[2] For a detailed account of the methods reference may be made to C. Ramsauer, *Ann. Physik.*, **72**, p. 345, 1923, and *Jahrb. Radioaktivität*, **19**, p. 345, 1923; R. B. Brode, *Physic. Rev.*, **25**, p. 636, 1925, and *Proc. of the Royal Soc.*, **109**, p. 397, 1925.

[3] F. Zwicky, *Physik. Zeitschr.*, **24**, p. 171, 1923.

[4] E. C. Stoner, *Phil. Mag.*, **48**, p. 719, 1924.

[5] F. Hund, *Zs. Physik*, **13**, p. 241, 1923; G. Wentzel, *Ibid.*, **15**, p. 172, 1923.

TRANSFER OF ENERGY FROM ELECTRONS TO ATOMS

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In the preceding paper[1] we have assumed that the electric dipole induced in an atom by a passing electron has the value corresponding to a static field which is equal to the instantaneous field of the electron. This is, however, not quite exact. Due to the rapid variation of the field there will be some sort of a time lag. The deflections will therefore be smaller than calculated in section A of preceding paper. But the most important result of the more rigorous calculation is that it yields an approximation to the transfer of energy which escaped our attention while we were using the approximation method A in the mentioned publication.[1] It is possible to solve also this problem on the basis of the perturbation theory. As the general treatment involves very complicated mathematics we will demonstrate our ideas with a simple model which gives all the principal effects. We choose for this purpose a linear oscillator with the characteristic frequency $\nu_0 = \omega/2\pi$. We then calculate the perturbation caused by an electron passing it with the velocity v. The axis of the oscillator will be taken normal to the unperturbed straight line orbit of the electron, and the distance from the oscillator to the intersection of both lines we call a. The elastic constant of the oscillator is f and m_0 its movable mass (with the coördinate x) which bears the charge ϵ. We have then, assuming the inverse square law

$$m_0 \frac{d^2x}{dt^2} + fx = \frac{e\epsilon a}{(a^2 + v^2t^2)^{3/2}} = F(t); \qquad \omega^2 = \frac{f}{m_0}. \tag{1}$$

The particular integral $\bar{x}$ of differential equations of this type can be found in a general way in putting

$$\bar{x} = A(t) \sin \omega t + B(t) \cos \omega t$$

where $A(t)$ and $B(t)$ have to be determined as functions of t, so as to satisfy

the equation 1. As 1 furnishes only one relation for A and B we still can restrict the two functions by an arbitrary condition, as which we choose

$$\frac{dA}{dt}\sin\omega t + \frac{dB}{dt}\cos\omega t \equiv 0. \qquad (2)$$

This combined with the equation 1 gives

$$\frac{dA}{dt}\cos\omega t - \frac{dB}{dt}\sin\omega t = \frac{F}{m_0\omega} \qquad (2')$$

so that it follows

$$A(t) = \int_{-\infty}^{t} \frac{F(t)}{m_0\omega}\cos\omega t\,dt, \qquad B(t) = -\int_{-\infty}^{t} \frac{F(t)}{m\omega}\sin\omega t\,dt.$$

With this expression we get the general solution of 1 in the form

$$x = e\epsilon a\left[\sin\omega t\int_{-\infty}^{t}\frac{\cos\omega t\,dt}{(a^2+v^2t^2)^{3/2}} - \cos\omega t\int_{-\infty}^{t}\frac{\sin\omega t\,dt}{(a^2+v^2t^2)^{3/2}}\right] + A_0\sin\omega t + B_0\cos\omega t \qquad (3)$$

A_0 and B_0 are constants.

We are especially interested in the particular integral $\bar{x}$ which represents the complete solution of our problem in case the oscillator has been at rest for $t = -\infty$. For $t < 0$, $\bar{x}$ can be transformed with help of the following relations.[2]

$$\text{for } t<0\quad \int_{-\infty}^{t}\frac{\cos\omega t\,dt}{(a^2+v^2t^2)^{3/2}} = -\frac{1}{a}\int_{-\infty}^{t}\int_{0}^{\infty} e^{vts}J_0'(as)\cos\omega t\,ds\,dt = -\frac{1}{a}\int_{0}^{\infty}\frac{e^{vst}}{\omega^2+v^2s^2}[vs\cos\omega t+\omega\sin\omega t]J_0'(as)\,ds$$

$$\text{and}\quad \int_{-\infty}^{t}\frac{\sin\omega t\,dt}{(a^2+v^2t^2)^{3/2}} = -\frac{1}{a}\int_{-\infty}^{t}\int_{0}^{\infty} e^{vts}J_0'(as)\sin\omega t\,ds\,dt = -\frac{1}{a}\int_{0}^{\infty}\frac{e^{vst}}{\omega^2+v^2s^2}[vs\sin\omega t-\omega\cos\omega t]J_0'(as)\,ds$$

J_0 is the Bessel function of the order zero. It follows then for the particular solution $\bar{x}$ that

$$\text{for } t<0 \qquad \bar{x} = -\epsilon e\int_{0}^{\infty}\frac{e^{vts}J_0'(as)\,s\,ds}{m_0v^2s^2+f}.$$

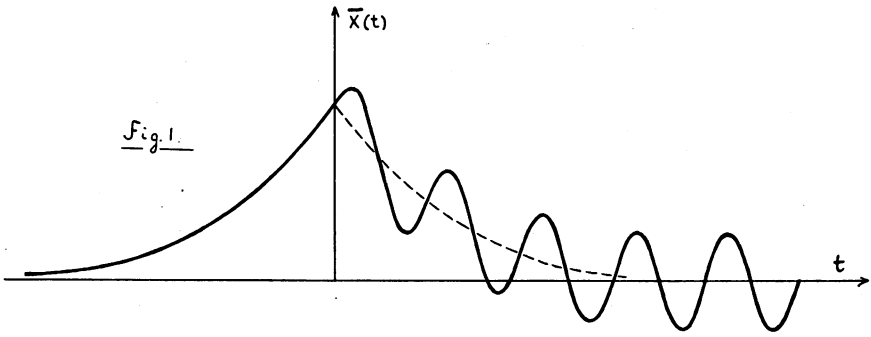
One obtains in the same way

$$\text{for } t>0\quad \bar{x} = -\epsilon e\int_{0}^{\infty}\frac{e^{-vts}J_0'(as)\,s\,ds}{m_0v^2s^2+f} + \frac{ea\epsilon}{m_0\omega}\sin\omega t\int_{-\infty}^{+\infty}\frac{\cos\omega s\,ds}{(a^2+v^2s^2)^{3/2}}$$

From this it is to be seen that

$$\bar{x}(t) = \bar{x}(-t) + \frac{ea\epsilon}{m_0\omega} \sin \omega t \int_{-\infty}^{+\infty} \frac{\cos \omega s \, ds}{(a^2 + v^2t^2)^{3/2}} = \bar{x}(-t) + A_1 \sin \omega t. \quad (4)$$

figure 1 shows the main features of the motion of the oscillator as given by the particular integral.



The sine function on the right represents the oscillation excited in the atom by the passing electron. The energy transfer is involved in this member, whereas the symmetrical part affects only the deflections without change of the kinetic energy of the electron. This will be discussed more fully in another paper. The electron loses, according to 3 and 4, the energy

$$\Delta U = f x_{\max.}^2 - f(A_0^2 + B_0^2) = f A_1^2 + 2 f A_0 A_1$$

which is transferred to the oscillator. (This would mean a transformation of the kinetic energy of the electron into scattered light of frequency ν_0.) We shall show that the magnitude of ΔU is in conspicuous contradiction with the facts, even if we take the lowest average value for ΔU assuming that the oscillator has been at rest before the interaction ($A_0 = 0$ and $B_0 = 0$). To obtain an approximate numerical calculation of $\overline{\Delta U}$, we replace the function $(a^2 + v^2s^2)^{-3/2}$ by $a^{-3}e^{-3/2\frac{v^2}{a^2}S^2}$ which is approximately equal. (It is also possible to evaluate the integral rigorously with the help of Hankel functions.) Then

$$\overline{\Delta U} = \frac{2\pi}{3} \frac{e^2\epsilon^2}{mv^2a^2} e^{-\frac{\omega^2a^2}{3v^2}}. \quad (5)$$

The maximum of $\overline{\Delta U}$ occurs for $\frac{\omega^2a^2}{3v^2} = 1$ and has the magnitude

$$\overline{\Delta U}_{\max.} = 0.73\pi \frac{e^2\alpha}{a^4} = -1.47\,\pi\, U(a)$$

where $U(a)$ is the potential energy of the electron in relation to the atom when their mutual distance is a. Experimentally we know that impacts of electrons (with energies below the resonance potential) with atoms of noble gases and metal vapors are completely elastic, the energy losses being only of the order $m/M < 1/1840$. It is easy to show that our ΔU is much too great. As an example we choose the data of Argon: $\omega = 19{,}8 \cdot 10^{15}$ ($2\pi \times$ resonance frequency) and $\alpha = 16{,}4 \cdot 10^{-25}$ cm.[3] With v corresponding to 3 volts one obtains for $\overline{\Delta U}$ at a distance $a = 2$ Å. and $a = 3$ Å., respectively, $\overline{\Delta U} = 0.7 \cdot 10^{-12}$ erg. and $\overline{\Delta U} = 0.1 \cdot 10^{-12}$ erg., whereas $mv^2/2 = 4{,}77 \cdot 10^{-12}$ erg. Passing at the distance of 2 Å. from the atom (oscillator) the electron would therefore still lose $1/5$ of its energy, which is in complete contradiction with the experimental facts. The energy losses in diatomic gases must be ascribed to the much lower and numerous critical potentials. An experimental investigation of the atoms with electron affinity would be of great interest. These atoms are able to catch a surplus electron much as ions do. In case a capture is not affected the impacts are probably completely elastic also, but this has not yet been proved experimentally.

Our considerations are also valid in Bohr's formulation of the quantum theory, so that our aperiodic case seems to illustrate a very serious difficulty for this theory. The true quantum laws evidently must be of such a kind that they are invariant against any mechanical perturbation below the resonance, and not only against very slow ones (extension of Ehrenfest's adiabatic principle). There does not exist a continuous dissipation of mechanical energy as we know it for light (scattering, dispersion). This contradiction between theory and facts which we have pointed out is especially clean cut, as it is deduced on purely mechanical basis, without any assumptions concerning the radiation. The computations being based on perturbation theory only, there seems to exist no serious difficulty in carrying through analogous ones in the new quantum theory, operating with matrices. And as it is an aperiodic case which could not be handled in a satisfactory way by the old theory, it will offer an excellent example to decide whether the new theory leads any farther or not.

It should be mentioned that Fermi[3] has published interesting calculations on the efficiency of inelastic impacts which have some relation to our considerations. He calculates the action of an electron on an atom by representing the field of the electron in form of a Fourier integral, which is mathematically equivalent to a continuous spectrum of light. In order to remain in agreement with the facts he has to assume that only those frequencies ν in this spectrum are effective for which $h\nu < \frac{mv^2}{2}$, whereas for higher frequencies nothing happens. This, however, is just the point, which has to be deduced by a treatment similar to that given

in the present paper, but based on a rational quantum theory, rather than on classical mechanics.

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[1] F. Zwicky, "The Quantum Theory and the Behavior of Slow Electrons in Gases." (In this issue of these PROCEEDINGS.)

[2] G. N. Watson, *Theory of Bessel Functions*, Cambridge University Press, **1922**, p. 384, equation 1.

[3] F. Fermi, *Zeitschr. Physik.*, **29**, p. 315, 1924.

FORM OF THE NUMBER OF THE PRIME POWER SUBGROUPS OF AN ABELIAN GROUP

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1. If the order g of a group G is divisible by p^α, p being any prime number, then the number of the subgroups of order p^α contained in G is always of the form $1 + kp$, where k is either 0 or a positive integer. The object of the present paper is to develop a useful extension of this well-known theorem for the case when G is abelian. In order to understand the nature of this extension it seems desirable to explain first a certain linear arrangement of the possible types of abelian subgroups of the same prime power order p^α, so that we can say definitely which of two subgroups of the same order is of the higher type whenever these subgroups are not of the same type.

Let H_1 and H_2 represent two abelian groups of the same order p^α, and of types $(\alpha_1, \alpha_2, \ldots., \alpha_\lambda)$, $(\beta_1, \beta_2, \ldots., \beta_\gamma)$, respectively, where both the α's and the β's are arranged in descending order of magnitude in case of a difference in magnitude. If the first α which is not equal to the corresponding β exceeds this β then H_1 is said to be of a higher type than H_2, and vice versa. The extension to which we referred above may now be stated as follows: *If any abelian group involves subgroups of order p^α which are of different types then the number of these subgroups of lowest type is always of the form $1 + kp$ while the number of those of every higher type but of the same order is always divisible by p.*

2. To simplify a proof of this theorem it will first be assumed that the order of the abelian group G is of the form p^m. If the number of the independent generators of G is at least equal to α the subgroups of order p^α which are of lowest type are composed of all the subgroups of this order