CORRELATION KERNELS FOR DISCRETE SYMPLECTIC AND ORTHOGONAL ENSEMBLES

ALEXEI BORODIN AND EUGENE STRAHOV

ABSTRACT. In [43] H. Widom derived formulae expressing correlation functions of orthogonal and symplectic ensembles of random matrices in terms of orthogonal polynomials. We obtain similar results for discrete ensembles with rational discrete logarithmic derivative, and compute explicitly correlation kernels associated to the classical Meixner and Charlier orthogonal polynomials.

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1. INTRODUCTION

The present paper addresses the following problem. Let $w(x)$ be a classical discrete weight function, and consider discrete symplectic and orthogonal ensembles associated to these weights, see Sections 2 and 3 for precise definitions. By methods of Random Matrix Theory (see e.g. Tracy and Widom [41]) we find m -point correlations in terms of Pfaffians with 2×2 matrix kernels. The problem is to express these kernels in terms of the orthogonal polynomials associated with our weight function.

It is well known that Pfaffian expressions including 2×2 matrix kernels appear in analysis of the orthogonal and the symplectic ensembles of Random Matrix Theory, see, for example, Mehta [26], Forrester [17], Tracy and Widom [41], Borodin and Strahov [9]. Such kernels also play a role in the works on the crossover between matrix symmetries, see Pandey and Mehta [37], Mehta and Pandey [39], Nagao and Forrester [32], superpositions of matrix ensembles, see Forrester and Rains [21, 22], and also in the multi-matrix models combining matrices of different symmetries, see Nagao [27, 29]. Forrester and Nagao [18], Borodin and Sinclair [8] give 2×2 matrix kernels for ensembles of asymmetric real matrices, in particular, for the eigenvalue statistics of Real Ginibre Ensemble. Vicious random walkers, random involutions, Pfaffian Schur process are examples of problems from combinatorics and statistical physics where Pfaffian formulas including 2×2 matrix kernels arise, see Nagao and Forrester [33], Nagao, Katori and Tanemura [34], Nagao [28], Forrester, Nagao and Rains [19], Borodin and Rains [4], Vuletić [42].

Often these kernels are constructed in terms of skew-orthogonal polynomials. Then a question arises how to compute the skew-orthogonal polynomials, and how to handle the Christoffel-Darboux sums involving them. In some cases explicit formulas for skew-orthogonal polynomials have been given in terms of related orthogonal polynomials, and matrix kernels and their asymptotic values have been computed. This approach is developed in Nagao and Wadati [35], Brézin and Neuberger [10], Adler, Forrester, Nagao and P. van Moerbeke [1], see also Forrester [17], Chapter 5. Nagao [29] provides the list of the cases when the expressions of skew orthogonal polynomials in terms of the classical orthogonal polynomials are explicitly known, see Table 1.

Our definitions of discrete symplectic and orthogonal ensembles in Sections 2 and 3 are motivated by relations with z -measures on Young diagrams with the Jack parameter $\theta = 2$, as it is described in Section 4. For ensembles obtained in Section 4 skew-orthogonal polynomials are not known, and we use a discrete version of the method developed by Widom in [43] in the context of orthogonal and symplectic ensembles of Hermitian matrices. Widom [43] gives general formulas expressing entries of 2×2 matrix kernels in terms of the scalar kernels for the corresponding unitary ensembles. Whenever the logarithmic derivative of the weight in the definition of orthogonal or symplectic ensemble under considerations is a rational function, the entries of the 2×2 matrix kernels are expressible in terms of orthogonal polynomials, and are equal to the scalar kernel plus extra terms. Similar results for ensembles with Laguerre-type weights were obtained in physical literature, see Sener and Verbaarschot [38], Klein and Verbaarschot [24]. These papers show

that the number of extra terms is finite, which leads to universality of correlation kernels for such ensembles.

Formulae obtained in Widom [43] are especially convenient for the asymptotic analysis since the asymptotics of polynomials associated to rather general classes of weights is known, see Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou [12, 13], Bleher and Its [2]. This enables one to use Widom's formulae in the proof of the universality for the orthogonal and symplectic ensembles, see Deift and Gioev [14, 15], Deift, Gioev, Kriecherbauer, and Vanlessen [16], Stojanovic [40].

The weight defining the families	The orthogonality set	The family of the orthogonal polynomials
$w(x) = e^{-x^2}$	$(-\infty, +\infty)$	Hermite polynomials (see [1, 10, 35])
$w(x) = x^a e^{-x}$	$(0, +\infty)$	Laguerre polynomials (see [1, 30])
$w(x) = (1-x)^a(1-x)^b$	$(-1, 1)$	Jacobi polynomials (see [1, 30, 35])
$w(x) = \left[\left(\frac{L}{2} + x\right)! \left(\frac{L}{2} - x\right)!\right]^{-2}$ L is an even integer	$\mathbb{Z}$	Hahn polynomials (see [33])
$w(x) = 1$	$\{0, 1, \dots, L\}$	Hahn polynomials (see [29])
$w(x) = q^x$	$\mathbb{Z}_{\geq 0}$	Meixner polynomials $M_n(x; c = 1, q)$ (see [19])

Table 1. The cases in which skew-orthogonal polynomials are explicitly known.

Our results for discrete symplectic and orthogonal ensembles are of the similar kind as those obtained in Widom [43]. Whenever a discrete analog of the logarithmic derivative of the weight is a rational function the matrix kernels are expressible in terms of the orthogonal polynomials associated to weight in the definition of the ensemble. This is used to work out the cases of the Charlier ensemble ($w(x) = \frac{a^x}{x!}$, $x \in \mathbb{Z}_{\geq 0}$), and the Meixner

ensemble ($w(x) = \frac{(\beta)_x}{x!} c^x$, $x \in \mathbb{Z}_{\geq 0}$). As an application, we compute the continuous limit of our formulas corresponding to the degeneration of the Meixner orthogonal polynomials to the Laguerre orthogonal polynomials.

Acknowledgements. We thank Percy Deift for discussions. This work was supported in part by BSF grant BSF-2006333. The first named author (A.B.) is grateful to Dimitri Gioev for helpful discussion during early stages of this work. He was also partially supported by the NSF grant DMS-0707163.

2. MAIN RESULTS FOR DISCRETE SYMPLECTIC ENSEMBLES

Let $w(x)$ be a strictly positive real valued function defined on $\mathbb{Z}_{\geq 0}$ with finite moments, i.e. the series $\sum_{x \in \mathbb{Z}_{\geq 0}} w(x)x^j$ converges for all $j = 0, 1, \dots$

Definition 2.1. The N -point discrete symplectic ensemble with the weight function w and the phase space $\mathbb{Z}_{\geq 0}$ is the random N -point configuration in $\mathbb{Z}_{\geq 0}$ such that the probability of a particular configuration $x_1 < \dots < x_N$ is given by

$$\text{Prob}\{x_1, \dots, x_N\} = Z_{N4}^{-1} \prod_{i=1}^N w(x_i) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 (x_i - x_j - 1)(x_i - x_j + 1).$$

Here Z_{N4} is a normalization constant which is assumed to be finite.

In what follows Z_{N4} is referred to as the partition function of the discrete symplectic ensemble under considerations.

Introduce a collection $\{P_n(\zeta)\}_{n=0}^{\infty}$ of complex polynomials which is the collection of orthogonal polynomials associated to the weight function w , and to the orthogonality set $\mathbb{Z}_{\geq 0}$. Thus

- P_n is a polynomial of degree n for all $n = 1, 2, \dots$, and $P_0 \equiv \text{const.}$
- If $m \neq n$, then

$$\sum_{x \in \mathbb{Z}_{\geq 0}} P_m(x) P_n(x) w(x) = 0.$$

For each $n = 0, 1, \dots$ set $\varphi_n(x) = (P_n, P_n)_w^{-1/2} P_n(x) w^{1/2}(x)$, where $(\cdot, \cdot)_w$ denotes the following inner product on the space $\mathbb{C}[\zeta]$ of all complex polynomials:

$$(f(\zeta), g(\zeta))_w := \sum_{x \in \mathbb{Z}_{\geq 0}} f(x) g(x) w(x).$$

We call φ_n the normalized functions associated to the orthogonal polynomials P_n .

Let $\mathcal{H}$ be the space spanned by the functions $\varphi_0, \varphi_1, \dots$

Definition 2.2. Suppose that there is a 2×2 matrix valued kernel $K_{N4}(x, y)$, $x, y \in \mathbb{Z}_{\geq 0}$, such that for a general finitely supported function η defined on $\mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} Z_{N4}^{-1} \sum_{(x_1 < \dots < x_N) \subset \mathbb{Z}_{\geq 0}} \prod_{i=1}^N w(x_i) (1 + \eta(x_i)) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 (x_i - x_j - 1)(x_i - x_j + 1) \\ = \sqrt{\det(I + \eta K_{N4})}, \end{aligned}$$

where K_{N4} is the operator associated to the kernel $K_{N4}(x, y)$, and η is the operator of multiplication by the function η . $K_{N4}(x, y)$ is called the correlation kernel of the discrete symplectic ensemble defined by the weight function $w(x)$ on the phase space $\mathbb{Z}_{\geq 0}$.

An explanation of the term “correlation kernel” can be found in Tracy and Widom [41], §2, 3.

We introduce the operators D_+ , D_- and ϵ which act on the elements of the space $\mathcal{H}$. The first and the second operators, D_+ and D_- , are defined by the expression:

$$(D_{\pm} f)(x) = \sum_{y \in \mathbb{Z}_{\geq 0}} D_{\pm}(x, y) f(y),$$

where the kernels $D_{\pm}(x, y)$ are given explicitly by

$$(2.1) \quad D_+(x, y) = \sqrt{\frac{w(x)}{w(x+1)}} \delta_{x+1, y}, \quad x, y \in \mathbb{Z}_{\geq 0},$$

$$(2.2) \quad D_-(x, y) = \sqrt{\frac{w(x-1)}{w(x)}} \delta_{x-1, y}, \quad x, y \in \mathbb{Z}_{\geq 0}.$$

The third operator, ϵ , is defined by the formula

$$\begin{aligned} (2.3) \quad (\epsilon \varphi)(2m) &= - \sum_{k=m}^{+\infty} \sqrt{\frac{w(2m)}{w(2k+1)}} \frac{w(2m+1)w(2m+3) \dots w(2k+1)}{w(2m)w(2m+2) \dots w(2k)} \varphi(2k+1), \\ (\epsilon \varphi)(2m+1) &= \sum_{k=0}^m \sqrt{\frac{w(2k)}{w(2m+1)}} \frac{w(2k+1)w(2k+3) \dots w(2m+1)}{w(2k)w(2k+2) \dots w(2m)} \varphi(2k), \end{aligned}$$

where $m = 0, 1, \dots$

To make sure that $\epsilon \varphi$ is well defined for any $\varphi \in \mathcal{H}$, we impose an additional assumption on the weight function w : we assume that

$$\frac{w(x-1)}{w(x)} = \frac{d_1(x)}{d_2(x)}, \quad x \geq 1,$$

for some polynomials d_1 and d_2 such that $\deg d_1 \geq \deg d_2$ and if $\deg d_1 = \deg d_2$ then $\lim_{x \rightarrow \infty} d_1(x)/d_2(x) > 1$. This implies, in particular, that $w(x-1)/w(x) > \text{const} > 1$ for

$x \gg 1$, and one easily verifies that the series defining $(\epsilon\varphi)(x)$ converges for any $\varphi \in \mathcal{H}$ and $x \in \mathbb{Z}_{\geq 0}$.

Let us also introduce the operator S_{N4} which acts in the same space $\mathcal{H}$, and whose kernel is $S_{N4}(x, y)$. To write down $S_{N4}(x, y)$ explicitly, introduce $2N \times 2N$ matrix $M^{(4)}$ whose j, k entry ($j, k = 0, 1, \dots, 2N - 1$) is

$$(2.4) \quad M_{jk}^{(4)} = \sum_{x \in \mathbb{Z}_{\geq 0}} \varphi_j(x) (D\varphi_k)(x),$$

where $D := D_+ - D_-$.

Proposition 2.3. *The matrix $M^{(4)}$ is invertible.*

All the proofs are delayed until Section 5.

Write $(M^{(4)})^{-1} = (\mu_{jk}^{(4)})$, and define the kernel $S_{N4}(x, y)$ by the formula:

$$(2.5) \quad S_{N4}(x, y) = \sum_{j,k=0}^{2N-1} \varphi_j(x) \mu_{jk}^{(4)} \varphi_k(y),$$

where $x, y \in \mathbb{Z}_{\geq 0}$.

Theorem 2.4. *The operator K_{N4} (see Definition 2.2) is expressible as*

$$K_{N4} = \begin{bmatrix} D_+ S_{N4} & -D_+ S_{N4} D_- \\ S_{N4} & -S_{N4} D_- \end{bmatrix}.$$

Remark 2.5. As it is clear from the proof, the operator K_{N4} for the discrete symplectic ensemble can be also represented as

$$(2.6) \quad K_{N4} = \begin{bmatrix} \nabla_+ S_{N4} & -\nabla_+ S_{N4} \nabla_- \\ S_{N4} & -S_{N4} \nabla_- \end{bmatrix}.$$

In the formula just written above the operators ∇_+, ∇_- are defined by

$$\begin{aligned} (\nabla_+ f)(x) &= \sqrt{\frac{w(x)}{w(x+1)}} (f(x+1) - f(x)), \\ (\nabla_- f)(x) &= \sqrt{\frac{w(x-1)}{w(x)}} (f(x) - f(x-1)). \end{aligned}$$

Let $\mathcal{H}_N$ be the subspace of $\mathcal{H}$ spanned by the functions $\varphi_0, \varphi_1, \dots, \varphi_{2N-1}$. Denote by K_N the projection operator onto $\mathcal{H}_N$. Its kernel is

$$K_N(x, y) = \sum_{k=0}^{2N-1} \varphi_k(x) \varphi_k(y).$$

It is convenient to enlarge the domains of D and ϵ , and to consider the operators

$$D : \mathcal{H} + \epsilon\mathcal{H} \rightarrow \mathcal{H} + D\mathcal{H},$$

$$\epsilon : \mathcal{H} + D\mathcal{H} \rightarrow \mathcal{H} + \epsilon\mathcal{H}.$$

It is not hard to check that these operators are mutual inverse. Denote by $D_{\mathcal{H}_N}$ the restriction of the operator D to $\mathcal{H}_N$.

Theorem 2.6. *The following operator identity holds true*

$$D_{\mathcal{H}_N} S_{N4} = (I_{\mathcal{H}_N + D\mathcal{H}_N} - [D, K_N] K_N \epsilon)^{-1} K_N.$$

The next Theorem gives the condition on the weight function $w(x)$ under which the operator S_{N4} can be written in an explicit form.

Theorem 2.7. *Let $w(x)$ be a weight function such that for $x \geq 1$*

$$\frac{w(x-1)}{w(x)} = \frac{d_1(x)}{d_2(x)},$$

where d_1, d_2 are polynomials of degree at most m satisfying the assumption above, and $d_1(0) = 0, d_2(0) \neq 0$. Then

$$(2.7) \quad [D, K_N] K_N = \sum_{i=1}^n \tilde{\psi}_i \otimes \psi_i,$$

where $a \otimes b$ denotes the operator with the kernel $a(x)b(y)$, $\psi_1, \dots, \psi_n$ are elements of $\mathcal{H}_N$, and $\tilde{\psi}_1, \dots, \tilde{\psi}_n$ are elements of $\mathcal{H}_N^\perp$. Assume in addition that the matrix $T_{ij} = \delta_{ij} + (\epsilon\psi_i, \tilde{\psi}_j)$, $i, j = 1, \dots, n$ is invertible. Then

$$(2.8) \quad S_{N4} = \epsilon K_N - \sum_{i,j=1}^n (T^{-1})_{ij} (\epsilon\tilde{\psi}_i) \otimes (K_N \epsilon\psi_j).$$

Set $d_2(x) = \text{const} \cdot (x - a_1)^{n_1} \dots (x - a_l)^{n_l}$, and let n_∞ be the order of $\frac{w(x-1)}{w(x)}$ at ∞ . As will be clear from the proof of Theorem 2.7, the number n from (2.7) is bounded by $n_\infty + \sum_{i=1}^l n_{a_i}$. In Proposition 10.1 we show that $n = 1$ implies $T = 1$.

Corollary 2.8. *If the commutation relation between the operators D and K_N takes the form*

$$[D, K_N] = \lambda(\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1),$$

where $\psi_1 \in \mathcal{H}_N, \psi_2 \in \mathcal{H}_N^\perp$, and λ is some constant, then

$$S_{N4} = \epsilon K_N - \lambda \epsilon \psi_2 \otimes \epsilon \psi_1.$$

The general formalism described above can be applied in particular to the Meixner and to the Charlier symplectic ensembles. The weight function for the Meixner symplectic ensemble is by definition the weight function associated to the Meixner orthogonal polynomials:

$$(2.9) \quad w_{\text{Meixner}}(x) = \frac{(\beta)_x}{x!} c^x, \quad x \in \mathbb{Z}_{\geq 0},$$

where β is a strictly positive real parameter, and $0 < c < 1$. The weight function of the Charlier symplectic ensemble is defined by

$$(2.10) \quad w_{\text{Charlier}}(x) = \frac{a^x}{x!}, \quad x \in \mathbb{Z}_{\geq 0},$$

where $a > 0$. w_{Charlier} is the weight function defining the classical Charlier orthogonal polynomials¹.

Theorem 2.9. *a) If $w(x)$ is the Meixner weight with the parameters c and β defined by equation (2.9), then the operator S_{N4} whose kernel is defined by equation (2.5) takes the following form:*

$$S_{N4} = \epsilon K_N + \frac{\sqrt{2N(2N + \beta - 1)}}{(1 - c)\sqrt{c}} (\epsilon\psi_2) \otimes (\epsilon\psi_1),$$

where the operator K_N has the kernel

$$(2.11) \quad K_N(x, y) = -\frac{\sqrt{2Nc(2N + \beta - 1)}}{1 - c} \frac{\varphi_{2N}(x)\varphi_{2N-1}(y) - \varphi_{2N-1}(x)\varphi_{2N}(y)}{x - y},$$

the functions $\{\varphi_k(x)\}_{k=0}^{\infty}$ are the normalized functions associated to the Meixner orthogonal polynomials, the operator ϵ acts by the formula

$$\begin{aligned} (\epsilon\varphi)(2m) &= -\sqrt{c} \sum_{l=0}^{+\infty} \sqrt{\frac{(\frac{\beta}{2} + m)_{l+1}}{(\frac{\beta+1}{2} + m)_l} \frac{(m+1)_l}{(m + \frac{1}{2})_{l+1}}} \varphi(2l + 2m + 1), \\ (\epsilon\varphi)(2m + 1) &= \sqrt{c} \sum_{l=0}^m \sqrt{\frac{(-\frac{\beta}{2} - m)_{l+1}}{(-\frac{\beta-1}{2} - m)_l} \frac{(-m)_l}{(-m - \frac{1}{2})_{l+1}}} \varphi(2m - 2l), \end{aligned}$$

where $m = 0, 1, \dots$, and the functions ψ_1, ψ_2 are defined by the expressions

$$(2.12) \quad \psi_1(x) = \sqrt{2Nc} \frac{\varphi_{2N}(x)}{x + \beta} - \sqrt{2N + \beta - 1} \frac{\varphi_{2N-1}(x)}{x + \beta},$$

$$(2.13) \quad \psi_2(x) = \sqrt{2N + \beta - 1} \frac{\varphi_{2N}(x)}{x + \beta - 1} - \sqrt{2Nc} \frac{\varphi_{2N-1}(x)}{x + \beta - 1}.$$

b) If $w(x)$ is the Charlier weight with the parameter a (see equation (2.10)), then the operator S_{N4} whose kernel is defined by equation (2.5) takes the following form:

$$S_{N4} = \epsilon K_N + \sqrt{\frac{2N}{a}} (\epsilon\varphi_{2N}) \otimes (\epsilon\varphi_{2N-1}),$$

where the operator K_N has the kernel

$$K_N(x, y) = -\sqrt{2Na} \frac{\varphi_{2N}(x)\varphi_{2N-1}(y) - \varphi_{2N-1}(x)\varphi_{2N}(y)}{x - y},$$

¹For definitions and basic properties of the classical discrete orthogonal polynomials see Ismail [23], and also Koekoek and Swarttouw [25]

the functions $\{\varphi_k(x)\}_{k=0}^\infty$ are the normalized functions associated to the Charlier orthogonal polynomials, and the operator ϵ acts as follows:

$$\begin{aligned} (\epsilon\varphi)(2m) &= -\sqrt{\frac{a}{2}} \sum_{l=0}^{+\infty} \frac{(m+1)_l}{(m+\frac{1}{2})_{l+1}} \varphi(2l+2m+1), \\ (\epsilon\varphi)(2m+1) &= \sqrt{\frac{a}{2}} \sum_{l=0}^m \frac{(-m)_l}{(-m-\frac{1}{2})_{l+1}} \varphi(2m-2l). \end{aligned}$$

3. MAIN RESULTS FOR DISCRETE ORTHOGONAL ENSEMBLES

Definition 3.1. The $2N$ -point discrete orthogonal ensemble with the weight function W and the phase space $\mathbb{Z}_{\geq 0}$ is the random $2N$ -point configuration in $\mathbb{Z}_{\geq 0}$ such that the probability of a particular configuration $x_1 < \dots < x_{2N}$ is given by

$$\text{Prob}\{x_1, \dots, x_{2N}\} = \begin{cases} Z_{N1}^{-1} \prod_{i=1}^{2N} W(x_i) \prod_{1 \leq i < j \leq 2N} (x_j - x_i), & \text{if } x_i - x_{i-1} \text{ is odd for any } i, \text{ and } x_1 \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Here Z_{N1} is a normalization constant.

In what follows we assume that the weight function $W(x)$ is such that

$$(3.1) \quad W(x-1)W(x) = w(x), \text{ for } x \geq 1, \text{ and } W(0) = w(0),$$

where $w(x)$ is a strictly positive real valued function on $\mathbb{Z}_{\geq 0}$ satisfying the same conditions as the weight function in the definition of the discrete symplectic ensemble in Section 2.

Definition 3.2. Suppose that there is a 2×2 matrix valued kernel $K_{N1}(x, y)$, $x, y \in \mathbb{Z}_{\geq 0}$, such that for an arbitrary finitely supported function η defined on $\mathbb{Z}_{\geq 0}$ we have

$$Z_{N1}^{-1} \sum_{(x_1 < \dots < x_{2N}) \subset \mathbb{Z}_{\geq 0}} \prod_{i=1}^{2N} W(x_i)(1 + \eta(x_i)) \prod_{1 \leq i < j \leq 2N} (x_j - x_i) = \sqrt{\det(I + \eta K_{N1})},$$

where K_{N1} is the operator associated with the kernel $K_{N1}(x, y)$, and η is the operator of multiplication by the function η . Then the kernel of K_{N1} is called the correlation kernel of the discrete orthogonal ensemble defined by the weight function $W(x)$ on the phase space $\mathbb{Z}_{\geq 0}$.

As in the symplectic case, details on the correlation functions can be found in Tracy and Widom [41], §2, 3.

Let ϵ and D be as in the previous section, and let $w(x)$ in the definitions of these operators be given in terms of the weight function $W(x)$ by formula (3.1). Introduce the operator S_{N1} which acts in the same space $\mathcal{H}$, and whose kernel is $S_{N1}(x, y)$. To

write down $S_{N1}(x, y)$ explicitly introduce $2N \times 2N$ matrix $M^{(1)}$ whose j, k entry ($j, k = 0, 1, \dots, 2N-1$) is

$$(3.2) \quad M_{jk}^{(1)} = \sum_{x, y \in \mathbb{Z}_{\geq 0}} \epsilon(x, y) \varphi_j(x) \varphi_k(y).$$

Proposition 3.3. *The matrix $M^{(1)}$ is invertible.*

Write $(M^{(1)})^{-1} = (\mu_{jk}^{(1)})$, and define the kernel $S_{N1}(x, y)$ by the formula:

$$(3.3) \quad S_{N1}(x, y) = \sum_{j, k=0}^{2N-1} \varphi_j(x) \mu_{jk}^{(1)} \varphi_k(y),$$

where $x, y \in \mathbb{Z}_{\geq 0}$.

Theorem 3.4. *The operator K_{N1} (see Definition 3.2) is expressible as*

$$K_{N1} = \begin{bmatrix} S_{N1}\epsilon & S_{N1} \\ \epsilon S_{N1}\epsilon - \epsilon & \epsilon S_{N1} \end{bmatrix}.$$

Denote by $\epsilon_{\mathcal{H}_N}$ the restriction of the operator ϵ to $\mathcal{H}_N$. Recall that K_N is the projection operator on $\mathcal{H}_N$.

Theorem 3.5. *The following operator identity holds true*

$$\epsilon_{\mathcal{H}_N} S_{N1} = (I_{\mathcal{H}_N + \epsilon \mathcal{H}_N} - [\epsilon, K_N] K_N D)^{-1} K_N.$$

Theorem 3.6. *Let $w(x)$ be as in Theorem 2.7. Then*

$$(3.4) \quad [\epsilon, K_N] K_N = \sum_{i=1}^n \tilde{\eta}_i \otimes \eta_i,$$

where $\eta_1, \dots, \eta_n$ are elements of $\mathcal{H}_N$, and $\tilde{\eta}_1, \dots, \tilde{\eta}_n$ are elements of $\mathcal{H}_N^\perp$. Assume in addition that the matrix $U_{ij} = \delta_{ij} + (D\eta_i, \tilde{\eta}_j)$, $i, j = 1, \dots, n$, is invertible. Then

$$(3.5) \quad S_{N1} = DK_N - \sum_{i,j=1}^n (U^{-1})_{ij} (D\tilde{\eta}_i) \otimes (K_N D\eta_j).$$

The number of terms in (3.4) is bounded in the same way as n in Theorem 2.7.

Corollary 3.7. *If the commutation relation between the operators D and K_N takes the form*

$$[D, K_N] = \lambda(\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1),$$

where $\psi_1 \in \mathcal{H}_N, \psi_2 \in \mathcal{H}_N^\perp$, and λ is some constant, then

$$S_{N1} = DK_N - \lambda \psi_2 \otimes \psi_1.$$

Theorem 3.8. *For Meixner or Charlier weight (see equations (2.10) and (2.9)), we have the following expressions for the operator S_{N1} :*

a) *For the Meixner orthogonal ensemble*

$$(3.6) \quad S_{N1} = DK_N + \frac{\sqrt{2N(2N + \beta - 1)}}{(1 - c)\sqrt{c}} \psi_2 \otimes \psi_1,$$

where the functions ψ_1 and ψ_2 are defined by equations (2.12) and (2.13) respectively.

b) *For the Charlier orthogonal ensemble*

$$(3.7) \quad S_{N1} = DK_N + \frac{\sqrt{2N}}{a} \varphi_{2N} \otimes \varphi_{2N-1}.$$

4. DISCRETE SYMPLECTIC AND ORTHOGONAL ENSEMBLES RELATED WITH z -MEASURES

Take $z, z' \in \mathbb{C}$, $\theta > 0$, $0 < \xi < 1$, and define a distribution on the set of all Young diagrams by

$$M_{z,z',\theta,\xi}(\lambda) = (1 - \xi)^{t\xi|\lambda|} \frac{(z)_{\lambda,\theta}(z')_{\lambda,\theta}}{H(\lambda,\theta)H'(\lambda,\theta)}.$$

We have used the following notation: $t = zz'/\theta$;

$$(z)_{\lambda,\theta} = \prod_{(i,j) \in \lambda} (z + (j - 1) - (i - 1)\theta),$$

where the product is taken over all boxes in a Young diagram λ , (i, j) stands for the box in i th row and j th column; $|\lambda|$ is the number of boxes in λ ;

$$H(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + 1),$$

$$H'(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + \theta),$$

where λ' denotes the transposed diagram. One can show that $\sum_{\lambda \in \mathbb{Y}} M_{z,z',\theta,\xi}(\lambda) = 1$, where the sum is over the $\mathbb{Y}$ of all Young diagrams. If, for example, $z' = \bar{z}$, then all the weights are nonnegative, and we obtain a probability distribution on the set of all Young diagrams. $M_{z,z',\theta,\xi}(\lambda)$ is called the z -measure. Details and explanations of importance of z -measures in representation theory can be found in Borodin and Olshanski [6], Olshanski [36].

Proposition 4.1. *For $N = 1, 2, \dots$ let $\mathbb{Y}(N) \subset \mathbb{Y}$ denote the set of diagrams λ with $l(\lambda) \leq N$. Under the bijection between diagrams $\lambda \in \mathbb{Y}(N)$ and N -point configurations on $\mathbb{Z}_{\geq 0}$ defined by*

$$\lambda \longleftrightarrow x_{N-i+1} = \lambda_i - 2i + 2N \quad (i = 1, \dots, N)$$

the z -measure with parameters $z = 2N$, $\theta = 2$, $z' = 2N + \beta - 2$ turns into

$$\text{Prob} \{x_1, \dots, x_N\} = \text{const} \cdot \prod_{i=1}^N \frac{(\beta)_{x_i}}{x_i!} \xi^{x_i} \prod_{1 \leq i \leq j \leq N} (x_i - x_j)^2 (x_i - x_j - 1)(x_i - x_j + 1),$$

which is precisely the discrete symplectic ensemble with the Meixner weight in the sense of Definition 2.1.

Proof. The proof is a straightforward computation based on the application of the explicit formulae for $H(\lambda; 2)H'(\lambda; 2)$, see the proof of Lemma 3.5 in [6], and $(z)_{\lambda, \theta}$, see Section 1 in [6]. $\square$

Proposition 4.2. For $N = 1, 2, \dots$ let $\mathbb{Y}'(2N) \subset \mathbb{Y}$ denote the set of diagrams λ with $l(\lambda') \leq 2N$. Under the bijection between diagrams $\lambda \in \mathbb{Y}'(2N)$ and $2N$ -point configurations on $\mathbb{Z}_{\geq 0}$ defined by

$$\lambda \leftrightarrow x_{2N-i+1} = 2\lambda'_i - i + 2N \quad (i = 1, \dots, 2N)$$

the z -measure with parameters $z = -2N$, $\theta = 2$, $z' = -2N - \beta$ turns into

$$\text{Prob} \{x_1, \dots, x_{2N}\} = \text{const} \cdot \prod_{i=1}^{2N} \frac{[\beta]_{x_i}}{x_i!!} \xi^{\frac{x_i}{2}} \prod_{1 \leq i \leq j \leq 2N} (x_j - x_i),$$

where

$$x_i!! = \begin{cases} 2 \cdot 4 \cdot \dots \cdot x_i, & x_i \text{ is even,} \\ 1 \cdot 3 \cdot \dots \cdot x_i, & x_i \text{ is odd,} \end{cases}$$

and

$$[\beta]_{x_i} = \begin{cases} (x_i + \beta - 1)(x_i + \beta - 3) \dots (\beta + 1), & x_i \text{ is even,} \\ (x_i + \beta - 1)(x_i + \beta - 3) \dots \beta, & x_i \text{ is odd.} \end{cases}$$

This is a discrete orthogonal ensemble in the sense of Definition 3.1.

Proof. The proof is also a straightforward computation based on the formula

$$\frac{1}{H(\lambda; 2)H'(\lambda; 2)} = \frac{\prod_{1 \leq i < j \leq l(\lambda')} (2\lambda'_i - i - 2\lambda'_j + j)}{\prod_{i=1}^{l(\lambda')} (2\lambda'_i - i + l(\lambda'))!}.$$

$\square$

5. THE DERIVATION OF THE CORRELATION KERNEL FOR DISCRETE SYMPLECTIC ENSEMBLES

Recall that the Pfaffian of a $2N \times 2N$ antisymmetric matrix $A = \|A_{jk}\|_{j,k=1}^{2N}$ is defined as

$$\text{Pf } A = \sum_{\substack{\sigma = (i_1, \dots, i_{2N}) \in S_{2N} \\ i_1 < i_2, \dots, i_{2N-1} < i_{2N} \\ i_1 < i_3 < \dots < i_{2N-1}}} \text{sgn}(\sigma) A_{i_1 i_2} \dots A_{i_{2N-1} i_{2N}}.$$

One has $(\text{Pf } A)^2 = \det A$.

Lemma 5.1. *Assume that $\varphi_1, \dots, \varphi_{2N}$ and $\psi_1, \dots, \psi_{2N}$ are arbitrary finitely supported functions on $\mathbb{Z}_{\geq 0}$. Set*

$$\underline{\varphi}(\cdot) = \begin{bmatrix} \varphi_1(\cdot) \\ \vdots \\ \varphi_{2N}(\cdot) \end{bmatrix}, \quad \underline{\psi}(\cdot) = \begin{bmatrix} \psi_1(\cdot) \\ \vdots \\ \psi_{2N}(\cdot) \end{bmatrix},$$

and introduce a $2N \times 2N$ antisymmetric matrix $A = [A_{ij}]_{i,j=1}^{2N}$ whose entries, A_{ij} , are given by

$$A_{ij} = \sum_{x \in \mathbb{Z}_{\geq 0}} [\varphi_i(x)\psi_j(x) - \psi_i(x)\varphi_j(x)].$$

We have

$$(5.1) \quad \sum_{\underline{x}=(x_1 < \dots < x_N) \subset \mathbb{Z}_{\geq 0}} \det [\underline{\varphi}(x_1), \underline{\psi}(x_1), \dots, \underline{\varphi}(x_N), \underline{\psi}(x_N)] = \text{Pf } A.$$

Proof. This is one of de Bruijn's formulas, see de Bruijn [11]. □

Lemma 5.2. *Let*

$$\pi_{i-1}(x) = x^{i-1} + \dots, \quad i = 1, \dots, 2N,$$

is an arbitrary system of monic polynomials of degrees $0, \dots, 2N-1$. Set

$$\varphi_i(x) = \pi_{i-1}(x), \quad \psi_i(x) = \pi_{i-1}(x+1).$$

Then

$$\prod_{1 \leq i < j \leq N} (x_i - x_j)^2 ((x_i - x_j)^2 - 1) = \det [\underline{\varphi}(x_1), \underline{\psi}(x_1), \dots, \underline{\varphi}(x_N), \underline{\psi}(x_N)].$$

Proof. We have

$$\begin{aligned} & \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 ((x_i - x_j)^2 - 1) \\ &= \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 (x_i - x_j + 1)(x_i - x_j - 1) \\ &= V(x_1 + 1, x_1, x_2 + 1, x_2, \dots, x_N + 1, x_N) \\ &= (-1)^N V(x_1, x_1 + 1, x_2, x_2 + 1, \dots, x_N, x_N + 1) \\ &= (-1)^N (-1)^{\frac{2N(2N-1)}{2}} \det (\pi_{i-1}(x_1), \pi_{i-1}(x_1 + 1), \dots, \pi_{i-1}(x_N), \pi_{i-1}(x_N + 1)) \\ &= \det [\underline{\varphi}(x_1), \underline{\psi}(x_1), \dots, \underline{\varphi}(x_N), \underline{\psi}(x_N)]. \end{aligned}$$

□

Proof of Proposition 2.3.

Applying Lemma 5.1 and Lemma 5.2 we obtain the identity

$$Z_{N4} = \text{Pf} [Q_{ij}]_{i,j=0}^{2N-1},$$

where

$$(5.2) \quad Q_{ij} = \sum_{x \in \mathbb{Z}_{\geq 0}} w(x) (\pi_i(x) \pi_j(x+1) - \pi_i(x+1) \pi_j(x)),$$

and $0 \leq i, j \leq 2N-1$. Since the partition function Z_{N4} of the discrete symplectic ensemble under considerations is strictly positive, the determinant of the matrix Q is nonzero, and the matrix Q is non-degenerate. The matrix $M^{(4)}$ defined by equation (2.4) is related to Q via $Q = \text{diag}(\|\pi_1\|^{1/2}, \dots, \|\pi_{2N-1}\|^{1/2}) \cdot M \cdot \text{diag}(\|\pi_1\|^{1/2}, \dots, \|\pi_{2N-1}\|^{1/2})$. $\square$

Proof of Theorem 2.4.

Let $\zeta(x)$ be some finitely supported function defined on $\mathbb{Z}_{\geq 0}$. The following identity holds true

$$(5.3) \quad \sum_{(x_1 < x_2 < \dots < x_N) \subset \mathbb{Z}_{\geq 0}} \prod_{i=1}^N w(x_i) \zeta(x_i) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 (x_i - x_j - 1)(x_i - x_j + 1) = \text{Pf} [A_{ij}(\zeta)]_{i,j=1}^{2N},$$

where the matrix elements $A_{ij}(\zeta)$ are defined by the formula

$$(5.4) \quad A_{ij}(\zeta) = \sum_{x \in \mathbb{Z}_{\geq 0}} [\pi_{i-1}(x) \pi_{j-1}(x+1) - \pi_{i-1}(x+1) \pi_{j-1}(x)] w(x) \zeta(x).$$

By the same argument as in Tracy and Widom [41], §8, we find

$$K_{N4}(x, y) = \begin{bmatrix} \sum_{i,j=0}^{2N-1} \pi_i(x+1) Q_{ij}^{-1} \pi_j(y) w^{1/2}(x) w^{1/2}(y) & - \sum_{i,j=0}^{2N-1} \pi_i(x+1) Q_{ij}^{-1} \pi_j(y+1) w^{1/2}(x) w^{1/2}(y) \\ \sum_{i,j=0}^{2N-1} \pi_i(x) Q_{ij}^{-1} \pi_j(y) w^{1/2}(x) w^{1/2}(y) & - \sum_{i,j=0}^{2N-1} \pi_i(x) Q_{ij}^{-1} \pi_j(y+1) w^{1/2}(x) w^{1/2}(y) \end{bmatrix},$$

where the variables x, y take values in $\mathbb{Z}_{\geq 0}$, Q_{ij} is defined by equation (5.2), and $\pi_i(x) = x^i + \dots$; $i = 0, 1, \dots$ is an arbitrary system of monic polynomials of the discrete variable x . It is now straightforward to check that the kernel $K_{N4}(x, y)$ just written above is exactly the kernel of the operator K_{N4} , where

$$K_{N4} = \begin{bmatrix} D_+ S_{N4} & -D_+ S_{N4} D_- \\ S_{N4} & -S_{N4} D_- \end{bmatrix}.$$

Finally observe that the matrix $A_{ij}(\zeta)$ remains unchanged if we replace $\pi_{j-1}(x+1)$ by $\pi_{j-1}(x+1) - \sqrt{\frac{w(x)}{w(x+1)}} \pi_{j-1}(x)$, and $\pi_{i-1}(x+1)$ by $\pi_{i-1}(x+1) - \sqrt{\frac{w(x)}{w(x+1)}} \pi_{i-1}(x)$ in equation (5.4). This results in formula (2.6). Theorem 2.4 is proved. $\square$

6. THE DERIVATION OF THE CORRELATION KERNEL FOR DISCRETE ORTHOGONAL ENSEMBLES

Lemma 6.1. *The probability of a particular configuration $x_1 < \dots < x_{2N}$ of the discrete orthogonal ensemble (see Definition 3.1) can be rewritten as*

$$(6.1) \quad \text{Prob} \{x_1, \dots, x_{2N}\} = \tilde{Z}_{N1}^{-1} \prod_{i=1}^{2N} w^{1/2}(x_i) \prod_{1 \leq i < j \leq 2N} (x_i - x_j) \text{Pf}[\epsilon(x_i, x_j)]_{i,j=1}^{2N}$$

where $w(x)$ is defined in terms of the weight function $W(x)$ by formula (3.1).

Proof. We will compute the Pfaffian in equation (6.1), and will show that (6.1) coincides with the expression for the probability of a particular configuration $x_1 < \dots < x_{2N}$ in Definition 3.1. Observe that the semi-infinite matrix ϵ defined by equation (2.3) is representable as follows

$$\epsilon = \mathcal{F} \Upsilon \mathcal{F},$$

where

$$\mathcal{F} = \begin{bmatrix} f(0) & 0 & 0 & 0 & \dots \\ 0 & f(1) & 0 & 0 & \dots \\ 0 & 0 & f(2) & 0 & \dots \\ 0 & 0 & 0 & f(3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$f(0), f(1), f(2), \dots$ are defined for $k = 0, 1, 2, \dots$ by

$$f(2k) = \frac{1}{\sqrt{w(2k)}} \frac{w(2)w(4)\dots w(2k)}{w(1)w(3)\dots w(2k-1)}, \quad f(2k+1) = \frac{1}{\sqrt{w(2k+1)}} \frac{w(1)w(3)\dots w(2k+1)}{w(2)w(4)\dots w(2k)},$$

and

$$\Upsilon = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

That is, Υ is an antisymmetric matrix whose entries are defined by the relations

$$\Upsilon(2i+1, 2j+1) = \Upsilon(2i, 2j) = 0, \text{ for any } i, j \geq 0,$$

$$\Upsilon(2i+1, 2j+2) = 0, \text{ for } 0 \leq i \leq j,$$

$$\Upsilon(2i, 2j+1) = -1, \text{ for } 0 \leq i \leq j.$$

Since relation (3.1) implies that

$$W(x) = \begin{cases} \frac{w(1)w(3)\dots w(x)}{w(2)w(4)\dots w(x-1)}, & x \text{ is odd,} \\ \frac{w(2)w(4)\dots w(x)}{w(1)w(3)\dots w(x-1)}, & x \text{ is even,} \end{cases}$$

we obtain

$$\begin{aligned}
& \prod_{i=1}^{2N} w^{1/2}(x_i) \operatorname{Pf}[\epsilon(x_i, x_j)]_{i,j=1}^{2N} \\
&= \left(\prod_{i=1}^{2N} w^{1/2}(x_i) \right) \left(\prod_{i=1}^{2N} f(x_i) \right) \operatorname{Pf} \left[\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N}) \right] \\
&= \left(\prod_{i=1}^{2N} W(x_i) \right) \operatorname{Pf} \left[\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N}) \right].
\end{aligned}$$

The proof is completed by the following Lemma, cf. Borodin and Strahov [9], Section 3.2. $\square$

Proposition 6.2.

$$\operatorname{Pf} \left[\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N}) \right] = \begin{cases} (-1)^N, & \text{if } x_i - x_{i-1} \text{ is odd, and } x_1 \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If x_1 is odd, then the first row of the matrix $\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N})$ consists of only zeros. Thus, if $\operatorname{Pf} \left[\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N}) \right] \neq 0$, x_1 must be even. Now assume that x_{2i-1} and x_{2i} have the same parity. In this case rows $2i-1$ and $2i$ of $\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N})$ are equal to each other. Therefore, if $\operatorname{Pf} \left[\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N}) \right] \neq 0$, the elements of the set $X = (x_1, \dots, x_{2N})$ are such that x_1 is even, x_2 is odd, x_3 is even, and so on. This proves the condition on the parity for the configurations $X = (x_1, \dots, x_{2N})$, for which $\operatorname{Pf} \left[\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N}) \right] \neq 0$. Moreover, using the definition of Pfaffian, it is not hard to conclude that $\operatorname{Pf} \left[\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N}) \right] = (-1)^N$ for the configurations for which $\operatorname{Pf} \left[\Upsilon(x_1, \dots, x_{2N} \middle| x_1, \dots, x_{2N}) \right] \neq 0$. $\square$

Lemma 6.3. *If $\phi_1, \dots, \phi_{2N}$ are arbitrary finitely supported functions on $\mathbb{Z}_{\geq 0}$, and $\epsilon(x, y)$ is an antisymmetric function defined on $\mathbb{Z}_{\geq 0}$, then*

$$(6.2) \quad \sum_{x_1 < \dots < x_{2N} \subset \mathbb{Z}_{\geq 0}} \det(\phi_j(x_k))_{j,k=1}^{2N} \operatorname{Pf}[\epsilon(x_i, x_j)]_{i,j=1}^{2N} = \operatorname{Pf} \left[\sum_{x,y \in \mathbb{Z}_{\geq 0}} \epsilon(x, y) \phi_j(x) \phi_k(y) \right].$$

Proof. This is another de Bruijn formula, see [11]. $\square$

Proof of Proposition 3.3

Consider the partition function $\tilde{Z}_{N1}$, which is defined by

$$\tilde{Z}_{N1} = \sum_{x_1 < \dots < x_{2N}} \prod_{i=1}^{2N} w^{1/2}(x_i) \prod_{1 \leq i < j \leq 2N} (x_i - x_j) \text{Pf}[\epsilon(x_i, x_j)]_{i,j=1}^{2N}.$$

Write

$$\prod_{i=1}^{2N} w^{1/2}(x_i) \prod_{1 \leq i < j \leq 2N} (x_i - x_j) = (-1)^N \det [w^{1/2}(x_k) \pi_j(x_k)],$$

where $0 \leq j \leq 2N - 1$, $1 \leq k \leq 2N$, and $\{\pi_j(x)\}$ is an arbitrary system of monic polynomials. Apply Lemma 6.3 and obtain

$$\tilde{Z}_{N1} = (-1)^N \det \left[\sum_{x,y \in \mathbb{Z}_{\geq 0}} \epsilon(x, y) \pi_j(x) w^{1/2}(x) \pi_k(y) w^{1/2}(y) \right]_{k,j=0}^{2N-1}.$$

Since $\tilde{Z}_{N1}$ is nonzero, the matrix with the j, k entry

$$\sum_{x,y \in \mathbb{Z}_{\geq 0}} \epsilon(x, y) \pi_j(x) w^{1/2}(x) \pi_k(y) w^{1/2}(y)$$

is non-degenerate. This implies that the matrix $M^{(1)}$ whose j, k entry is given by formula (3.2) is non-degenerate as well. $\square$

Proof of Theorem 3.4

Use Lemma 6.1 to rewrite the formula in definition 3.2 as follows

$$\tilde{Z}_{N1}^{-1} \sum_{x_1 < \dots < x_{2N}} \prod_{i=1}^{2N} w^{1/2}(x_i) (1 + \eta(x_i)) \prod_{1 \leq i < j \leq 2N} (x_i - x_j) \text{Pf}[\epsilon(x_i, x_j)]_{i,j=1}^{2N} = \sqrt{\det(I + \eta K_{N1})}.$$

The rest of the argument is very similar to the derivation of the correlation kernel in §9 of Tracy and Widom [41], and we omit the details. $\square$

7. THE GENERAL IDENTITIES

Lemma 7.1. *We have for $0 \leq i \leq 2N - 1$*

$$(7.1) \quad (S_{N4} K_N D\varphi_i)(x) = \varphi_i(x), \quad (S_{N1} K_N \epsilon\varphi_i)(x) = \varphi_i(x).$$

$$(7.2) \quad S_{N4} |_{\mathcal{H}_N^\perp} = 0, \quad S_{N1} |_{\mathcal{H}_N^\perp} = 0.$$

where $\mathcal{H}_N^\perp$ denotes the complement of $\mathcal{H}_N$ in $\mathcal{H}$.

Proof. a) For $0 \leq i \leq 2N - 1$, and $x \in \mathbb{Z}_{\geq 0}$ we have

$$(K_N D\varphi_i)(x) = \sum_{j=0}^{2N-1} \varphi_j(x) \sum_{y \in \mathbb{Z}_{\geq 0}} \varphi_j(y) (D\varphi_i)(y).$$

The second sum in the equation above can be rewritten as follows

$$\begin{aligned}
\sum_{y \in \mathbb{Z}_{\geq 0}} \varphi_j(y) (D\varphi_i)(y) &= \sum_{y \in \mathbb{Z}_{\geq 0}} \varphi_j(y) \left(\sqrt{\frac{w(y)}{w(y+1)}} \varphi_i(y+1) - \sqrt{\frac{w(y-1)}{w(y)}} \varphi_i(y-1) \right) \\
&= \sum_{y \in \mathbb{Z}_{\geq 0}} p_j(y) p_i(y+1) w(y) - \sum_{y \in \mathbb{Z}_{\geq 0}} p_j(y) p_i(y-1) w(y-1) \\
&= \sum_{y \in \mathbb{Z}_{\geq 0}} (p_j(y) p_i(y+1) - p_j(y+1) p_i(y)) w(y) \\
&= M_{ji}^{(4)},
\end{aligned}$$

where $\{p_j\}$ is the family of polynomials orthonormal with respect to the weight w . Therefore,

$$(K_N D\varphi_i)(x) = \sum_{j=0}^{2N-1} \varphi_j(x) M_{ji}^{(4)}.$$

The action by the operator S_{N4} from the left gives

$$\begin{aligned}
(S_{N4} K_N D\varphi_i)(x) &= \sum_{y \in \mathbb{Z}_{\geq 0}} S_{N4}(x, y) \sum_{j=0}^{2N-1} \varphi_j(y) M_{ji}^{(4)} \\
&= \sum_{y \in \mathbb{Z}_{\geq 0}} \sum_{k,l=0}^{2N-1} \varphi_k(x) \mu_{kl}^{(4)} \varphi_l(y) \sum_{j=0}^{2N-1} \varphi_j(y) M_{ji}^{(4)} = \varphi_i(x).
\end{aligned}$$

b) It is clear from the definition of the operator S_{N4} that $(S_{N4}\varphi_j)(x) = 0$ for $j = 2N, 2N+1, \dots$. In other words, $S_{N4} |_{\mathcal{H}_N^\perp} = 0$.

c) The formulas for S_{N1} can be proved by the same procedure, replacing D everywhere by ϵ . $\square$

Lemma 7.1 shows that the operators S_{N4} , S_{N1} , K_N , D and ϵ satisfy the same algebraic relations as the corresponding operators in the continuous case considered in Widom [43], Sections 2-4. Therefore Theorem 2.6 and Theorem 3.5 can be established by arguments from Widom [43], Sections 2-4. Here we only give a simple "matrix" explanation why formulae in Theorem 2.6 and Theorem 3.5 indeed hold true.

Let us write each operator in the basis $\varphi_0, \varphi_1, \varphi_2, \dots$. Then each operator A which acts in $\mathcal{H}$ has the representation

$$A = \begin{pmatrix} (A)_N & (A)_1 \\ (A)_0 & (A)_2 \end{pmatrix}$$

where the upper-left corner corresponds to the subspace $\mathcal{H}_N$ spanned by $\varphi_0, \dots, \varphi_{2N-1}$. In particular, K_N is representable as

$$K_N = \begin{pmatrix} I_N & 0 \\ 0 & 0 \end{pmatrix},$$

and we have

$$(A)_{\mathcal{H}_N} = A \cdot K - N = \begin{pmatrix} (A)_N & 0 \\ (A)_0 & 0 \end{pmatrix}, \quad (A)_{\mathcal{H}_N^\perp} = A \cdot (I - K_N) = \begin{pmatrix} 0 & (A)_1 \\ 0 & (A)_2 \end{pmatrix}.$$

In what follows we formally manipulate with operators. Equation (7.1) implies

$$(K_N S_{N4} K_N)(K_N D K_N) = K_N.$$

We can rewrite the above identity as follows

$$\begin{pmatrix} (S_{N4})_N & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (D)_N & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_N & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $(S_{N4})_{\mathcal{H}_N} = \begin{pmatrix} (D)_N^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, and we obtain

$$(D)_{\mathcal{H}_N} (S_{N4})_{\mathcal{H}_N} = \begin{pmatrix} I_N & 0 \\ (D)_0 (D)_N^{-1} & 0 \end{pmatrix}.$$

In order to express $(D)_0 (D)_N^{-1}$ use the relation

$$\epsilon D = I.$$

This relation can be rewritten as

$$\begin{pmatrix} (\epsilon)_N & (\epsilon)_1 \\ (\epsilon)_0 & (\epsilon)_2 \end{pmatrix} \begin{pmatrix} (D)_N & (D)_1 \\ (D)_0 & (D)_2 \end{pmatrix} = \begin{pmatrix} I_N & 0 \\ 0 & I \end{pmatrix},$$

and we obtain $(\epsilon)_N (D)_N = I_N - (\epsilon)_1 (D)_0$. Multiplying this equation by $(D)_0$ from the left we have

$$(D)_0 (\epsilon)_N (D)_N = (D)_0 (I_N - (\epsilon)_1 (D)_0) = (I_N - (D)_0 (\epsilon)_1) (D)_0,$$

therefore

$$(D)_0 (D)_N^{-1} = (I - (D)_0 (\epsilon)_1)^{-1} (D)_0 (\epsilon)_N.$$

We conclude that the following formula holds

$$(D)_{\mathcal{H}_N} (S_{N4})_{\mathcal{H}_N} = \begin{pmatrix} I_N & 0 \\ (I - (D)_0 (\epsilon)_1)^{-1} (D)_0 (\epsilon)_N & 0 \end{pmatrix}.$$

Now we are going to show that

$$(7.3) \quad \begin{pmatrix} I_N & 0 \\ (I - (D)_0 (\epsilon)_1)^{-1} (D)_0 (\epsilon)_N & 0 \end{pmatrix} = (I - (DK_N - K_N DK_N) \epsilon)^{-1} K_N.$$

Indeed,

$$DK_N - K_N DK_N = \begin{pmatrix} 0 & 0 \\ (D)_0 & 0 \end{pmatrix},$$

$$I - (DK_N - K_N DK_N)\epsilon = \begin{pmatrix} 0 & 0 \\ -(D)_0(\epsilon)_N & I - (D)_0(\epsilon)_1 \end{pmatrix},$$

and using the formula $\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -b^{-1}a & b^{-1} \end{pmatrix}$ we get equation (7.3). Thus we have shown that

$$(D)_{\mathcal{H}_N}(S_{N4})_{\mathcal{H}_N} = [I - (DK_N - K_N DK_N)\epsilon]^{-1} K_N.$$

Taking into account equation (7.2), we obtain the formula in Theorem 2.6. The formula in Theorem 3.5 can be deduced by similar computations.

8. PROOF OF THEOREM 2.7.

The idea of the proof is the same as in Widom [44], section 3.

Recall that the operator D acts on the elements of $\mathcal{H}_N$ according to the formula

$$(D\varphi)(x) = \sqrt{\frac{w(x)}{w(x-1)}}\varphi(x+1) - \sqrt{\frac{w(x-1)}{w(x)}}\varphi(x-1).$$

Set $L_D = (I - K_N)D$, and agree that the domain of this operator is $\mathcal{H}_N$. Denote by $\mathcal{N}_D$ the null space of L_D , i.e.

$$\mathcal{N}_D = \{\varphi | L_D\varphi = 0, \varphi \in \mathcal{H}_N\}.$$

We want to find $\mathcal{N}_D^\perp$.

Let us find the null space $\mathcal{N}_D$ of $L_D = (I - K_N)D$. The general element of $\mathcal{H}_N$ is of the form $\varphi = pw^{1/2}$, where $\deg p \leq 2N - 1$. Such φ is an element of $\mathcal{N}_D$ if and only if $D\varphi$ is an element of $\mathcal{H}_N$. We have

$$(D\varphi)(x) = \sqrt{w(x)} \left[p(x+1) - \frac{w(x-1)}{w(x)} p(x-1) \right].$$

It follows that φ is an element of $\mathcal{N}_D$ if and only if $\frac{w(x-1)}{w(x)} p(x-1)$ is a polynomial of degree less or equal to $2N - 1$.

We will denote by $a_1, a_2, \dots, a_l$ finite poles of $\frac{w(x-1)}{w(x)}$. Let $(\mathcal{H}_N)_{a_i}$ be the subspace of those φ , for which $\frac{w(x-1)}{w(x)} p(x-1) = O(1)$ in a neighborhood of a_i , and let $(\mathcal{H}_N)_\infty$ be the subspace of those φ , for which $\frac{w(x-1)}{w(x)} p(x-1) = O(x^{2N-1})$ as $x \rightarrow \infty$. Then

$$\mathcal{N}_D = \left(\bigcap_{i=1}^l (\mathcal{H}_N)_{a_i} \right) \cap (\mathcal{H}_N)_\infty,$$

and

$$\mathcal{N}_D^\perp = \left(\sum_{i=1}^l (\mathcal{H}_N)_{a_i}^\perp \right) + (\mathcal{H}_N)_\infty^\perp.$$

Let a_i be the finite pole of $\frac{w(x-1)}{w(x)}$ of order n_i . Observe that $p(x-1)\frac{w(x-1)}{w(x)} = O(1)$ if and only if

$$(8.1) \quad p(a_i - 1) = 0, \quad p'(a_i - 1) = 0, \quad \dots, \quad p^{(n_i-1)}(a_i - 1) = 0.$$

Therefore φ is an element of $(\mathcal{H}_N)_{a_i}$ if and only if condition (8.1) is satisfied.

Now expand φ in terms of basis elements $\varphi_1, \dots, \varphi_{2N-1}$ of $\mathcal{H}_N$

$$\varphi = \sum_{j=0}^{2N-1} B_j \varphi_j.$$

Thus

$$p(x) = \sum_{j=0}^{2N-1} B_j p_j(x),$$

and condition (8.1) implies that φ is an element of $(\mathcal{H}_N)_{a_i}$ if and only if

$$(8.2) \quad \sum_{j=0}^{2N-1} B_j p_j^{(k)}(a_i - 1) = 0$$

for $0 \leq k \leq n_i - 1$. Set

$$\xi_k = \sum_{j=0}^{2N-1} \varphi_j p_j^{(k)}(a_i - 1).$$

Condition (8.2) implies that $\varphi \in (\mathcal{H}_N)_{a_i}$ if and only if φ is orthogonal to all ξ_k . Thus, ξ_k 's span $(\mathcal{H}_N)_{a_i}^\perp$.

Clearly, $(\mathcal{H}_N)_\infty$ is the span of φ_k for $0 \leq k \leq 2N - n_\infty - 1$, where n_∞ is the order of $\frac{w(x-1)}{w(x)}$ at ∞ . Therefore $(\mathcal{H}_N)_\infty^\perp$ is the span of φ_k for $k \geq 2N - n_\infty$. We conclude that the dimension of the span of all $(\mathcal{H}_N)_{a_i}^\perp$ (including $(\mathcal{H}_N)_\infty^\perp$) is at most $n_\infty + \sum_{i=1}^l n_{a_i}$. Pick an

orthonormal basis $\psi_1, \dots, \psi_n$ of this span. Note that $n \leq n_\infty + \sum_{i=1}^l n_{a_i}$.

Since $\psi_1, \dots, \psi_n$ is a basis of $\mathcal{N}_D^\perp$, we have $L_D = \sum_{i=1}^n L_D \psi_i \otimes \psi_i$. Therefore we obtain

$$L_D = (I - K_N)D = \sum_{i=1}^n (I - K_N)D\psi_i \otimes \psi_i.$$

Set $\tilde{\psi}_i = (I - K_N)D\psi_i$. Since $L_D\psi_i \neq 0$ we conclude that $\tilde{\psi}_i \in \mathcal{H}_N^\perp$, and we have

$$[D, K_N]K_N = \sum_{i=1}^n \tilde{\psi}_i \otimes \psi_i, \quad \tilde{\psi}_i \in \mathcal{H}_N^\perp, \psi_i \in \mathcal{H}_N.$$

Thus the first part of Theorem 2.7 is proved.

By the antisymmetry of ϵ we obtain

$$[D, K_N]K_N\epsilon = - \sum_{i=1}^n \tilde{\psi}_i \otimes \epsilon\psi_i.$$

In order to find $D_{\mathcal{H}_N}S_{N4}$ we need to compute the inverse of $I - [D, K_N]K_N\epsilon$. Using the formula

$$(I + \sum a_i \otimes b_i)^{-1} = I - \sum_{i,j} (T^{-1})_{ij} a_i \otimes b_j,$$

where T is the identity matrix plus the matrix of inner products (b_i, a_j) (assuming T is invertible), and $\sum a_i \otimes b_i$ is a finite rank operator, we obtain the formula for $D_{\mathcal{H}_N}S_{N4}$ stated in Theorem 2.7. $\square$

9. PROOF OF THEOREM 3.6.

Set $L_\epsilon = (I - K_N)\epsilon$, and agree that the domain of this operator is $\mathcal{H}_N$. Denote by $\mathcal{N}_\epsilon$ the null space of L_ϵ . Observe that

$$\mathcal{N}_\epsilon = \{v | v \in \mathcal{H}_N, \epsilon v \in \mathcal{H}_N\}.$$

Assume that $\psi \in \mathcal{N}_D^\perp$, and $u \in \mathcal{N}_\epsilon$. Since $u \in \mathcal{H}_N$, and K_N is a symmetric operator, we have $(u, K_N\epsilon\psi) = (u, \epsilon\psi)$. Since $u \in \mathcal{N}_\epsilon$, ϵu is an element of $\mathcal{H}_N$ (see above). But $D\epsilon u = u \in \mathcal{H}_N$, which implies that ϵu belongs to $\mathcal{N}_D$, see previous Section. Therefore, $(u, \epsilon\psi) = -(\epsilon u, \psi) = 0$, and

$$(u, K_N\epsilon\psi) = 0, \text{ for all } \psi \in \mathcal{N}_D^\perp \text{ and } u \in \mathcal{N}_\epsilon.$$

We conclude that $K_N\epsilon K_N$ takes $\mathcal{N}_D^\perp$ into $\mathcal{N}_\epsilon^\perp$.

Let us show that $\mathcal{N}_\epsilon^\perp$ has the same dimension as $\mathcal{N}_D^\perp$. The operators $K_N\epsilon K_N$ and $K_N D K_N$ are invertible because we have already shown that their matrices $M_{jk}^{(1)}$ and $M_{jk}^{(4)}$ in the basis $\varphi_0, \dots, \varphi_{2N-1}$ are invertible, see Propositions 2.3 and 3.3. We just saw that $K_N\epsilon K_N$ takes $\mathcal{N}_D^\perp$ to $\mathcal{N}_\epsilon^\perp$, and very similar arguments show that $K_N D K_N$ takes $\mathcal{N}_\epsilon^\perp$ to $\mathcal{N}_D^\perp$. Therefore $\dim \mathcal{N}_D^\perp = \dim \mathcal{N}_\epsilon^\perp$.

Now take $\psi_1, \dots, \psi_n$ found in the previous section and orthonormalize the functions $K_N\epsilon\psi_j$. As a result obtain an orthonormal basis in $\mathcal{N}_\epsilon^\perp$. Denote this basis $\eta_1, \dots, \eta_n$.

This gives

$$L_\epsilon = (I - K_N)\epsilon = \sum_{i=1}^n (I - K_N)\epsilon\eta_i \otimes \eta_i.$$

Set $\tilde{\eta}_i = (I - K_N)\epsilon\eta_i$. By the same argument as in the previous section we find (assuming invertibility of U) that

$$\epsilon_{\mathcal{H}_N} S_{N1} = K_N - \sum_{i,j=1}^n (U^{-1})_{ij} (\tilde{\eta}_i) \otimes (K_N D\eta_j),$$

where the matrix U is defined by its matrix elements by

$$U_{ij} = \delta_{ij} + (D\eta_i, \tilde{\eta}_j).$$

This immediately gives the formula in the statement of Theorem 3.6. $\square$

10. PROOFS OF COROLLARY 2.8 AND COROLLARY 3.7

Proposition 10.1. *If the commutation relation between the operators D and K_N takes the form as in the statement of Corollary 2.8, with $\psi_1 \in \mathcal{H}_N$, and $\psi_2 \in \mathcal{H}_N^\perp$, then $\epsilon\psi_1$ is an element of $\mathcal{H}_N$.*

Proof. The explicit form of $[D, K_N]$ in the statement of Corollary 2.8 implies

$$(10.1) \quad [\epsilon, K_N] = \lambda(\epsilon\psi_1 \otimes \epsilon\psi_2 + \epsilon\psi_2 \otimes \epsilon\psi_1).$$

(We have used the fact that $\epsilon D = 1$, and the antisymmetry of ϵ). Acting by $[\epsilon, K_N]$ on ψ_1 we find

$$[\epsilon, K_N]\psi_1 = \lambda\epsilon\psi_1(\epsilon\psi_2, \psi_1).$$

This gives

$$K_N\epsilon\psi_1 = (1 + \lambda(\psi_2, \epsilon\psi_1))\epsilon\psi_1.$$

Therefore, either $\epsilon\psi_1$ is in $\mathcal{H}_N$, or $K_N\epsilon\psi_1 = 0$. To rule out the second possibility observe that $[D, K_N]K_N = \lambda\psi_2 \otimes \psi_1$, as it follows from the expression for $[D, K_N]$ in the hypothesis of Corollary 2.8. Here $\psi_2 \in \mathcal{H}_N^\perp$, and $\psi_1 \in \mathcal{H}_N$, and Theorem 2.7 implies that ψ_1 is an element of $\mathcal{N}_D^\perp$. It was shown in the proof of Theorem 3.6 that $K_N\epsilon$ bijectively maps $\mathcal{N}_D^\perp$ into $\mathcal{N}_\epsilon^\perp$. Therefore, $K_N\epsilon\psi_1$ must be an element of $\mathcal{N}_\epsilon^\perp$, so $K_N\epsilon\psi_1$ cannot be zero. $\square$

Proof of Corollary 2.8. If the conditions of Corollary 2.8 are satisfied we have

$$[D, K_N]K_N\epsilon = -\lambda\psi_2 \otimes \epsilon\psi_1.$$

Then Theorem 2.6 implies

$$D_{\mathcal{H}_N} S_{N4} = (I_{\mathcal{H}_N + D\mathcal{H}_N} + \lambda\psi_2 \otimes \epsilon\psi_1)^{-1} K_N.$$

Now we use the formula

$$(I + a \otimes b)^{-1} = I - T^{-1}a \otimes b,$$

where $a = \lambda\psi_2$, $b = \epsilon\psi_1$, and $T = 1 + \lambda(\psi_2, \epsilon\psi_1)$. By Proposition 10.1 $\epsilon\psi_1 \in \mathcal{H}_N$, and we conclude that $T = 1$. Therefore

$$D_{\mathcal{H}_N} S_{N4} = K_N - \lambda\psi_2 \otimes \epsilon\psi_1,$$

or

$$S_{N4} = \epsilon K_N - \lambda\epsilon\psi_2 \otimes \epsilon\psi_1.$$

□

Proof of Corollary 3.7.

By Theorem 3.6 we need to compute inverse of

$$I - [\epsilon, K_N]K_N D.$$

From equation (10.1) we find

$$[\epsilon, K_N]K_N D = -\lambda(\epsilon\psi_1 \otimes DK_N\epsilon\psi_2 + \epsilon\psi_2 \otimes DK_N\epsilon\psi_1).$$

Now we compute $DK_N\epsilon\psi_1$ as follows

$$\begin{aligned} DK_N\epsilon\psi_1 &= [D, K_N]\epsilon\psi_1 + K_N D\epsilon\psi_1 = [D, K_N]\epsilon\psi_1 + \psi_1 \\ &= \lambda(\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1)\epsilon\psi_1 + \psi_1 = \psi_1. \end{aligned}$$

(We have used $(\psi_1, \epsilon\psi_1) = 0$, and the fact that by Proposition 10.1 $(\psi_2, \epsilon\psi_1) = 0$). In a similar way we compute $DK_N\epsilon\psi_2$

$$\begin{aligned} DK_N\epsilon\psi_2 &= [D, K_N]\epsilon\psi_2 + K_N D\epsilon\psi_2 = [D, K_N]\epsilon\psi_2 \\ &= \lambda(\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1)\epsilon\psi_2 = 0. \end{aligned}$$

Therefore we have obtained

$$I - [\epsilon, K_N]K_N D = I + \lambda\epsilon\psi_2 \otimes \psi_1.$$

Using the formula

$$(I + a \otimes b)^{-1} = I - T^{-1}a \otimes b$$

where now $a = \lambda\epsilon\psi_2$, $b = \psi_1$,

$$T = 1 + (b, a) = 1 + (\psi_1, \lambda\epsilon\psi_2) = 1 - \lambda(\epsilon\psi_1, \psi_2) = 1,$$

we obtain

$$(I - [\epsilon, K_N]K_N D)^{-1} = I - \lambda\epsilon\psi_2 \otimes \psi_1.$$

This gives

$$\epsilon_{\mathcal{H}_N} S_{N1} = K_N - \lambda\epsilon\psi_2 \otimes \psi_1,$$

and we finally arrive to the formula

$$S_{N1} = DK_N - \lambda\psi_2 \otimes \psi_1.$$

□

11. DISCRETE RIEMANN-HILBERT PROBLEMS (DRHP) AND DIFFERENCE EQUATIONS FOR ORTHOGONAL POLYNOMIALS.

In this section we recall a few claims from Borodin and Boyarchenko [5]. Let $\mathfrak{X}$ be a discrete locally finite subset of $\mathbb{C}$. Let $w : \mathfrak{X} \rightarrow \mathbb{C}$ be a function. Assume that all moments of w are finite. Denote by $\mathbb{C}[\zeta]$ the space of polynomials in the complex variable ζ , and introduce the inner product in $\mathbb{C}[\zeta]$

$$(f(\zeta), g(\zeta))_w := \sum_{x \in \mathfrak{X}} f(x)g(x)w(x).$$

If the restriction of $(\cdot, \cdot)_w$ to the space $\mathbb{C}[\zeta]^{\leq d}$ of the polynomials of degree at most d is non-degenerate for all $d \geq 0$, then there exists a unique collection of monic orthogonal polynomials $\{P_n(\zeta)\}_{n=0}^{\infty}$ associated to w such that if $m \neq n$, then $(P_n(\zeta), P_m(\zeta))_w = 0$, and $(P_n(\zeta), P_n(\zeta))_w \neq 0$ for all n . If this condition holds, we say that the weight function w is non-degenerate. Note that if $\mathfrak{X}$ is finite, and consists of $N + 1$ points, the inner product $(\cdot, \cdot)_w$ is necessarily degenerate on $\mathbb{C}[\zeta]^{\leq d}$ for all $d > N$. In this case we require that $(\cdot, \cdot)_w$ be non-degenerate on $\mathbb{C}[\zeta]^{\leq d}$ for $0 \leq d \leq N$, and we are only interested in a collection of orthogonal polynomials of degrees up to N .

As in Refs. [3, 4, 5], we say that an analytic function

$$m : \mathbb{C} \setminus \mathfrak{X} \rightarrow \text{Mat}(2, \mathbb{C})$$

solves the DRHP $(\mathfrak{X}, w)$ if m has simple poles at the points of $\mathfrak{X}$ and its residues at these points are given by the jump (or residue) condition

$$\text{Res}_{\zeta=x} m(\zeta) = \lim_{\zeta \rightarrow x} (m(\zeta) \varpi(x)), \quad x \in \mathfrak{X},$$

where

$$\varpi(x) = \begin{pmatrix} 0 & w(x) \\ 0 & 0 \end{pmatrix}.$$

Theorem 11.1. *Let $\{P_n(\zeta)\}_{n=0}^N$ be the collection of monic orthogonal polynomials associated to w , where $N = \text{card}(\mathfrak{X}) - 1 \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. For any $k = 1, 2, \dots, N$ the DRHP $(\mathfrak{X}, w)$ has a unique solution $m_{\mathfrak{X}}(\zeta)$ satisfying the asymptotic condition*

$$m_{\mathfrak{X}}(\zeta) \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} = I + O\left(\frac{1}{\zeta}\right),$$

where I is the identity matrix. If we write

$$m_{\mathfrak{X}}(\zeta) = \begin{pmatrix} m_{\mathfrak{X}}^{11}(\zeta) & m_{\mathfrak{X}}^{12}(\zeta) \\ m_{\mathfrak{X}}^{21}(\zeta) & m_{\mathfrak{X}}^{22}(\zeta) \end{pmatrix},$$

then $m_{\mathfrak{X}}^{11}(\zeta) = P_k(\zeta)$, $m_{\mathfrak{X}}^{21}(\zeta) = (P_{k-1}, P_{k-1})_w^{-1} P_{k-1}(\zeta)$, $m_{\mathfrak{X}}^{21}(\zeta) = \sum_{x \in \mathfrak{X}} \frac{P_k(\zeta) w(x)}{\zeta - x}$, and $m_{\mathfrak{X}}^{22}(\zeta) = (P_{k-1}, P_{k-1})_w^{-1} \sum_{x \in \mathfrak{X}} \frac{P_{k-1}(\zeta) w(x)}{\zeta - x}$.

Proof. See Borodin and Boyarchenko [5], Section 2.3. □

Let $\mathfrak{X}$ be a finite or a locally finite subset of $\mathbb{R}$, $\text{card}(\mathfrak{X}) = N + 1$, where $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Assume that $\mathfrak{X}$ is parameterized as $\mathfrak{X} = \{\pi_x\}_{x=0}^N$, and that there exists an affine transformation $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ such that $\sigma \pi_{x+1} = \pi_x$ for all $0 \leq x < N$. Denote the derivative of σ (which is constant) by η , so that

$$\sigma(\zeta_1) - \sigma(\zeta_2) = \eta(\zeta_1 - \zeta_2) \quad \text{for all } \zeta_1, \zeta_2 \in \mathbb{C}.$$

Let w be a strictly positive real valued function defined on $\mathfrak{X}$. Then w is non-degenerate.

Theorem 11.2. *Assume that there exist entire functions $d_1(\zeta)$ and $d_2(\zeta)$ such that*

$$\frac{w(\pi_{x-1})}{w(\pi_x)} = \eta \cdot \frac{d_1(\pi_x)}{d_2(\pi_x)}, \quad 1 \leq x \leq N,$$

$$d_1(\pi_0) = 0,$$

$$d_2(\sigma^{-1}\pi_N) = 0, \text{ if } N \text{ is finite.}$$

a) *We have*

$$P_k(\sigma\zeta) = (M^{11}(\zeta)P_k(\zeta) + cM^{12}(\zeta)P_{k-1}(\zeta)) d_1(\zeta)^{-1}$$

$$cP_{k-1}(\sigma\zeta) = (M^{21}(\zeta)P_k(\zeta) + cM^{22}(\zeta)P_{k-1}(\zeta)) d_1^{-1}(\zeta)$$

where $c = (P_{k-1}, P_{k-1})_w^{-1}$, and $M^{11}(\zeta)$, $M^{12}(\zeta)$, $M^{21}(\zeta)$, $M^{22}(\zeta)$ are entire functions. Moreover, the matrix $M(\zeta)$ whose entries are $M^{11}(\zeta)$, $M^{12}(\zeta)$, $M^{21}(\zeta)$, $M^{22}(\zeta)$ is given by

$$M(\zeta) = m_{\mathfrak{X}}(\sigma\zeta)D(\zeta)m_{\mathfrak{X}}^{-1}(\zeta),$$

where

$$D(\zeta) = \begin{pmatrix} d_1(\zeta) & 0 \\ 0 & d_2(\zeta) \end{pmatrix},$$

and $m_{\mathfrak{X}}(\zeta)$ is the solution of the DRHP $(\mathfrak{X}, w)$ with the asymptotic condition

$$m_{\mathfrak{X}}(\zeta) \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} = I + O\left(\frac{1}{\zeta}\right).$$

b) *If it is known that $d_1(\zeta)$ and $d_2(\zeta)$ are polynomials of degree at most n in ζ , and*

$$d_1(\zeta) = \lambda_1 \zeta^n + (\text{lower terms}),$$

$$d_2(\zeta) = \lambda_2 \zeta^n + (\text{lower terms}),$$

then 2×2 matrix $M(\zeta)$ is a polynomial of degree at most n in ζ , and

$$\begin{pmatrix} M^{11}(\zeta) & M^{12}(\zeta) \\ M^{21}(\zeta) & M^{22}(\zeta) \end{pmatrix} = \begin{pmatrix} \eta^k \lambda_1 & 0 \\ 0 & \eta^{-k} \lambda_2 \end{pmatrix} \zeta^k + (\text{lower terms}).$$

Proof. For the proof of these statements see Borodin and Boyarchenko [5], Sections 3.1-3.3. $\square$

12. COMMUTATION RELATIONS

In this section set $\mathfrak{X} = \mathbb{Z}_{\geq 0}$, and $\sigma\zeta = \zeta - 1$ for any $\zeta \in \mathbb{C}$. Assume that $w(x)$ is a strictly positive function on $\mathbb{Z}_{\geq 0}$, which satisfies the condition in Theorem 11.2, i.e. the ratio of $w(x-1)$ and $w(x)$ equals the ratio of entire functions $d_1(x)$ and $d_2(x)$, and $d_1(0) = 0$.

Proposition 12.1. *We have for $n = 1, 2, \dots$*

$$(D_- \varphi_n)(x) = \frac{M^{11}(x)}{d_2(x)} \varphi_n(x) + \frac{M^{12}(x)}{d_2(x)(P_n, P_n)_w^{1/2}(P_{n-1}, P_{n-1})_w^{1/2}} \varphi_{n-1}(x),$$

$$(D_- \varphi_{n-1})(x) = \frac{M^{21}(x)(P_n, P_n)_w^{1/2}(P_{n-1}, P_{n-1})_w^{1/2}}{d_2(x)} \varphi_n(x) + \frac{M^{22}(x)}{d_2(x)} \varphi_{n-1}(x).$$

Proof. These equations are equivalent to the difference equations for the monic orthogonal polynomials in Theorem 11.2, a). $\square$

Proposition 12.2. *Introduce the 2×2 matrices $D_{\pm}(x)$ by the formula*

$$(12.1) \quad \begin{pmatrix} D_{\pm} \varphi_{2N}(x) \\ D_{\pm} \varphi_{2N-1}(x) \end{pmatrix} = \begin{pmatrix} D_{\pm}^{11}(x) & D_{\pm}^{12}(x) \\ D_{\pm}^{21}(x) & D_{\pm}^{22}(x) \end{pmatrix} \begin{pmatrix} \varphi_{2N}(x) \\ \varphi_{2N-1}(x) \end{pmatrix}.$$

The matrix elements of $D_+(x)$ are related with the matrix elements of $D_-(x+1)$ as

$$(12.2) \quad \begin{aligned} D_+^{11}(x) &= D_-^{22}(x+1), & D_+^{12}(x) &= -D_-^{12}(x+1), \\ D_+^{21}(x) &= -D_-^{21}(x+1), & D_+^{22}(x) &= D_-^{11}(x+1). \end{aligned}$$

Proof. The definition of $D_{\pm}$ implies that

$$(12.3) \quad D_+(x)D_-(x+1) = \frac{w(x)}{w(x+1)}.$$

Observe that $\det M(x) = d_1(x) \cdot d_2(x)$ (see [5], §3). This fact together with the assumption that the ratio of $w(x-1)$ and $w(x)$ equals the ratio of $d_1(x)$ and $d_2(x)$ imply that the determinant of the matrix $D_-(x+1)$ equals $\frac{w(x)}{w(x+1)}$, which (together with equation (12.3)) leads to the relation between the matrix elements in the statement of the Proposition. $\square$

Proposition 12.3. *We have*

$$\begin{aligned} & [D, K_N](x, y) \\ &= a_{2N}(\varphi_{2N}(x), \varphi_{2N-1}(x)) \begin{pmatrix} \frac{D_-^{21}(x+1) - D_-^{21}(y)}{x+1-y} & \frac{D_-^{22}(x+1) - D_-^{22}(y)}{x+1-y} \\ -\frac{D_-^{11}(x+1) - D_-^{11}(y)}{x+1-y} & -\frac{D_-^{12}(x+1) - D_-^{12}(y)}{x+1-y} \end{pmatrix} \begin{pmatrix} \varphi_{2N}(y) \\ \varphi_{2N-1}(y) \end{pmatrix} \\ &+ a_{2N}(\varphi_{2N}(x), \varphi_{2N-1}(x)) \begin{pmatrix} \frac{D_-^{21}(x) - D_-^{21}(y+1)}{x-y-1} & -\frac{D_-^{11}(x) - D_-^{11}(y+1)}{x-y-1} \\ \frac{D_-^{22}(x) - D_-^{22}(y+1)}{x-y-1} & -\frac{D_-^{12}(x) - D_-^{12}(y+1)}{x-y-1} \end{pmatrix} \begin{pmatrix} \varphi_{2N}(y) \\ \varphi_{2N-1}(y) \end{pmatrix}, \end{aligned}$$

where a_{2N} is the coefficient in the Christoffel-Darboux formula for the kernel $K(x, y)$,

$$K(x, y) = a_{2N} \frac{\varphi_{2N}(x)\varphi_{2N-1}(y) - \varphi_{2N-1}(x)\varphi_{2N}(y)}{x - y}.$$

Proof. A direct computation of the commutation relation between the operators D and K gives

$$\begin{aligned} [D, K](x, y) = a_{2N} & \left[\frac{(D_+ \varphi_{2N})(x) \varphi_{2N-1}(y) - (D_+ \varphi_{2N-1})(x) \varphi_{2N}(y)}{x+1-y} \right. \\ & - \frac{(D_- \varphi_{2N-1})(y) \varphi_{2N}(x) - (D_- \varphi_{2N})(y) \varphi_{2N-1}(x)}{x+1-y} \\ & - \frac{(D_- \varphi_{2N})(x) \varphi_{2N-1}(y) - (D_- \varphi_{2N-1})(x) \varphi_{2N}(y)}{x-1-y} \\ & \left. + \frac{(D_+ \varphi_{2N-1})(y) \varphi_{2N}(x) - (D_+ \varphi_{2N})(y) \varphi_{2N-1}(x)}{x-1-y} \right] \end{aligned}$$

Express $D_{\pm} \varphi_{2N}$, $D_{\pm} \varphi_{2N-1}$ as linear combinations of φ_{2N} and φ_{2N-1} whose coefficients are entries of matrices $D_+(x)$, $D_-(x)$. In the resulting formula replace the matrix elements $D_+^{11}(x)$, $D_+^{12}(x)$, $D_+^{21}(x)$, $D_+^{22}(x)$ by the matrix elements $D_-^{11}(x+1)$, $D_-^{12}(x+1)$, $D_-^{21}(x+1)$, $D_-^{22}(x+1)$ in accordance with formula (12.2). This gives the formula in the statement of the Proposition. $\square$

Proposition 12.4. *Assume that the functions d_1 , d_2 are polynomials of degree at most m . Furthermore, assume that d_2 has zeros at points $a_1, \dots, a_l$ of degrees $n_1, \dots, n_l$ correspondingly. Then each of the expressions $\frac{D_-^{21}(x+1) - D_-^{21}(y)}{x+1-y}$, $\frac{D_-^{22}(x+1) - D_-^{22}(y)}{x+1-y}$, $\frac{D_-^{11}(x+1) - D_-^{11}(y)}{x+1-y}$, $\frac{D_-^{12}(x+1) - D_-^{12}(y)}{x+1-y}$ is a finite sum of expressions of the form*

$$\left(\sum_{k=0}^{m-1-n_1-\dots-n_l} A_k x^k + \sum_{i=1}^l \sum_{k_i=1}^{n_i} \frac{B_{k_i}}{(x+1-a_i)^{k_i}} \right) \left(\sum_{k=0}^{m-1-n_1-\dots-n_l} C_k y^k + \sum_{i=1}^l \sum_{k_i=1}^{n_i} \frac{D_{k_i}}{(y-a_i)^{k_i}} \right),$$

and each of the expressions $\frac{D_-^{21}(x) - D_-^{21}(y+1)}{x-y-1}$, $\frac{D_-^{11}(x) - D_-^{11}(y+1)}{x-y-1}$, $\frac{D_-^{22}(x) - D_-^{22}(y+1)}{x-y-1}$, $\frac{D_-^{12}(x) - D_-^{12}(y+1)}{x-y-1}$ is a finite sum of expressions of the form

$$\left(\sum_{k=0}^{m-1-n_1-\dots-n_l} \tilde{A}_k x^k + \sum_{i=1}^l \sum_{k_i=1}^{n_i} \frac{\tilde{B}_{k_i}}{(x-a_i)^{k_i}} \right) \left(\sum_{k=0}^{m-1-n_1-\dots-n_l} \tilde{C}_k y^k + \sum_{i=1}^l \sum_{k_i=1}^{n_i} \frac{\tilde{D}_{k_i}}{(y+1-a_i)^{k_i}} \right),$$

where A_k , C_k , B_{k_i} and D_{k_i} , $\tilde{A}_k$, $\tilde{C}_k$, $\tilde{B}_{k_i}$ and $\tilde{D}_{k_i}$ are some constant coefficients.

Proof. If $d_1(x)$ and $d_2(x)$ are polynomials of degree at most m , then $M^{11}(x)$, $M^{12}(x)$, $M^{21}(x)$, and $M^{22}(x)$ are polynomials of degree at most m , see Theorem 11.2, b). By Proposition 12.1, and by the very definition of the matrix $D_-(x)$ each $D_-^{11}(x)$, $D_-^{22}(x)$, $D_-^{12}(x)$, $D_-^{21}(x)$ is a polynomial of degree at most m divided by $d_2(x)$. Set

$$d_2(x) = \text{const} \cdot (x - a_1)^{n_1} \dots (x - a_l)^{n_l},$$

where

$$n_1 + n_2 + \dots + n_l \leq m.$$

Denote $D_-^{11}(x)d_2(x) = A_mx^m + \dots + A_0$. Then we have

$$\begin{aligned} \frac{D_-^{11}(x+1) - D_-^{11}(y)}{x+1-y} &= \frac{A_m(x+1)^m + \dots + A_1(x+1) + A_0}{(x+1-y)(x+1-a_1)^{n_1} \dots (x+1-a_l)^{n_l}} \\ &\quad - \frac{A_my^m + \dots + A_1y + A_0}{(x+1-y)(y-a_1)^{n_1} \dots (y-a_l)^{n_l}}. \end{aligned}$$

Since

$$\frac{(x+1)^k - y^k}{x+1-y} = \sum_{a+b \leq k-1} (x+1)^a y^b$$

we arrive at the result. The same considerations are applicable to all expressions in the statement of the Proposition. $\square$

13. DIFFERENCE EQUATIONS FOR THE ORTHONORMAL FUNCTIONS ASSOCIATED TO THE MEIXNER AND TO THE CHARLIER WEIGHTS

Proposition 13.1. *Set*

$$\varphi_n(x) = (P_n, P_n)_{w_{Meixner}}^{-1} P_n(x) \sqrt{w_{Meixner}(x)}$$

where $w_{Meixner}(x)$ is the Meixner weight defined by equation (2.9), and $\{P_j(x)\}$ is the family of the monic Meixner polynomials. The functions $\{\varphi_n(x)\}_{n=0}^\infty$ satisfy the following system of difference equations

$$\begin{aligned} \sqrt{cx(x+\beta-1)}\varphi_n(x-1) &= (x-n)\varphi_n(x) + \sqrt{cn(n+\beta-1)}\varphi_{n-1}(x), \\ \sqrt{cx(x+\beta-1)}\varphi_{n-1}(x-1) &= c(x+(n+\beta-1))\varphi_{n-1}(x) - \sqrt{cn(n+\beta-1)}\varphi_n(x). \end{aligned}$$

Proof. Let $m(\zeta)$ denote the solution of DRHP($\mathbb{Z}_{\geq 0}, w_{Meixner}$) satisfying the asymptotic condition

$$m(\zeta) \begin{pmatrix} \zeta^{-n} & 0 \\ 0 & \zeta^n \end{pmatrix} = I + O(\zeta^{-1}),$$

see Section 11. The matrix $m(\zeta)$ has a full asymptotic expansion in ζ as $\zeta \rightarrow \infty$, and we can write

$$(13.1) \quad m(\zeta) \begin{pmatrix} \zeta^{-n} & 0 \\ 0 & \zeta^n \end{pmatrix} = I + \begin{pmatrix} \alpha & \tau \\ \gamma & \delta \end{pmatrix} \zeta^{-1} + O(\zeta^{-2}).$$

By Theorem 11.2 there exists an entire matrix-valued function $M(\zeta)$ such that

$$(13.2) \quad M(\zeta) = m(\zeta-1)D(\zeta)m^{-1}(\zeta),$$

where the matrix $D(\zeta)$ has the following form

$$D(\zeta) = \begin{pmatrix} \zeta & 0 \\ 0 & c\zeta + c(\beta-1) \end{pmatrix}.$$

We can rewrite equation (13.2) as

$$M(\zeta) = \left[m(\zeta - 1) \begin{pmatrix} (\zeta - 1)^{-n} & 0 \\ 0 & (\zeta - 1)^n \end{pmatrix} \right] \\ \times \left[\begin{pmatrix} (1 - \frac{1}{\zeta})^n & 0 \\ 0 & (1 - \frac{1}{\zeta})^{-n} \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & c\zeta + c(\beta - 1) \end{pmatrix} \right] \left[m(\zeta) \begin{pmatrix} \zeta^{-n} & 0 \\ 0 & \zeta^n \end{pmatrix} \right]^{-1}.$$

The last multiplier in the expression above has the form

$$\begin{pmatrix} 1 - \alpha\zeta^{-1} & -\tau\zeta^{-1} \\ -\gamma\zeta^{-1} & 1 - \delta\zeta^{-1} \end{pmatrix} + O(\zeta^{-2}),$$

the multiplication of the two matrices in the middle gives

$$\begin{pmatrix} \zeta - n & 0 \\ 0 & c\zeta + cn + c(\beta - 1) \end{pmatrix} + O(\zeta^{-1}),$$

and the result of the multiplication of the first two matrices in the expression for $M(\zeta)$ is

$$\begin{pmatrix} 1 + \alpha\zeta^{-1} & \tau\zeta^{-1} \\ \gamma\zeta^{-1} & 1 + \delta\zeta^{-1} \end{pmatrix} + O(\zeta^{-2}),$$

as it is evident from (13.1). The straightforward calculation gives

$$M(\zeta) = \begin{pmatrix} \zeta - n & (c - 1)\tau \\ -(c - 1)\gamma & c\zeta + c(n + \beta - 1) \end{pmatrix} + O(\zeta^{-1}).$$

Since $M(\zeta)$ is entire the last term $O(\zeta^{-1})$ is identically zero by Liouville's theorem. By Theorem 11.2 the monic Meixner polynomials satisfy the following system of the difference equations

$$(13.3) \quad \begin{aligned} \zeta P_n(\zeta - 1) &= (\zeta - n)P_n(\zeta) + c_{n-1}(c - 1)\tau P_{n-1}(\zeta) \\ c_{n-1}\zeta P_{n-1}(\zeta - 1) &= -(c - 1)\gamma P_n(\zeta) + c_{n-1}(c\zeta + c(n + \beta - 1))P_{n-1}(\zeta) \end{aligned}$$

where $c_n = (P_n, P_n)_w^{-1}$. Moreover, we also have the condition $\det M(\zeta) = \det D(\zeta)$ which follows from the fact that the determinant of the solution of the Discrete Riemann-Hilbert problem identically equals 1, and from the relation between $M(\zeta)$ and $D(\zeta)$, equation (13.2). This condition implies a relation between parameters,

$$(13.4) \quad (c - 1)^2 \tau \gamma = cn(n + \beta - 1).$$

Replacing $n - 1$ by n in the second equation of system (13.3), and inserting the result into the first equation we obtain the recurrence relation for the monic Meixner polynomials. Since the recurrence equation for the Meixner polynomials is known (see, for example, equation (1.9.3) in Ref. [25]) this (together with formula (13.4)) determines coefficients γ and τ in system (13.3). Finally, rewriting (13.3) in terms of functions $\varphi_n(x)$ and $\varphi_{n-1}(x)$ we obtain the difference equations in the statement of the Proposition. $\square$

Proposition 13.2. *Let $\{\varphi_n(x)\}_{n=0}^\infty$ be the family of the orthonormal functions associated with the Charlier weight defined by equation 2.10. The functions $\{\varphi_n(x)\}_{n=0}^\infty$ satisfy the following system of difference equations*

$$\begin{aligned}\sqrt{ax}\varphi_n(x-1) &= (x-n)\varphi_n(x) + \sqrt{an}\varphi_{n-1}(x), \\ \sqrt{ax}\varphi_{n-1}(x-1) &= -\sqrt{an}\varphi_n(x) + a\varphi_{n-1}(x).\end{aligned}$$

Proof. The system of difference equations for the orthonormal functions $\{\varphi_n(x)\}_{n=0}^\infty$ associated to the Charlier weight follows from the corresponding system of difference equations for the case of the Meixner weight. Indeed, we have the following limiting relation

$$(13.5) \quad \lim_{n \rightarrow \infty} M_n \left(x; \beta, \frac{a}{\beta + a} \right) = C_n(x; a)$$

between the n th Charlier polynomial $C_n(x; a)$ and the n th Meixner polynomial $M_n(x; \beta, c)$, see, for example, Ref. [23]. $\square$

14. THE MEIXNER AND CHARLIER SYMPLECTIC AND ORTHOGONAL ENSEMBLES

By Propositions 12.3, 12.4 the kernel of the operator $[D, K_N]$ is expressible in terms of the following functions:

$$(14.1) \quad x^k \varphi_{2N-1}(x), \quad x^k \varphi_{2N}(x); \quad 0 \leq k \leq m-1 - \sum_{i=1}^l n_i,$$

and, for each zero a_i of $d_2(x)$, the functions

$$(14.2) \quad (x+1-a_i)^{-k_i} \varphi_{2N-1}(x), \quad (x+1-a_i)^{-k_i} \varphi_{2N}(x); \quad 1 \leq k_i \leq n_i,$$

$$(14.3) \quad (x-a_i)^{-k_i} \varphi_{2N-1}(x), \quad (x-a_i)^{-k_i} \varphi_{2N}(x); \quad 1 \leq k_i \leq n_i.$$

Proposition 14.1. *In the case of the Meixner weight we have*

$$(14.4) \quad [D, K_N](x, y) = \frac{\sqrt{2N(2N+\beta-1)}}{c-1} (\zeta_{2N}(x), \zeta_{2N-1}(x), \eta_{2N}(x), \eta_{2N-1}(x))$$

$$\begin{pmatrix} 0 & 0 & \sqrt{2N(2N+\beta-1)} & -2N\sqrt{c} \\ 0 & 0 & -\frac{2N-1+\beta}{\sqrt{c}} & \sqrt{2N(2N+\beta-1)} \\ \sqrt{2N(2N+\beta-1)} & -\frac{2N-1+\beta}{\sqrt{c}} & 0 & 0 \\ -2N\sqrt{c} & \sqrt{2N(2N+\beta-1)} & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_{2N}(y) \\ \zeta_{2N-1}(y) \\ \eta_{2N}(y) \\ \eta_{2N-1}(y) \end{pmatrix}$$

where we have introduced $\zeta_j(x) = \frac{\varphi_j(x)}{x+\beta}$ and $\eta_j(x) = \frac{\varphi_j}{x+\beta-1}$.

Proof. We use Proposition 12.3 which determines the commutation relation between the operators D and K_N in terms of the matrix elements of the matrix $D_-(x)$ defined by equation (12.1). This matrix can be written explicitly in the case of the Meixner weight

since we have obtained in Proposition 13.1 the system of difference equations for the orthonormal functions $\varphi_{2N-1}(x)$ and $\varphi_{2N}(x)$. Namely,

$$D_-(x) = \begin{bmatrix} \frac{x-2N}{c(x+\beta-1)} & \frac{\sqrt{2N(2N+\beta-1)}}{\sqrt{c(x+\beta-1)}} \\ -\frac{\sqrt{2N(2N+\beta-1)}}{\sqrt{c(x+\beta-1)}} & \frac{x+(2N+\beta-1)}{x+\beta-1} \end{bmatrix}$$

Inserting the matrix elements of the matrix $D_-(x)$ into the formula in the statement of Proposition 12.3, and taking into account that the coefficient a_{2N} corresponding to the case of the Meixner weight equals $-\frac{\sqrt{2Nc(\beta+2N-1)}}{1-c}$ we obtain the formula for $[D, K_N]$. $\square$

We want to construct linear combinations $\psi_1, \psi_2, \psi_3, \psi_4$ of $\zeta_{2N}, \zeta_{2N-1}, \eta_{2N}, \eta_{2N-1}$ in such a way that ψ_1 and ψ_2 will be lying in $\mathcal{H}_N$, and ψ_3, ψ_4 will be lying in $\mathcal{H}_N^\perp$.

Proposition 14.2. *Set*

$$(14.5) \quad \psi_1 = \sqrt{2cN}\zeta_{2N} - \sqrt{\beta + 2N - 1}\zeta_{2N-1},$$

$$(14.6) \quad \psi_2 = \sqrt{2N}F(-2N + 1, \beta - 1; \beta; c)\eta_{2N} - \sqrt{c(\beta + 2N - 1)}F(-2N, \beta - 1; \beta; c)\eta_{2N-1},$$

$$(14.7) \quad \psi_3 = (\beta + 2N)F(\beta, \beta + 2N - 1; \beta + 2N; c)\zeta_{2N} - \sqrt{2cN(\beta + 2N - 1)}F(\beta, \beta + 2N; \beta + 2N + 1; c)\zeta_{2N-1},$$

$$(14.8) \quad \psi_4 = \sqrt{\beta + 2N - 1}\eta_{2N} - \sqrt{2cN}\eta_{2N-1}.$$

Then $\psi_1, \psi_2, \psi_3, \psi_4$ are linear independent, ψ_1, ψ_2 are elements of $\mathcal{H}_N$, and ψ_3, ψ_4 are elements of $\mathcal{H}_N^\perp$.

Proof. a) Suppose that the linear combination

$$\psi_1 = A\frac{\varphi_{2N}(x)}{x + \beta} + B\frac{\varphi_{2N-1}(x)}{x + \beta}$$

is an element of $\mathcal{H}_N$. Here A, B are some coefficients. If ψ_1 is lying in $\mathcal{H}_N$, then

$$(14.9) \quad Ap_{2N}(-\beta) + Bp_{2N-1}(-\beta) = 0$$

Therefore, we need to check that the equality

$$(14.10) \quad \frac{p_{2N-1}(-\beta)}{p_{2N}(-\beta)} = \sqrt{\frac{2cN}{\beta + 2N - 1}}$$

is satisfied in order to conclude that ψ_1 defined by (14.5) is lying in $\mathcal{H}_N$. To check (14.10) note that

$$p_n(x) = \frac{M_n(x; \beta, c)}{\|M_n(\cdot; \beta, c)\|},$$

where $M_n(x; \beta, c)$ is the n^{th} Meixner polynomial. We have $M_n(x; \beta, c) = F(-n, -x; \beta; \frac{c-1}{c})$, see, for example, Refs. [23, 25] for the basic properties of the Meixner polynomials. The norm, $\|M_n(\cdot; \beta, c)\|$, is equal to

$$\|M_n(\cdot; \beta, c)\| = \sqrt{\frac{\Gamma(\beta)\Gamma(n+1)}{\Gamma(\beta+n)}} c^{-n/2} (1-c)^{-\beta/2}.$$

Taking into account the formula

$$F(-n, \beta; \beta; -z) = (1+z)^n,$$

we see that relation (14.10) indeed holds.

b) In the same way we show that ψ_2 defined by equation (14.6) is an element of $\mathcal{H}_N$.

c) Now we need to prove that the functions ψ_3, ψ_4 defined by equations (14.7), (14.8) are elements of $\mathcal{H}^\perp$. To show this we note that the space $\mathcal{H}_N$ in the case of the Meixner weight can be understood as that spanned by the functions

$$\varphi_0; \varphi_1; (x+\beta)(x+\beta-1)x^k \sqrt{w_{\text{Meixner}}(x)}, \quad 0 \leq k \leq 2N-3.$$

All linear combinations of $\frac{\varphi_{2N-1}(x)}{x+\beta}$, $\frac{\varphi_{2N}(x)}{x+\beta}$, $\frac{\varphi_{2N-1}(x)}{x+\beta-1}$ and $\frac{\varphi_{2N}(x)}{x+\beta-1}$ are certainly orthogonal to $(x+\beta)(x+\beta-1)x^k \sqrt{w_{\text{Meixner}}(x)}$, $0 \leq k \leq 2N-3$. Therefore, we can take as ψ_3, ψ_4 linear combinations of the form

$$C \frac{\varphi_{2N-1}(x)}{x+\beta} + D \frac{\varphi_{2N}(x)}{x+\beta} + E \frac{\varphi_{2N-1}(x)}{x+\beta-1} + F \frac{\varphi_{2N}(x)}{x+\beta-1},$$

where the coefficients C, D, E and F are subjected to the conditions

(14.11)

$$C \left(\varphi_0, \frac{\varphi_{2N-1}(x)}{x+\beta} \right) + D \left(\varphi_0, \frac{\varphi_{2N}(x)}{x+\beta} \right) + E \left(\varphi_0, \frac{\varphi_{2N-1}(x)}{x+\beta-1} \right) + F \left(\varphi_0, \frac{\varphi_{2N}(x)}{x+\beta-1} \right) = 0,$$

(14.12)

$$C \left(\varphi_1, \frac{\varphi_{2N-1}(x)}{x+\beta} \right) + D \left(\varphi_1, \frac{\varphi_{2N}(x)}{x+\beta} \right) + E \left(\varphi_1, \frac{\varphi_{2N-1}(x)}{x+\beta-1} \right) + F \left(\varphi_1, \frac{\varphi_{2N}(x)}{x+\beta-1} \right) = 0.$$

In particular, we can take

$$\psi_3 = C \frac{\varphi_{2N-1}}{x+\beta} + D \frac{\varphi_{2N}}{x+\beta}, \quad \psi_4 = E \frac{\varphi_{2N-1}}{x+\beta-1} + F \frac{\varphi_{2N}}{x+\beta-1},$$

provided that the following conditions on the coefficients C, D, E and F are satisfied

$$(14.13) \quad \frac{C}{D} = -\frac{\left(\varphi_0, \frac{\varphi_{2N}}{x+\beta} \right)}{\left(\varphi_0, \frac{\varphi_{2N-1}}{x+\beta} \right)}, \quad \frac{E}{F} = -\frac{\left(\varphi_0, \frac{\varphi_{2N}}{x+\beta-1} \right)}{\left(\varphi_0, \frac{\varphi_{2N-1}}{x+\beta-1} \right)}.$$

(It is not hard to check that for $n \geq 1$

(14.14)

$$\left(\varphi_1, \frac{\varphi_n}{x+\beta}\right) = \sqrt{\frac{\beta}{c}} \left(\varphi_0, \frac{\varphi_n}{x+\beta}\right), \quad \left(\varphi_1, \frac{\varphi_n}{x+\beta-1}\right) = \frac{c+\beta-1}{\sqrt{\beta c}} \left(\varphi_0, \frac{\varphi_n}{x+\beta-1}\right).$$

Taking into account these relations we see that if conditions (14.13) are satisfied, equations (14.11), (14.12) hold as well).

Thus it remains to check that for $n \geq 1$ the following relations are valid:

$$(14.15) \quad \frac{\left(\varphi_0, \frac{\varphi_n}{x+\beta}\right)}{\left(\varphi_0, \frac{\varphi_{n-1}}{x+\beta}\right)} = \frac{\sqrt{cn(\beta+n-1)}F(\beta, \beta+n; \beta+n+1; c)}{(\beta+n)F(\beta, \beta+n-1; \beta+n; c)},$$

$$(14.16) \quad \frac{\left(\varphi_0, \frac{\varphi_n}{x+\beta-1}\right)}{\left(\varphi_0, \frac{\varphi_{n-1}}{x+\beta-1}\right)} = \frac{\sqrt{cn}}{(\beta+n-1)}.$$

Let us compute the scalar product in the left-hand side of equation (14.15). We have

$$\left(\varphi_0, \frac{\varphi_n}{x+\beta}\right) = (1-c)^\beta \sqrt{\frac{(\beta)_n c^n}{n!}} \sum_{x=0}^{+\infty} \frac{M_n(x; \beta, c)}{x+\beta} \frac{(\beta)_x c^x}{x!}.$$

It is convenient to exploit the discrete Rodrigues formula for the Meixner polynomials (see, for example, Ismail [23], Section 6.1):

$$M_n(x; \beta, c) \frac{(\beta)_x c^x}{x!} = \nabla^n \left[\frac{(\beta+n)_x c^x}{x!} \right],$$

where

$$(\nabla f)(x) = f(x) - f(x-1).$$

This gives

$$(14.17) \quad \left(\varphi_0, \frac{\varphi_n}{x+\beta}\right) = (1-c)^\beta \sqrt{\frac{(\beta)_n c^n}{n!}} \sum_{x=0}^{+\infty} \frac{1}{x+\beta} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{(\beta+n)_{x-k} c^{x-k}}{(x-k)!},$$

where we have used the formula

$$(\nabla^n f)(x) = \sum_{k=0}^n \binom{n}{k} (-1)^k f(x-k).$$

Two sums in the righthand side of equation (14.17) can be rewritten further as

$$\sum_{k=0}^n \binom{n}{k} \sum_{x=0}^{+\infty} \frac{1}{x+\beta+k} \frac{(\beta+n)_x c^x}{x!}.$$

Taking into account the formula

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{x+k} = \frac{n!}{x(x+1)\dots(x+n)}$$

we sum up over k and obtain

$$\begin{aligned} \left(\varphi_0, \frac{\varphi_n}{x+\beta} \right) &= (1-c)^\beta \sqrt{\frac{(\beta)_n c^n}{n!}} \sum_{x=0}^{+\infty} \frac{n!}{(x+\beta)(x+\beta+1)\dots(x+\beta+n)} \frac{(\beta+n)_x c^x}{x!} \\ &= \frac{(1-c)^\beta}{\beta+n} c^{n/2} \sqrt{\frac{\Gamma(\beta)\Gamma(n+1)}{\Gamma(\beta+n)}} F(\beta, \beta+n; \beta+n+1; c). \end{aligned}$$

This formula implies that relation (14.15) indeed holds. Relation (14.16) can be checked in the same way. $\square$

Proposition 14.3. *In the case of the Meixner weight the commutation relation between D and K_N can be expressed as*

$$(14.18) \quad [D, K_N] = \frac{\sqrt{2N(2N+\beta-1)}}{(c-1)\sqrt{c}} (\psi_1, \psi_4) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_4 \end{pmatrix},$$

where $\psi_1 \in \mathcal{H}_N$ and $\psi_4 \in \mathcal{H}_N^\perp$ are defined by equations (14.5), (14.8) correspondingly.

Proof. Equations (14.5)-(14.8) can be rewritten as

$$(14.19) \quad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} \zeta_{2N} \\ \zeta_{2N-1} \\ \eta_{2N} \\ \eta_{2N-1} \end{pmatrix},$$

where

$$\begin{aligned} m_{11} &= \sqrt{2cN}, \quad m_{12} = -\sqrt{\beta+2N-1}; \\ m_{21} &= (\beta+2N)F(\beta, \beta+2N-1; \beta+2N; c); \\ m_{22} &= -\sqrt{2cN(\beta+2N-1)}F(\beta, \beta+2N; \beta+2N+1; c); \\ m_{33} &= \sqrt{2N}F(-2N+1, \beta-1; \beta; c), \quad m_{34} = -\sqrt{c(\beta+2N-1)}F(-2N, \beta-1; \beta; c); \\ m_{43} &= \sqrt{\beta+2N-1}, \quad m_{44} = -\sqrt{2cN}. \end{aligned}$$

Introduce the following notation

$$\begin{aligned} M_1 &= \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad M_2 = \begin{pmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{pmatrix}, \\ B &= \frac{\sqrt{2N(2N+\beta-1)}}{c-1} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \end{aligned}$$

where $b_{11} = \sqrt{2N(2N + \beta - 1)}$, $b_{12} = -2N\sqrt{c}$, $b_{21} = -\frac{2N-1+\beta}{\sqrt{c}}$, $b_{22} = \sqrt{2N(2N + \beta - 1)}$, and

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using equation (14.19) we express ζ_{2N} , ζ_{2N-1} , η_{2N} and η_{2N-1} in terms of ψ_1, ψ_2, ψ_3 and ψ_4 , and rewrite formula (14.4) as follows

$$[D, K_N] = (\psi_1, \psi_2, \psi_3, \psi_4) Q \begin{pmatrix} 0 & (M_1^T)^{-1} B M_2^{-1} \\ (M_2^T)^{-1} B^T M_1^{-1} & 0 \end{pmatrix} Q \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

Now we compute the matrix $(M_1^T)^{-1} B M_2^{-1}$ explicitly. We have

$$(M_1^T)^{-1} B M_2^{-1} = \frac{\sqrt{2N(2N + \beta - 1)}}{(c-1)\Delta_1\Delta_2} \begin{pmatrix} m_{22} & -m_{21} \\ -m_{12} & m_{11} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} m_{44} & -m_{34} \\ -m_{43} & m_{33} \end{pmatrix},$$

where $\Delta_1 = \det M_1$, $\Delta_2 = \det M_2$. We observe that

$$\begin{aligned} b_{11}m_{44} - b_{12}m_{43} &= 0, \\ b_{22}m_{43} - b_{21}m_{44} &= 0, \\ b_{11}m_{12} - b_{21}m_{11} &= 0, \\ b_{22}m_{11} - b_{12}m_{12} &= 0. \end{aligned}$$

Taking the relations above into account we find

$$(M_1^T)^{-1} B M_2^{-1} = \frac{\sqrt{2N(2N + \beta - 1)}}{(c-1)\Delta_1\Delta_2} \begin{pmatrix} 0 & -(m_{22}b_{11} - m_{21}b_{21})m_{34} + (m_{22}b_{12} - m_{21}b_{22})m_{33} \\ 0 & 0 \end{pmatrix}.$$

Now we will show that

$$-(m_{22}b_{11} - m_{21}b_{21})m_{34} + (m_{22}b_{12} - m_{21}b_{22})m_{33} = \frac{\Delta_1\Delta_2}{\sqrt{c}}.$$

Indeed, the straightforward algebra gives

$$\begin{aligned} & -(m_{22}b_{11} - m_{21}b_{21})m_{34} + (m_{22}b_{12} - m_{21}b_{22})m_{33} \\ &= \sqrt{\beta + 2N - 1} [(\beta + 2N)F(\beta, \beta + 2N - 1; \beta + 2N; c) - 2cNF(\beta, \beta + 2N; \beta + 2N + 1; c)] \\ &\times [(\beta + 2N - 1)F(-2N, \beta - 1; \beta; c) - 2NF(-2N + 1, \beta - 1; \beta; c)] \end{aligned}$$

From the other hand,

$$\begin{aligned} \Delta_1 &= \sqrt{\beta + 2N - 1} [(\beta + 2N)F(\beta, \beta + 2N - 1; \beta + 2N; c) - 2cNF(\beta, \beta + 2N; \beta + 2N + 1; c)], \\ \Delta_2 &= \sqrt{c} [(\beta + 2N - 1)F(-2N, \beta - 1; \beta; c) - 2NF(-2N + 1, \beta - 1; \beta; c)]. \end{aligned}$$

Thus

$$(M_1^T)^{-1} B M_2^{-1} = \frac{\sqrt{2N(2N + \beta - 1)}}{(c - 1)\sqrt{c}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Formula (14.18) follows after simple algebra. $\square$

Proof of Theorem 2.9, a) and Theorem 3.8, a).

Formulae in the statements of the Theorem 2.9, a) and Theorem 3.8, a) follow immediately from Corollary 2.8, Corollary 3.7, and Proposition 14.3. $\square$

Proposition 14.4. *In the case of the Charlier weight defined by equation (2.10) we have*

$$(14.20) \quad [D, K_N] = \sqrt{\frac{2N}{a}} (\varphi_{2N-1} \otimes \varphi_{2N} + \varphi_{2N} \otimes \varphi_{2N-1}).$$

Proof. According to Proposition (12.3) the commutation relation between the operators D and K_N is determined by matrix elements of 2×2 matrix D_- defined by equation (12.1). This matrix can be found explicitly from the system of difference equations for the orthonormal functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ associated with the Charlier weight obtained in Proposition 13.2. The result is

$$D_-(x) = \begin{pmatrix} \frac{x-2N}{a} & \frac{\sqrt{2N}}{a} \\ -\sqrt{\frac{2N}{a}} & 1 \end{pmatrix}.$$

Inserting the matrix elements of the matrix $D_-(x)$ into the formula in the statement of Proposition (12.3), and taking into account that the coefficient a_{2N} in this formula equals $-\sqrt{2Na}$ we obtain the desired result. $\square$

Proof of Theorem 2.9, b) and Theorem 3.8, b).

Since $\varphi_{2N-1} \in \mathcal{H}_N$, and $\varphi_{2N} \in \mathcal{H}_N^\perp$ we can directly apply Corollary 2.8 and Corollary 3.7. $\square$

15. A LIMITING RELATION BETWEEN MEIXNER SYMPLECTIC AND ORTHOGONAL ENSEMBLES, AND THE CHARLIER SYMPLECTIC AND ORTHOGONAL ENSEMBLES

Theorem 15.1. *As $\beta \rightarrow \infty$ and $c = \frac{a}{\beta+a}$ the correlation kernels for the Meixner symplectic and orthogonal ensembles with weight $\frac{(\beta)_x}{x!} c^x$ (given in Theorem 2.9, a), Theorem 3.8, a) respectively) turn into the correlation kernels for the Charlier symplectic and orthogonal ensembles with weight $\frac{a^x}{x!}$ (given in Theorem 2.9, b), Theorem 3.8, b) respectively).*

Proof. In order to prove Theorem 15.1 we apply the formula (13.5). If $\varphi_n^{\text{Meixner}}$ denotes the n th orthonormal function associated with the Meixner polynomials, and if $\varphi_n^{\text{Charlier}}$ denotes the n th orthonormal function associated with the Charlier polynomials formula (13.5) implies

$$\lim_{\beta \rightarrow \infty} \varphi_n^{\text{Meixner}}(x; \beta, \frac{a}{\beta+a}) = \varphi_n^{\text{Charlier}}(x; a).$$

Therefore if $c = \frac{a}{\beta+a}$, and $\beta \rightarrow \infty$,

$$K_N^{Meixner}(x, y) \simeq K_N^{Charlier}(x, y).$$

Now the second statement of the Theorem can be checked immediately, taking into account that the kernels of the operators ϵ and D in the case of the Meixner ensemble become indistinguishable from the kernels of the corresponding operators in the case of the Charlier ensemble, as $\beta \rightarrow \infty$ and $c = \frac{a}{\beta+a}$. $\square$

16. A LIMITING RELATION BETWEEN THE CORRELATION FUNCTIONS OF THE MEIXNER AND THE LAGUERRE SYMPLECTIC ENSEMBLES

Fix the measure $\alpha_{Laguerre}$ on $\mathbb{R}_{\geq 0}$ defined by

$$\alpha_{Laguerre}(dx) = x^\alpha e^{-x} dx, \quad \alpha > -1.$$

Consider the set $\text{Conf}_N(\mathbb{R}_{\geq 0})$ consisting of N -point configurations $X = (x_1, \dots, x_N)$. On this set we define a probability measure $P_{N4}^{(\alpha)}$ as follows

$$P_{N4}^{(\alpha)}(dX) = \text{const}_{N4} |V(X)|^4 \alpha_{Laguerre}^{\otimes}(dX),$$

where $\alpha_{Laguerre}^{\otimes}(dX) = \prod_{i=1}^N \alpha_{Laguerre}(dx_i)$, $V(X) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$ is the Vandermonde determinant, and const_{N4} is the normalization constant.

If we interpret the points $x_1, x_2, \dots, x_N$ of the random point configuration X as the eigenvalues of a $N \times N$ quaternion real matrix then the measure $P_{N4}^{(\alpha)}$ determines the symplectic Laguerre ensemble of Random Matrix Theory, see, for example, Mehta [26], Forrester [17].

Let $\{L_n^{(\alpha)}\}_{n=0}^\infty$ be the family of the Laguerre polynomials defined by the orthogonality relation

$$\int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \frac{\Gamma(\alpha + n + 1)}{n!} \delta_{m,n},$$

and set

$$(16.1) \quad \varphi_n^{(\alpha)}(x) = \sqrt{\frac{n!}{\Gamma(\alpha + n + 1)}} L_n^{(\alpha)}(x) x^{\frac{\alpha}{2}} e^{-\frac{x}{2}}.$$

For the symplectic Laguerre ensemble the correlation kernel is the kernel of the operator $K_{N4}^{(\alpha)}$, which is of the form (see, for example, Widom [43])

$$(16.2) \quad K_{N4}^{(\alpha)} = \frac{1}{2} \begin{pmatrix} \mathcal{D} S_{N4}^{(\alpha)} & \mathcal{D} S_{N4}^{(\alpha)} \mathcal{D} \\ S_{N4}^{(\alpha)} & S_{N4}^{(\alpha)} \mathcal{D} \end{pmatrix},$$

Here the operator $\mathcal{D}S_{N4}^{(\alpha)}$ has the kernel

$$(16.3) \quad \mathcal{D}S_{N4}^{(\alpha)}(x, y) = K_{N2}^{(\alpha)}(x, y) + \frac{\sqrt{2N(2N + \alpha)}}{2} (\sqrt{2N + \alpha} \zeta_{2N}^{(\alpha)}(x) - \sqrt{2N} \zeta_{2N-1}^{(\alpha)}(x)) \\ \times (\sqrt{2N} \mathcal{E} \zeta_{2N}^{(\alpha)}(y) - \sqrt{2N + \alpha} \mathcal{E} \zeta_{2N-1}^{(\alpha)}(y)),$$

and the kernels of $S_{N4}^{(\alpha)}$, $S_{N4}^{(\alpha)}\mathcal{D}$, and $\mathcal{D}S_{N4}^{(\alpha)}\mathcal{D}$ can be obtained by action of $\mathcal{D}$ and $\mathcal{E}$. In the formulae written above the function $K_{N2}^{(\alpha)}(x, y)$ (which is the correlation kernel for the unitary Laguerre ensemble of Random Matrix Theory) is given by the formula

$$K_{N2}^{(\alpha)}(x, y) = -\sqrt{2N(2N + \alpha)} \frac{\varphi_{2N}^{(\alpha)}(x)\varphi_{2N-1}^{(\alpha)}(y) - \varphi_{2N-1}^{(\alpha)}(x)\varphi_{2N}^{(\alpha)}(y)}{x - y},$$

$\mathcal{D}$ is the operator of the differentiation, $\mathcal{E}$ is the operator with the kernel $\mathcal{E}(x, y) = \frac{1}{2} \text{sgn}(x - y)$, and $\zeta_k^{(\alpha)}(x) = x^{-1} \varphi_k^{(\alpha)}(x)$. Note that our notation is slightly different from that of Widom [43].

It is clear from the definition of the symplectic Laguerre ensemble, and from the definition of the symplectic Meixner ensemble that there is a limiting relation between correlation functions of these ensembles. Namely, denote by $\rho(x_1, \dots, x_m)$ the correlation function of the Meixner symplectic ensemble with weight $w_{\text{Meixner}}(x)$ given by formula 2.9, and by $\rho_m^{(\alpha)}(x_1, \dots, x_m)$ the correlation function of the Laguerre symplectic ensemble defined by weight $x^\alpha e^{-x}$, $x > 0$. Set $\beta = 1 + \alpha$. Then

$$(16.4) \quad \lim_{c \rightarrow 1^-} \frac{1}{(1 - c)^m} \rho_m \left(\frac{x_1}{1 - c}, \dots, \frac{x_m}{1 - c} \right) = \rho_m^{(\alpha)}(x_1, \dots, x_m).$$

Our aim here is to check equation (16.4) on the level of the correlation kernels.

Theorem 16.1. *Take $\beta = \alpha + 1$. Then for any strictly positive integers x, y we have*

$$\lim_{c \rightarrow 1^-} \frac{1}{1 - c} \mathcal{D}S_{N4} \left(\frac{x}{1 - c}, \frac{y}{1 - c} \right) = \mathcal{D}S_{N4}^{(\alpha)}(x, y), \\ \lim_{c \rightarrow 1^-} S_{N4} \left(\frac{x}{1 - c}, \frac{y}{1 - c} \right) = \frac{1}{2} S_{N4}^{(\alpha)}(x, y), \\ \lim_{c \rightarrow 1^-} \frac{1}{1 - c} (\nabla_+ S_{N4}) \left(\frac{x}{1 - c}, \frac{y}{1 - c} \right) = \frac{1}{2} \mathcal{D}S_{N4}^{(\alpha)}(x, y), \\ \lim_{c \rightarrow 1^-} -\frac{1}{1 - c} (S_{N4} \nabla_-) \left(\frac{x}{1 - c}, \frac{y}{1 - c} \right) = \frac{1}{2} \mathcal{D}S_{N4}^{(\alpha)}(y, x) = \frac{1}{2} S_{N4}^{(\alpha)} \mathcal{D}(x, y), \\ \lim_{c \rightarrow 1^-} -\frac{1}{(1 - c)^2} (\nabla_+ S_{N4} \nabla_-) \left(\frac{x}{1 - c}, \frac{y}{1 - c} \right) = \frac{1}{2} \mathcal{D}S_{N4}^{(\alpha)} \mathcal{D}(x, y).$$

Sketch of the proof. The Meixner polynomials

$$M_n(x; \beta, c) = F(-n, -x; \beta; 1 - c^{-1})$$

are related with the Laguerre polynomials $L_n^{(\alpha)}(x)$ via the formula

$$\lim_{c \rightarrow 1^-} M_n \left(\frac{x}{1-c}; \alpha + 1, c \right) = \frac{n!}{(\alpha + 1)_n} L_n^{(\alpha)}(x).$$

In addition, noting that

$$\frac{(\beta)_{\frac{x}{1-c}}}{\Gamma(1 + \frac{x}{1-c})} = \frac{\Gamma(\beta + \frac{x}{1-c})}{\Gamma(\beta)\Gamma(1 + \frac{x}{1-c})} \simeq \left(\frac{x}{1-c} \right)^{\beta-1} \frac{1}{\Gamma(\beta)} \quad \text{as } c \rightarrow 1,$$

it is not hard to check that

$$(16.5) \quad \lim_{c \rightarrow 1^-} \frac{1}{\sqrt{1-c}} \varphi_n \left(\frac{x}{1-c}; \alpha + 1, c \right) = \varphi_n^{(\alpha)}(x),$$

where $\varphi_n^{(\alpha)}(x)$ is defined by formula (16.1). Now observe that $\int_0^{+\infty} \psi_1^{(\alpha)}(x) dx = 0$, i.e.

$$(\mathcal{E}\psi_1^{(\alpha)})(y) = \frac{1}{2} \int_0^y \psi_1^{(\alpha)}(x) dx - \frac{1}{2} \int_y^{+\infty} \psi_1^{(\alpha)}(x) dx = \int_0^y \psi_1^{(\alpha)}(x) dx = - \int_y^{+\infty} \psi_1^{(\alpha)}(x) dx.$$

Indeed, for $\alpha > 0$ $\mathcal{E}\psi_1^{(\alpha)}$ is an element of $\mathcal{H}_N^{(\alpha)}$ ($\mathcal{H}_N^{(\alpha)}$ is spanned by $\varphi_0^{(\alpha)}, \varphi_1^{(\alpha)}, \dots, \varphi_{2N-1}^{(\alpha)}$), and we conclude that $\mathcal{E}\psi_1^{(\alpha)}(0) = 0$ for $\alpha > 0$. But $\mathcal{E}\psi_1^{(\alpha)}(0) = -\frac{1}{2} \int_0^{+\infty} \psi_1^{(\alpha)}(x) dx$. To check that $\int_0^{+\infty} \psi_1^{(\alpha)}(x) dx = 0$ holds true for $-1 < \alpha \leq 0$ we can use the analytic continuation with respect to the parameter α . Using these formulae we arrive to the following

Lemma 16.2. *For any two positive integers x, y*

$$\lim_{c \rightarrow 1^-} \frac{1}{1-c} K_N \left(\frac{x}{1-c}, \frac{y}{1-c} \right) = K_{N2}^{(\alpha)}(x, y),$$

$$\lim_{c \rightarrow 1^-} \frac{1}{(1-c)^{3/2}} \psi_1 \left(\frac{x}{1-c} \right) = \sqrt{2N} \zeta_{2N}^{(\alpha)}(x) - \sqrt{2N+\alpha} \zeta_{2N-1}^{(\alpha)}(x) = \psi_1^{(\alpha)}(x),$$

$$\lim_{c \rightarrow 1^-} \frac{1}{(1-c)^{3/2}} \psi_2 \left(\frac{x}{1-c} \right) = \sqrt{2N+\alpha} \zeta_{2N}^{(\alpha)}(x) - \sqrt{2N} \zeta_{2N-1}^{(\alpha)}(x) = \psi_2^{(\alpha)}(x),$$

$$\lim_{c \rightarrow 1^-} \frac{1}{\sqrt{1-c}} \left(\epsilon \psi_1 \right) \left(\frac{y}{1-c} \right) = \frac{1}{2} \left(\mathcal{E}\psi_1^{(\alpha)} \right) (y).$$

Using the limiting relations in the Lemma just stated above, and the expression for DS_{N4} in Theorem 2.9, a) we obtain

$$\lim_{c \rightarrow 1^-} \frac{1}{1-c} DS_{N4} \left(\frac{x}{1-c}, \frac{y}{1-c} \right) = \mathcal{DS}_{N4}^{(\alpha)}(x, y).$$

This is the first limiting relation between the kernels of the Meixner symplectic ensemble and the Laguerre symplectic ensemble stated in the Theorem. Other limiting relations can be obtained by action of operators $\nabla_{\pm}$ and ϵ , and by application of Lemma 16.2. $\square$

17. CORRELATION FUNCTIONS FOR THE MEIXNER ORTHOGONAL ENSEMBLE AND THE PARITY RESPECTING CORRELATIONS FOR THE LAGUERRE ORTHOGONAL ENSEMBLE

Consider the Laguerre orthogonal ensemble. This ensemble can be defined by the probability density function

$$\text{const} \cdot \prod_{i=1}^{2N} e^{-\frac{z_i}{2}} z_i^{\frac{\alpha}{2}} \prod_{1 \leq j < k \leq 2N} (z_j - z_k),$$

where $0 \leq z_1 < z_2 < \dots < z_{2N}$. Denote by $x_1, \dots, x_N$ the odd labelled particles with respect to this ordering, and by $y_1, \dots, y_N$ the even labelled particles with respect to this ordering. We will also denote the probability density function for this ensemble by $p^{(\alpha)}(x_1, \dots, x_N; y_1, \dots, y_N)$. Thus

$$p^{(\alpha)}(x_1, \dots, x_N; y_1, \dots, y_N) = \text{const} \cdot \prod_{i=1}^{2N} e^{-\frac{z_i}{2}} z_i^{\frac{\alpha}{2}} \prod_{1 \leq j < k \leq 2N} (z_j - z_k).$$

The (k_1, k_2) -point correlation function for k_1 odd labelled particles, and k_2 of even-labelled particles is defined as

$$\begin{aligned} \rho_{(k_1, k_2)}^{(\alpha)}(x_1, \dots, x_{k_1}; y_1, \dots, y_{k_2}) &= \frac{N!}{(N - k_1)!} \frac{N!}{(N - k_2)!} \\ &\times \int_{(0, +\infty)^{k_1}} \int_{(0, \infty)^{k_2}} p^{(\alpha)}(x_1, \dots, x_N; y_1, \dots, y_N) \prod_{l=k_1+1}^N \prod_{s=k_2+1}^N dx_l dy_s. \end{aligned}$$

As it is explained in Forrester and Rains [22], $\rho_{(k_1, k_2)}^{(\alpha)}(x_1, \dots, x_{k_1}; y_1, \dots, y_{k_2})$ describes parity respecting correlations of particles from the Laguerre orthogonal ensemble, see [22] for details and further references. Let $\rho_{(k_1, k_2)}(x_1, \dots, x_{k_1}; y_1, \dots, y_{k_2})$ be similarly defined correlation function for the Meixner orthogonal ensemble. Let $c \rightarrow 1^-$ and assume that

$$\left[\frac{x_1}{1-c} \right], \dots, \left[\frac{x_{k_1}}{1-c} \right] \text{ are even,} \quad \left[\frac{y_1}{1-c} \right], \dots, \left[\frac{y_{k_2}}{1-c} \right] \text{ are odd.}$$

Then definitions of the Meixner and Laguerre orthogonal ensembles imply

$$(17.1) \quad \lim_{c \rightarrow 1^-} \frac{1}{(1-c)^{k_1+k_2}} \rho_{(k_1, k_2)} \left(\left[\frac{x_1}{1-c} \right], \dots, \left[\frac{x_{k_1}}{1-c} \right]; \left[\frac{y_1}{1-c} \right], \dots, \left[\frac{y_{k_2}}{1-c} \right] \right) \\ = \rho_{(k_1, k_2)}^{(\alpha)}(x_1, \dots, x_{k_1}; y_1, \dots, y_{k_2}).$$

Theorem 17.1. *We have*

$$\rho_{(k_1, k_2)}^{(\alpha)}(x_1, \dots, x_{k_1}; y_1, \dots, y_{k_2}) \\ = \text{Pf} \begin{bmatrix} \left[K_{N1}^{(\alpha)}(x_j, x_l) \right]_{j,l=1,\dots,k_1}^{ee} & \left[K_{N1}^{(\alpha)}(x_j, y_l) \right]_{j=1,\dots,k_1; l=1,\dots,k_2}^{eo} \\ \left[K_{N1}^{(\alpha)}(y_j, x_l) \right]_{j=1,\dots,k_2; l=1,\dots,k_1}^{oe} & \left[K_{N1}^{(\alpha)}(y_j, y_l) \right]_{j,l=1,\dots,k_2}^{oo} \end{bmatrix},$$

where

$$\begin{aligned} \left[K_{N1}^{(\alpha)}(x, y) \right]^{ee} &= \begin{bmatrix} \left[\mathcal{E} S_{N1}^{(\alpha)} \right]^{ee}(x, y) & \left[S_{N1}^{(\alpha)} \right]^{ee}(x, y) \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{ee}(x, y) & \left[S_{N1}^{(\alpha)} \mathcal{E} \right]^{ee}(x, y) \end{bmatrix}, \\ \left[K_{N1}^{(\alpha)}(x, y) \right]^{oe} &= \begin{bmatrix} \left[\mathcal{E} S_{N1}^{(\alpha)} \right]^{oe}(x, y) & \left[S_{N1}^{(\alpha)} \right]^{oe}(x, y) \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{oe}(x, y) & \left[S_{N1}^{(\alpha)} \mathcal{E} \right]^{oe}(x, y) \end{bmatrix}, \\ \left[K_{N1}^{(\alpha)}(x, y) \right]^{eo} &= \begin{bmatrix} \left[\mathcal{E} S_{N1}^{(\alpha)} \right]^{eo}(x, y) & \left[S_{N1}^{(\alpha)} \right]^{eo}(x, y) \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{eo}(x, y) & \left[S_{N1}^{(\alpha)} \mathcal{E} \right]^{eo}(x, y) \end{bmatrix}, \\ \left[K_{N1}^{(\alpha)}(x, y) \right]^{oo} &= \begin{bmatrix} \left[\mathcal{E} S_{N1}^{(\alpha)} \right]^{oo}(x, y) & \left[S_{N1}^{(\alpha)} \right]^{oo}(x, y) \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{oo}(x, y) & \left[S_{N1}^{(\alpha)} \mathcal{E} \right]^{oo}(x, y) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} \left[\mathcal{E} S_{N1}^{(\alpha)} \right]^{ee}(x, y) &= K_N^{(\alpha)}(x, y) + \sqrt{2N(2N+\alpha)} \left(\mathcal{E}^e \psi_2^{(\alpha)} \right)(x) \psi_1^{(\alpha)}(y) = \left[\mathcal{E} S_{N1}^{(\alpha)} \right]^{eo}(x, y), \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \right]^{oe}(x, y) &= K_N^{(\alpha)}(x, y) + \sqrt{2N(2N+\alpha)} \left(\mathcal{E}^o \psi_2^{(\alpha)} \right)(x) \psi_1^{(\alpha)}(y) = \left[\mathcal{E} S_{N1}^{(\alpha)} \right]^{oo}(x, y), \\ \left[S_{N1}^{(\alpha)} \mathcal{E} \right]^{ee}(x, y) &= K_N^{(\alpha)}(x, y) + \sqrt{2N(2N+\alpha)} \left(\mathcal{E}^e \psi_2^{(\alpha)} \right)(y) \psi_1^{(\alpha)}(x) = \left[S_{N1}^{(\alpha)} \mathcal{E} \right]^{oe}(x, y), \\ \left[S_{N1}^{(\alpha)} \mathcal{E} \right]^{eo}(x, y) &= K_N^{(\alpha)}(x, y) + \sqrt{2N(2N+\alpha)} \left(\mathcal{E}^o \psi_2^{(\alpha)} \right)(y) \psi_1^{(\alpha)}(x) = \left[S_{N1}^{(\alpha)} \mathcal{E} \right]^{oo}(x, y), \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{ee}(x, y) &= \left[\mathcal{E}^e K_N^{(\alpha)} \right](x, y) - \mathcal{E}^e(x, y) + \sqrt{2N(2N+\alpha)} \left(\mathcal{E}^e \psi_1^{(\alpha)} \right)(x) \left(\mathcal{E}^e \psi_2^{(\alpha)} \right)(y), \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{eo}(x, y) &= \left[\mathcal{E}^e K_N^{(\alpha)} \right](x, y) - \mathcal{E}^e(x, y) + \sqrt{2N(2N+\alpha)} \left(\mathcal{E}^e \psi_1^{(\alpha)} \right)(x) \left(\mathcal{E}^o \psi_2^{(\alpha)} \right)(y), \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{oe}(x, y) &= \left[\mathcal{E}^o K_N^{(\alpha)} \right](x, y) - \mathcal{E}^o(x, y) + \sqrt{2N(2N+\alpha)} \left(\mathcal{E}^o \psi_1^{(\alpha)} \right)(x) \left(\mathcal{E}^e \psi_2^{(\alpha)} \right)(y), \\ \left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{oo}(x, y) &= \left[\mathcal{E}^o K_N^{(\alpha)} \right](x, y) - \mathcal{E}^o(x, y) + \sqrt{2N(2N+\alpha)} \left(\mathcal{E}^o \psi_1^{(\alpha)} \right)(x) \left(\mathcal{E}^o \psi_2^{(\alpha)} \right)(y), \end{aligned}$$

$$\begin{aligned}
\left[\mathcal{E} S_{N1}^{(\alpha)} \mathcal{E} - \mathcal{E} \right]^{oo} (x, y) &= \left[\mathcal{E}^o K_N^{(\alpha)} \right] (x, y) - \mathcal{E}^o (x, y) + \sqrt{2N(2N + \alpha)} \left(\mathcal{E}^o \psi_1^{(\alpha)} \right) (x) \left(\mathcal{E}^o \psi_2^{(\alpha)} \right) (y), \\
\left[S_{N1}^{(\alpha)} \right]^{ee} (x, y) &= \frac{\partial}{\partial x} K_N^{(\alpha)} (x, y) + \frac{\sqrt{2N(2N + \alpha)}}{2} \psi_2^{(\alpha)} (x) \psi_1^{(\alpha)} (y), \\
\left[S_{N1}^{(\alpha)} \right]^{ee} (x, y) &= \left[S_{N1}^{(\alpha)} \right]^{oe} (x, y) = \left[S_{N1}^{(\alpha)} \right]^{eo} (x, y) = \left[S_{N1}^{(\alpha)} \right]^{oo} (x, y).
\end{aligned}$$

In the formulae above the operators $\mathcal{E}^e$ and $\mathcal{E}^o$ are defined by the relations

$$(\mathcal{E}^e f)(x) = -\frac{1}{2} \int_x^{+\infty} f(y) dy, \quad (\mathcal{E}^o f)(x) = \frac{1}{2} \int_0^x f(y) dy.$$

Proof. The correlation kernel K_{N1} for the Meixner orthogonal ensemble is obtained explicitly in Theorem 3.8, a). To find the kernel for the correlation function $\rho_{(k_1, k_2)}^{(\alpha)}$ we need to compute $K_{N1}(\left[\frac{x}{1-c}\right], \left[\frac{y}{1-c}\right])$ as $c \rightarrow 1^-$. This can be done exploiting formulae in Lemma 16.2, and in addition the following limiting relations

$$\begin{aligned}
\lim_{c \rightarrow 1^-} \frac{1}{\sqrt{1-c}} (\epsilon \psi_2) \left(\left[\frac{x}{1-c} \right] \right) &= \begin{cases} \left(\mathcal{E}^e \psi_2^{(\alpha)} \right) (x), & \left[\frac{x}{1-c} \right] \text{ is even,} \\ \left(\mathcal{E}^o \psi_2^{(\alpha)} \right) (x), & \left[\frac{x}{1-c} \right] \text{ is odd,} \end{cases} \\
\lim_{c \rightarrow 1^-} (\epsilon K_N) \left(\left[\frac{x}{1-c} \right], \left[\frac{y}{1-c} \right] \right) &= \begin{cases} \left(\mathcal{E}^e K_N^{(\alpha)} \right) (x, y), & \left[\frac{x}{1-c} \right] \text{ is even,} \\ \left(\mathcal{E}^o K_N^{(\alpha)} \right) (x, y), & \left[\frac{x}{1-c} \right] \text{ is odd.} \end{cases}
\end{aligned}$$

□

Observe that the structure of the parity respecting correlation functions in Theorem 17.1 is very similar to that in Section 3.1 of [22]. For $\alpha = 0$ the kernels of Theorem 17.1 and the case $A = 0$ of [22] must be equivalent. However, direct verification of this fact is not an easy task, see e.g. Section 5.1 of [22], and we postpone the discussion of this equivalence until a later publication.

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DEPARTMENT OF MATHEMATICS, 253-37, CALTECH, PASADENA, CA 91125

E-mail address: `borodin@caltech.edu`

DEPARTMENT OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, GIVAT RAM, JERUSALEM 91904

E-mail address: `strahov@math.huji.ac.il`