty)$ is continuous at $\chi = \mathbf{1}$ (i.e., we know that $\chi_j \to \mathbf{1} \Rightarrow Q_\chi(\infty) \to 1$), then $\chi \mapsto Q_\chi(\infty)$ is continuous on $\pi_1(\Omega)^*$.*

Proof. Suppose $\chi_j \to c$. Then $\chi_j/c \to \mathbf{1}$. Since $Q_c Q_{\chi_j/c}$ is a trial function for the χ_j dual maximizer problem, we have that

$$Q_c(\infty) Q_{\chi_j/c}(\infty) \le Q_{\chi_j}(\infty) \tag{3.2}$$

By hypothesis, $Q_{\chi_j/c}(\infty) \to 1$, so (3.2) implies that

$$Q_c(\infty) \le \liminf_{j\to\infty} Q_{\chi_j}(\infty). \tag{3.3}$$

This and (3.1) imply that $Q_{\chi_j}(\infty) \to Q_c(\infty)$. □

Lemma 3.5. *Let $\mathfrak{e}$ be a regular PW set. Suppose $n \mapsto \|F_n\|_\infty$ is a bounded almost periodic function and that $\chi_\mathfrak{e}^{n_j} \to \mathbf{1}$. Then $Q_{\chi_\mathfrak{e}^{n_j}} \to 1$.*

Proof. By hypothesis, there exists a compact additive group $\mathbb{K}$ and a bounded continuous function, B, on $\mathbb{K}$ so that $\mathbb{Z}$ is a dense subgroup in $\mathbb{K}$ and $B(n) = \|F_n\|_\infty$. Let $A(\alpha) = B(\alpha)^{-1}$ which is also continuous on $\mathbb{K}$, bounded away from 0 (and bounded above by 1) with

$$Q_{\chi_\mathfrak{e}^n}(\infty) = A(n) \tag{3.4}$$

By passing to a subsequence, we can suppose that $n_j \to \alpha \in \mathbb{K}$ and that $Q_{\chi_\mathfrak{e}^{n_j}}(\infty)$ has a limit q.

Fix n_s. By passing to a further subsequence, we can suppose that $Q_{\chi_\mathfrak{e}^{n_s - n_j}}$ has a limit, g, on the universal cover. Since $\chi_\mathfrak{e}^{n_j} \to \mathbf{1}$, g is a trial function for the $\chi_\mathfrak{e}^{n_s}$ problem so

$$Q_{\chi_\mathfrak{e}^{n_s}}(\infty) \geq g(\infty) = \lim_{n_j \to \infty} A(n_s - n_j) = A(n_s - \alpha) \tag{3.5}$$

by the continuity of A. Now take $n_s \to \infty$. By definition of q, we have

$$q = \lim_{n_s \to \infty} Q_{\chi_\mathfrak{e}^{n_s}}(\infty) \geq \limsup_{n_s \to \infty} A(n_s - \alpha) = A(0) = 1$$

since $n_s \to \alpha$ and $A(0) = 1$ by (3.4). Thus $q \geq 1$. Since $Q_\chi(\infty) \in (0, 1]$, we conclude that $q = 1$, i.e., 1 is the only limit point of $Q_{\chi_\mathfrak{e}^{n_j}}(\infty)$ proving the lemma. $\square$

Lemma 3.6. *Let $\mathfrak{e}$ be a regular PW set. Suppose that $n \to \|F_n\|_\infty$ is a bounded almost periodic function and that $\mathfrak{e}$ has a canonical generator. Then $\chi \mapsto Q_\chi(\infty)$ is continuous at $\chi = \mathbf{1}$, i.e.,*

$$\chi_j \to \mathbf{1} \;\Rightarrow\; \lim_{j \to \infty} Q_{\chi_j}(\infty) = 1 \tag{3.6}$$

Proof. $\pi_1(\Omega)^*$ is a compact, separable group, so metrizable. Let d be a metric on $\pi_1(\Omega)^*$ yielding the usual topology. Since $\{\chi_\mathfrak{e}^m\}$ is dense, we can pick integers $m_j(\ell)$ for each j and $\ell = 1, 2, \dots$ so that $d(\chi_j, \chi_\mathfrak{e}^{m_j(\ell)}) \leq 2^{-\ell}$.

By Lemma 3.3, we can pick $\ell_j \geq j$ so that

$$Q_{\chi_\mathfrak{e}^{m_j(\ell_j)}}(\infty) \leq Q_{\chi_j}(\infty) + 2^{-j} \tag{3.7}$$

Let $k(j) = m_j(\ell_j)$. Since $d(\mathbf{1}, \chi_\mathfrak{e}^{k(j)}) \leq d(\mathbf{1}, \chi_j) + 2^{-j}$, we see that $\chi_\mathfrak{e}^{k(j)} \to \mathbf{1}$, so by Lemma 3.5, $Q_{\chi_\mathfrak{e}^{k(j)}}(\infty) \to 1$. By (3.7), we conclude that $\liminf Q_{\chi_j}(\infty) \geq 1$. Since $Q_{\chi_j}(\infty) \in (0, 1]$, we conclude that the limit is 1. $\square$

Proof of Theorem 1.5. By the hypothesis, Lemma 3.6 applies, so we conclude that $\chi \mapsto Q_\chi(\infty)$ is continuous at $\mathbf{1}$. By Lemma 3.4, $\chi \mapsto Q_\chi(\infty)$ is continuous on all of $\pi_1(\Omega)^*$, so, by Theorem 1.2, the set $\mathfrak{e}$ is DCT. $\square$

4. Limit Points of M_n are Blaschke Products

In this section and the next, we consider the functions $M_n(z) = [B_{\mathfrak{e}}(z)/B_n(z)]^n$ of (1.29). Since $\mathfrak{e} \subset \mathfrak{e}_n$, we have that $G_n(z) \le G_{\mathfrak{e}}(z)$ so

$$|M_n(z)| \le 1 \tag{4.1}$$

$M_n(z)$ is analytic on the universal cover of $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}_n$. Since the harmonic measures of components of $\mathfrak{e}_n$ are j/n, $B_n(z)^n$ is single valued analytic on $\mathbb{C} \setminus \mathfrak{e}_n$, so $M_n(z)$ has character $\chi_n \equiv \chi_{\mathfrak{e}}^n$ for curves in $\widetilde{\Omega}$ that avoid $\mathfrak{e}_n$.

In this section, we'll prove that limit points of M_n (after removing some removable potential singular points) are Blaschke products analytic on $\widetilde{\Omega}$ and, in the next, that these Blaschke products are dual Widom maximizers. This section will only require that $\mathfrak{e} \subset \mathbb{R}$ is regular for potential theory and obeys a PW condition while the next will also require the DCT condition.

$\mathbb{R} \setminus \mathfrak{e}$ is a disjoint union of bounded open components (plus two unbounded components), $K \in \mathcal{G}$. We'll call these the gaps and $\mathcal{G}$ the set of gaps. A *gap collection* is a subset $\mathcal{G}_0 \subset \mathcal{G}$. A *gap set* is a gap collection, $\mathcal{G}_0$, and for each $K_k \in \mathcal{G}_0$ a point $x_k \in K_k$. For any gap $K = (\beta - \alpha, \beta + \alpha)$, we define

$$K^{(\epsilon)} = (\beta - (1-\epsilon)\alpha, \beta + (1-\epsilon)\alpha)$$

so that $K^{(\epsilon)} \subset K$ and $|K^{(\epsilon)}| = (1-\epsilon)|K|$.

For any gap set, S, we define the associated Blaschke product

$$B_S(z) = \prod_{K_k \in \mathcal{G}_0} B_{\mathfrak{e}}(z, x_k) \tag{4.2}$$

Lifted to $\mathbb{D}$, each $B_{\mathfrak{e}}(z, x_k)$ is a product of elementary Blaschke factors and thus, so is the product in (4.2). It is known ([18, Theorem 9.9.4]) that such products either converge to 0 uniformly on compacts, or else converge to an analytic function vanishing only at the individual zeros and, in the latter case, the product has $\lim_{r\uparrow 1} |B_S(\mathbf{x}(re^{i\theta}))| = 1$ for a.e. θ ([19, Theorem 5.3.1]). Since $\sum_{K\in\mathcal{G}} \sup_{y\in K} G_{\mathfrak{e}}(\infty, y) < \infty$ by the PW condition, we see that the product in (4.2) converges to a nonzero value at $z = \infty$. Thus $B_S(z)$ is an analytic function on $\widetilde{\Omega}$ which vanishes exactly at points w with $\pi(w) \in \{x_j\}_{K_j \in \mathcal{G}_0}$. Moreover, for a.e. point $y \in \mathfrak{e}$,

$$\lim_{\epsilon\downarrow 0} |B_S(y + i\epsilon)| = 1 \tag{4.3}$$

Recall ([4, (b) following Theorem 1.1]) that any Chebyshev polynomial, T_n, has at most one zero in any gap $K \in \mathcal{G}$. Our main result in this section is

Theorem 4.1. *Let $n_j \to \infty$ so that for some gap set, S, we have that if $K_k \in \mathcal{G}_0$, then for large j, $T_{n_j}(z)$ has a zero $z_j^{(k)}$ in K_k which converges to x_k as $j \to \infty$ and so that for any $K \in \mathcal{G} \backslash \mathcal{G}_0$, and for all $\epsilon > 0$, $T_{n_j}(z)$ has no zero in $K^{(\epsilon)}$ for all large j. Then, as $j \to \infty$, $M_{n_j}(z) \to B_S(z)$ uniformly on compact subsets of $\widetilde{\Omega} \setminus \{w \,|\, \pi(w) \in \{x_k\}\}$.*

Remarks. 1. The points w with $\pi(w) = x_k$ for some k are removable singular points for B_S. In fact, it is easy to see that while $M_{n_j}(x_k + i0)$ and $M_{n_j}(x_k - i0)$ may be different, both values converge to 0, so, in a certain sense, one has convergence on all of $\widetilde{\Omega}$.

2. By Montel's Theorem and (4.1), the functions M_n lie in a compact set in the Fréchet topology of uniform convergence on compact subsets. We can therefore make multiple demands and one might guess that, as in [4], we want to also demand that χ_{n_j} has a limit as does $[C(\mathfrak{e}_{n_j})/C(\mathfrak{e})]^{n_j}$ and the M_{n_j}. It turns out that the single condition on the limits of zeros will automatically imply these other objects converge.

We will prove this result by controlling convergence for z near ∞ using

Proposition 4.2. *Let Υ be a Riemann surface and U_n open sets so that for any compact set $K \subset \Upsilon$, eventually, $K \subset U_n$. Let f_n be analytic functions on U_n so that*

$$\sup_n \sup_{z \in U_n} |f_n(z)| < \infty \tag{4.4}$$

Let f_∞ be analytic on Υ so that for some $z_0 \in \Upsilon$ and some neighborhood, V, of z_0, we have that

$$\lim_{n\to\infty} |f_n(z)| = |f_\infty(z)| \text{ for all } z \in V \tag{4.5}$$

$$f_n(z_0) > 0, \qquad f_\infty(z_0) > 0 \tag{4.6}$$

$$z \in V \Rightarrow \forall n : f_n(z) \neq 0 \text{ and } f_\infty(z) \neq 0 \tag{4.7}$$

Then $f_n \to f$ uniformly on compact subsets of Υ.

Proof. By shrinking V, we can suppose that it is simply connected and $\overline{V}$ is compact. By (4.6)/(4.7), we can define $g_n(z) = \log f_n(z)$ uniquely if we demand that

$$\text{Im} g_n(z_0) = 0 \tag{4.8}$$

By (4.5), $\mathrm{Re} g_n \to \mathrm{Re} g_\infty$ on V so by the Cauchy–Riemann equations, $\nabla(\mathrm{Im} g_n) \to \nabla(\mathrm{Im} g_\infty)$. By (4.8), $\mathrm{Im} g_n \to \mathrm{Im} g_\infty$, so $f_n \to f_\infty$ on V. By Vitali's Theorem ([18, Section 6.2]) and (4.4), $f_n \to f_\infty$ uniformly on compacts. □

Thus instead of $M_n(z)$, we can look at

$$|M_n(z)| = \exp(-nh_n(z)), \qquad h_n(z) = G_{\mathfrak{e}}(z) - G_n(z) \tag{4.9}$$

Let $d\rho_n$ be the potential theoretic equilibrium measure of $\mathfrak{e}_n$ (see [19, Section 3.6–3.7] for background on potential theory). Then

Proposition 4.3. *One has that*

$$h_n(z) = \int_{\bigcup_{K_j \in \mathcal{G}} K_j} G_{\mathfrak{e}}(x, z) d\rho_n(x) \tag{4.10}$$

Remark. In [4], we proved the Totik–Widom bound (1.12) for PW sets, $\mathfrak{e} \subset \mathbb{R}$, by using this when $z = \infty$, i.e.,

$$h_n(\infty) = \int_{\bigcup_{K_j \in \mathcal{G}} K_j} G_{\mathfrak{e}}(x) d\rho_n(x)$$

We proved this by thinking of $d\rho_n$ as harmonic measure at ∞, i.e., if H is harmonic on $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}_n$ with boundary values $H(x)$ on $\mathfrak{e}_n$, then

$$H(\infty) = \int_{\mathfrak{e}_n} H(x) d\rho_n(x)$$

If we wrote the analog of this for general z, we'd get

$$H(z) = \int_{\mathfrak{e}_n} H(x) d\rho_n(x, z)$$

varying the harmonic measure. Instead we think of (4.10) with $G_{\mathfrak{e}}$ arising as the Green's function for solving Poisson's equation with zero boundary values on $\mathfrak{e}$ and $d\rho_n$ occurs as the Laplacian of G_n.

Proof. Both sides of (4.10) are continuous functions of $z \in \mathbb{C} \cup \{\infty\}$ (by regularity of $\mathfrak{e}$ and $\mathfrak{e}_n$) and both sides vanish on $\mathfrak{e}$. Off $\mathfrak{e}$, they have the same distributional Laplacian, namely $d\rho_n \restriction (\mathfrak{e}_n \setminus \mathfrak{e})$. Thus the difference is harmonic on $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$, continuous on $\mathbb{C} \cup \{\infty\}$, vanishing on $\mathfrak{e}$ and bounded near ∞. The boundedness means the difference is also harmonic at ∞ ([19, Theorem 3.1.26]) and then the maximum principle implies that the difference is 0. □

The final step in the proof of Theorem 4.1 involves the form as $n \to \infty$ of $d\rho_n \restriction K$ for $K \in \mathcal{G}$. Recall that $\mathfrak{e}_n$ is a union of n bands which are closures of the connected components of $T_n^{-1}[(-t_n, t_n)]$. On each of these, as x increases, T_n is either strictly monotone increasing or strictly decreasing from $-t_n$ to t_n or vice-versa. Recall also that each of the bands has ρ_n measure exactly $1/n$ (see [4, Thm. 2.3]). In [4], it is proven that each gap, K, contains all or part of a single band so that

$$n\rho_n(K) \le 1 \tag{4.11}$$

If there is $x_\infty \in K$ which is a limit as $j \to \infty$ of zeros, x_{n_j} of T_{n_j}, then for j large, $\mathfrak{e}_{n_j} \cap K$ is a complete band of exponentially small width so, in that case

$$n_j \rho_{n_j} \restriction K \to \delta_{x_\infty} \tag{4.12}$$

weakly. If for each ϵ, there is a large J_ϵ so if $j \ge J_\epsilon$, then T_{n_j} has no zero in $K^{(\epsilon)}$, then for all sufficiently large j, $\rho_{n_j}(K^{(\epsilon)}) = 0$. Since $G_\mathfrak{e}$ vanishes at the edges of K (and so $\sup_{x \in K \setminus K^{(\epsilon)}} G_\mathfrak{e}(x,z) \to 0$ as $\epsilon \downarrow 0$ uniformly as z runs through compact sets), we conclude that

$$n \int_K G_\mathfrak{e}(x,z) d\rho_n(x) \to \begin{cases} G_\mathfrak{e}(x_\infty, z), & \text{if } K \in \mathcal{G}_0 \\ 0, & \text{if } K \notin \mathcal{G}_0 \end{cases} \tag{4.13}$$

By the PW condition, $\sum_{K \in \mathcal{G}} \sup_{y \in K} G_\mathfrak{e}(z,y) < \infty$ uniformly in z on compacts, we can go from pointwise limits in (4.12) to limits on sums. We conclude that:

Proposition 4.4. *Under the hypotheses of Theorem 4.1, uniformly for z in compact subsets of $\Omega \setminus \{x_k\}_{K_k \in \mathcal{G}_0}$, we have that*

$$n \int_{\bigcup_{K_k \in \mathcal{G}} K_k} G_\mathfrak{e}(x,z) d\rho_n(x) \to \sum_{K_k \in \mathcal{G}_0} G_\mathfrak{e}(x_k, z) \tag{4.14}$$

Proof of Theorem 4.1. By (4.9), (4.10) and (4.14),

$$\lim_{n_j \to \infty} |M_{n_j}(z)| = \prod_{K_k \in \mathcal{G}_0} |B_\mathfrak{e}(z, x_k)| = |B_S(z)| \tag{4.15}$$

That $M_{n_j} \to B_S$ then follows from Proposition 4.2. □

5. Blaschke Products are Dual Widom Maximizers

Given the setup of Theorem 4.1, the function $B_S(z)$ is character automorphic with some character β. In this section, we'll prove that B_S is a dual Widom maximizer for character β. One can deduce this from results of Volberg–Yuditskii [23, Lemma 6.4]. Instead, we'll follow an approach of Eichinger–Yuditskii [5] (who study an Ahlfors problem rather than a dual Widom problem) that relies on results of Sodin–Yuditskii [21].

A basic technique of Sodin–Yuditskii is to consider the space, $\mathcal{H}_\alpha$, of all functions on $\widetilde{\Omega}$ which are in $H^2(\mathbb{D})$ when moved to $\mathbb{D}$ and which are character automorphic with character $\alpha \in \pi_1(\Omega)^*$. $\mathcal{H}_\alpha$ is a family of functions on $\widetilde{\Omega}$ which is a reproducing kernel Hilbert space ([17, Problems 4–11 of Section 3.3]) under the inner product of H^2. In particular, there is a function $K^\alpha \in \mathcal{H}_\alpha$ so that for all $f \in \mathcal{H}_\alpha$

$$f(\infty) = \langle K^\alpha, f\rangle \tag{5.1}$$

Note: Our inner products are linear in the second factor and anti-linear in the first as in [17].

We will prove

Theorem 5.1. *For any gap set, S, if B_S is the associated Blaschke product and β its character, then B_S is a dual Widom maximizer for β, i.e.,*

$$\|B_S\|_\infty = 1 \tag{5.2}$$

and if $f \in H^\infty(\Omega, \beta)$ with $\|f\|_\infty \leq 1$, then

$$|f(\infty)| \leq B_S(\infty) \tag{5.3}$$

(5.2) is, of course, true for any (convergent) Blaschke product. We prove (5.3) by proving two facts:

(1) For any character, γ, and $f \in H^\infty(\Omega, \beta)$ with $\|f\|_\infty \leq 1$, one has that

$$|f(\infty)|^2 \leq \frac{K^{\gamma\beta}(\infty)}{K^\gamma(\infty)} \tag{5.4}$$

(2) There exists at least one α_0 with

$$|B_S(\infty)|^2 = \frac{K^{\alpha_0\beta}(\infty)}{K^{\alpha_0}(\infty)} \tag{5.5}$$

Lemma 5.2. (5.4) *holds.*

Proof. Since $f \in H^\infty(\Omega, \beta)$ and $K^\gamma \in \mathcal{H}_\gamma$, we have that $fK^\gamma \in \mathcal{H}_{\gamma\beta}$. Thus

$$\begin{aligned} |f(\infty)K^\gamma(\infty)|^2 &= |\langle K^{\gamma\beta}, fK^\gamma\rangle|^2 \\ &\leq \|fK^\gamma\|_2^2 \|K^{\gamma\beta}\|_2^2 && (5.6) \\ &\leq \|K^\gamma\|_2^2 \|K^{\gamma\beta}\|_2^2 && (5.7) \\ &= \langle K^\gamma, K^\gamma\rangle \langle K^{\gamma\beta}, K^{\gamma\beta}\rangle \\ &= K^\gamma(\infty) K^{\gamma\beta}(\infty) && (5.8) \end{aligned}$$

which is (5.4) since $K^\gamma(\infty) > 0$. In the above, (5.6) is the Schwarz inequality, (5.7) uses $\|f\|_\infty \leq 1$ and (5.8) is (5.1). □

For step 2, we need a deep result of Sodin–Yuditskii. For each gap $K \in \mathcal{G}$, we define C_K to be two copies glued together at the ends, i.e., we take two copies $\{(y,+),(y,-) \mid y \in \overline{K}\}$ and for $y \in \partial K$ (two points), we set $(y,+) = (y,-)$ so C_K is topologically a circle. According to Sodin–Yuditskii [21], there is a map, $\mathfrak{A}$, the Abel map, from $\prod_{K\in\mathcal{G}} C_K$ to the character group, so that, in particular, the inner part of $K^{\mathfrak{A}(y,\sigma)}$ is B_S where S is the gap set with

$$\mathcal{G}_0 = \{K \mid (y_K, \sigma_K) \text{ has } \sigma_K = + \text{ and } y_K \in K\}$$

(i.e., $y_K \notin \partial K$) and for $K \in \mathcal{G}_0$, the point in K is y_K.

In particular, if S is given and $(y,\sigma) = \{(y_K,\sigma_K)\}_{K\in\mathcal{G}}$ is picked so that for $K_k \in \mathcal{G}_0$, we have that $(y_{K_k}, \sigma_{K_k}) = (x_k, +)$ (and for $K \notin \mathcal{G}_0$, (y_K, σ_K) is arbitrary in C_K), then the inner factor of $K^{\mathfrak{A}(y,\sigma)}$ is divisible by B_S, i.e., if $\alpha_1 = \mathfrak{A}(y,\sigma)$, then K^{α_1}/B_S is in $\mathcal{H}_{\alpha_0}$ where $\alpha_0 = \alpha_1\beta^{-1}$. If $g \in \mathcal{H}_{\alpha_0}$, then because multiplication by B_S is an isometry on H^2, we have that

$$\begin{aligned} \langle K^{\alpha_0\beta} B_S^{-1}, g\rangle &= \langle K^{\alpha_0\beta}, B_S g\rangle \\ &= B_S(\infty) g(\infty) && (5.9) \\ &= B_S(\infty)\langle K^{\alpha_0}, g\rangle && (5.10) \\ &= \langle \overline{B_S(\infty)} K^{\alpha_0}, g\rangle && (5.11) \end{aligned}$$

Since g is arbitrary in $\mathcal{H}_{\alpha_0}$ and both K^{α_0} and $K^{\alpha_0\beta}B_S^{-1}$ lie in $\mathcal{H}_{\alpha_0}$, we conclude that

$$K^{\alpha_0\beta}(z) B_S(z)^{-1} = \overline{B_S(\infty)} K^{\alpha_0}(z) \tag{5.12}$$

Evaluating at $z = \infty$, we find that

Lemma 5.3. (5.5) *holds for* $\alpha_0 = \alpha_1\beta^{-1}$ *where* α_1 *is the image under the Abel map of data* $\{(y_K,\sigma_K)\}_{K\in\mathcal{G}}$ *which has* $(y_{K_k}, \sigma_{K_k}) = (x_k, +)$ *if* $K_k \in \mathcal{G}_0$.

Proof of Theorem 5.1. By Lemmas 5.2 and 5.3, if $g \in H^\infty(\Omega, \beta)$ with $\|g\|_\infty \le 1$, then

$$|g(\infty)|^2 \le \frac{K^{\alpha_0\beta}(\infty)}{K^{\alpha_0}(\infty)} = |B_S(\infty)|^2 \tag{5.13}$$

Thus, if $g(\infty) > 0$, we have that

$$0 < g(\infty) \le B_S(\infty) \tag{5.14}$$

so B_S is a dual Widom maximizer. □

6. Proof of the Main Theorem

In this section, we'll prove Theorem 1.3.

Proposition 6.1. *Under the hypotheses of Theorem 4.1, we have that $L_{n_j}(z)$ (given by (1.28)) converges uniformly on compact subsets of $\widetilde{\Omega}$ to the Widom minimizer for the character, β, of B_S.*

Remark. M_n only converge away from the $\{x_k\}_{K_k \in \mathcal{G}_0}$ because the M_n's aren't analytic on $\widetilde{\Omega}$ but only on those points whose images under $\mathbf{x}$ aren't in $\mathfrak{e}_n$. But L_n is analytic on all of $\widetilde{\Omega}$ so we can hope for convergence at the x_k's too. Indeed, the x_k's are limit points of zeros and the Widom minimizers vanish at those points.

Proof. We have that $M_{n_j}(\infty) = \left[C(\mathfrak{e})/C(\mathfrak{e}_{n_j})\right]^{n_j}$, so by Theorem 4.1,

$$B_S(\infty) = \lim_{j\to\infty} \left[C(\mathfrak{e})/C(\mathfrak{e}_{n_j})\right]^{n_j} \tag{6.1}$$

Thus, if H_n is given by (1.31), then

$$H_{n_j}(z) \to B_S(z)/B_S(\infty) \tag{6.2}$$

for z near ∞ (in fact on compact subsets of $\widetilde{\Omega} \setminus \{w \mid \pi(w) \in \{x_k\}\}$).

Since B_S is the dual Widom maximizer for β, $B_S(z)/B_S(\infty)$ is F_β, the Widom minimizer for β. By the first equation in (1.31), we get that $L_{n_j}(z)$ converges to $F_\beta(z)$ for z near ∞.

By the Totik–Widom bound, $\|L_{n_j}\|_\infty$ are uniformly bounded, so by Vitali's Theorem, L_{n_j} converges to F_β uniformly on compact subsets of $\widetilde{\Omega}$. □

Proposition 6.2. *Under the hypotheses of Theorem 4.1, we have that*

$$\lim_{j\to\infty} \|L_{n_j}\|_\infty = 2\|F_\beta\|_\infty \tag{6.3}$$

Proof. Since $\log|L_{n_j}(z)|$ is harmonic on Ω away from those zeros of T_{n_j} in the gaps where it goes to $-\infty$, its maximum occurs at limit points on $\mathfrak{e}$. Since $|B_\mathfrak{e}(x)| = 1$ for $x \in \mathfrak{e}$, we conclude that

$$\|L_{n_j}\|_\infty = \frac{t_{n_j}}{C(\mathfrak{e})^{n_j}} = \frac{2C(\mathfrak{e}_{n_j})^{n_j}}{C(\mathfrak{e})^{n_j}} \tag{6.4}$$

by (1.8)

By (6.1), we conclude that

$$\lim_{j\to\infty} \|L_{n_j}\|_\infty = 2\left[B_S(\infty)\right]^{-1} \tag{6.5}$$

and by (1.20), noting that $Q_\beta = B_S$,

$$\left[B_S(\infty)\right]^{-1} = \|F_\beta\|_\infty \tag{6.6}$$

proving (6.3). □

Proof of Theorem 1.3. By Theorem 3.2, we have the required almost periodicity of $F_n(z)$ and $\|F_n\|_\infty$. By continuity of $\|F_\chi\|_\infty$ and the Totik–Widom bound, the functions on the left of (1.22) lie in a compact set, so if the limit is not zero, by passing to suitable subsequences, we can find one whose limit is zero for which the hypotheses of Theorem 4.1 hold. But then the limit is zero by Proposition 6.1. We conclude that (1.22) holds.

Again, by continuity of $\|F_\chi\|_\infty$ and the Totik–Widom bound, the numbers on the left side of (1.23) are bounded above and away from zero, so if (1.23) fails we can find a subsequence for which the limit is not 2 and for which the hypotheses of Theorem 4.1 hold. This violates Proposition 6.2 so we conclude that (1.23) holds. □

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