

From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion

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We combine tools from effective field theory and generalized unitarity to construct a map between on-shell scattering amplitudes and the classical potential for interacting spinless particles. For general relativity, we obtain analytic expressions for the classical potential of a binary black hole system at second order in the gravitational constant and all orders in velocity. Our results exactly match all known results up to fourth post-Newtonian order, and offer a simple check of future higher order calculations. By design, these methods should extend to higher orders in perturbation theory.

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Introduction.—The theory of scattering amplitudes has revealed unique insights into the structure of quantum field theory (QFT) and inspired powerful new tools for calculation. While phenomenological applications have largely centered on high-energy colliders, an effort has emerged to connect the amplitudes program to the physics of gravitational waves, which were recently discovered at LIGO [1].

Unfortunately, any attempt at bridging these subjects is immediately confounded by the fact that a binary black hole inspiral is quite dissimilar from black hole scattering. The latter is a transient interaction of widely separated black holes which are effectively free before and after the event. The former describes objects bound in quasicircular orbit by a classical conservative potential, together with the dissipative radiation-reaction force induced by gravitational wave emission.

There is a long history of mapping scattering observables to the classical gravitational potential; e.g., see the seminal work of Refs. [2,3] as well as more recent treatments [4–11]. In this Letter we unify ideas from effective field theory (EFT) and generalized unitarity to systematize this procedure for a general QFT of spinless particles [4,5]. To begin, we construct an EFT for two nonrelativistic (NR) scalars which interact via the classical potential V . Since the two-particle on-shell amplitudes in the EFT and full theory are equal, i.e., $M_{\text{EFT}} = M$, we can determine V order by order in perturbation theory.

Of course, on-shell methods like generalized unitarity vastly simplify amplitude calculations (see Refs. [12,13]

and references therein). In this approach, M is expressed not in terms of Feynman diagrams but rather as a sum of scalar integrals weighted by scalar integral coefficients which are rational functions of the external momenta.

Our main results are summarized in Eq. (23), which recasts the coefficients c of the classical potential in terms of the scalar integral coefficients d in a general QFT at leading and next-to-leading order in the interaction strength. For general relativity (GR), we obtain Eqs. (26) and (27), which are new analytic expressions for the potential at second post-Minkowskian (2PM) order, i.e., at $\mathcal{O}(G^2)$ and at *all* orders in velocity. These equations are physically equivalent to all state-of-the-art results, which extend to fourth post-Newtonian (4PN) order [14–16]. Since our results include information at all orders in the PN expansion, they may be useful for checking future higher order calculations. The present work goes beyond previous calculations of the 2PM amplitude [5,8,11] by deriving an explicit mapping to the 2PM potential.

This work introduces several new methods. First, we show how calculations are drastically simplified when the classical limit is taken at the earliest possible stage of the computation. This is implemented by a simple power counting scheme in large angular momentum $J \gg 1$, together with a restriction on loop momenta to the so-called potential region of kinematics. Copious quantum mechanical contributions are thus truncated at the integrand level, while complicated four-dimensional integrals are reduced to far simpler three-dimensional ones.

Second, we introduce the method of “integrand subtraction” to effectively eliminate three-dimensional integrals which can be quite complex due to infrared singularities. In this approach, the *difference* of the integrands in the full theory and EFT are similar to those encountered in NR GR [17,18] and easily integrate to purely rational functions of the external kinematics.

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Third, we show how gauge-dependent quantities like the classical potential can be compared by computing gauge-invariant on-shell scattering amplitudes without the need for constructing explicit coordinate transformations or wrangling with equations of motion ambiguities.

Effective field theory.—*Definition:* An EFT for NR scalar fields A and B is described by the action $S = \int dt(L_{\text{kin}} + L_{\text{int}})$, where

$$L_{\text{kin}} = \int_{\mathbf{k}} A^\dagger(-\mathbf{k}) \left(i\partial_t - \sqrt{\mathbf{k}^2 + m_A^2} \right) A(\mathbf{k}) + \int_{\mathbf{k}} B^\dagger(-\mathbf{k}) \left(i\partial_t - \sqrt{\mathbf{k}^2 + m_B^2} \right) B(\mathbf{k}) \quad (1)$$

is the kinetic term and the interaction term is [4]

$$L_{\text{int}} = - \int_{\mathbf{k}, \mathbf{k}'} V(\mathbf{k}, \mathbf{k}') A^\dagger(\mathbf{k}') A(\mathbf{k}) B^\dagger(-\mathbf{k}') B(-\mathbf{k}). \quad (2)$$

Here, $\int_{\mathbf{k}_1 \dots \mathbf{k}_n} = \int [d^3\mathbf{k}_1/(2\pi)^3] \dots [d^3\mathbf{k}_n/(2\pi)^3]$ and the Feynman vertex $V(\mathbf{k}, \mathbf{k}')$ is the potential in the center of mass frame.

Classical limit.—The above EFT is obtained from the full theory by integrating out massless force carriers mediating near-instantaneous interactions and taking the NR limit, $|\mathbf{k}|, |\mathbf{k}'| \ll m_{A,B}$. By definition, these potential modes have energies parametrically less than their momenta, so $|k_0 - k'_0| \ll |\mathbf{k} - \mathbf{k}'|$. (While it may seem peculiar to integrate out massless states, the potential modes are off shell. Moreover, the EFT contains ultrasoft modes with energy and momenta of order $|\mathbf{k} - \mathbf{k}'|$, but these encode dissipative effects irrelevant to the conservative potential.) For a classical system, the NR particles are separated by a distance $|\mathbf{r}| \sim 1/|\mathbf{k} - \mathbf{k}'|$ that is parametrically larger than the Compton wavelengths of the particles, $|\mathbf{k}|, |\mathbf{k}'|$. The resulting hierarchy, $|\mathbf{k} - \mathbf{k}'| \ll |\mathbf{k}|, |\mathbf{k}'|$, corresponds to an expansion in large angular momentum, $J \sim |\mathbf{k} \times \mathbf{r}| \gg 1$, as utilized by Damour [7,10]. The classical component of any quantity is then extracted via the scaling

$$J^{-1} \propto \mathbf{k} - \mathbf{k}' \propto \kappa^{-1}, \quad (3)$$

where $\mathbf{k}, \mathbf{k}' \propto 1 + J^{-1}$. The first relation holds because angular momentum scales linearly with distance while the second relation holds due to the virial theorem. Here κ is the coupling constant, which, for example, in gravity is the gravitational constant, $\kappa = 4\pi G$. The classical potential has the same scaling as the leading Coulomb interaction, $\kappa/|\mathbf{k} - \mathbf{k}'|^2 \propto J^3$.

Higher order potential terms are parametrized by arbitrary Hermitian combinations of the rotational invariants $\mathbf{k}^2, \mathbf{k}'^2$, and $\mathbf{k} \cdot \mathbf{k}'$. However, since $\mathbf{k}^2 - \mathbf{k}'^2$ vanishes on shell, it can be eliminated by a field redefinition. Similarly,

Eq. (2) has no energy dependence since energy can also be traded for \mathbf{k}^2 and \mathbf{k}'^2 via the equations of motion. We thus choose a field basis in which V only depends on $\mathbf{k}^2 + \mathbf{k}'^2$ and $|\mathbf{k} - \mathbf{k}'|$, so [4]

$$V(\mathbf{k}, \mathbf{k}') = \frac{\kappa}{|\mathbf{k} - \mathbf{k}'|^2} (c_1 + c_2 \kappa |\mathbf{k} - \mathbf{k}'| + \dots), \quad (4)$$

where we have only included terms that are classical and thus scale as J^3 in accordance with Eq. (3), and the ellipsis denotes terms higher order in κ . (Higher order classical terms odd in κ include factors of $\log|\mathbf{k} - \mathbf{k}'|$.) $c_i(\mathbf{k}^2 + \mathbf{k}'^2/2)$ are momentum-dependent functions characterizing contributions at the i th order in the coupling constant and all orders in velocity. Here we make the usual assumption that there is a convergent velocity expansion.

Amplitudes.—From Eq. (1) and Eq. (2) it is straightforward to obtain the Feynman rules,

$$\begin{aligned} \overrightarrow{(k_0, \mathbf{k})} &= \frac{i}{k_0 - \sqrt{\mathbf{k}^2 + m_{A,B}^2} + i0}, \\ \begin{array}{c} \mathbf{k} \quad \mathbf{k}' \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ -\mathbf{k} \quad -\mathbf{k}' \end{array} &= -iV(\mathbf{k}, \mathbf{k}'), \end{aligned} \quad (5)$$

where from here on the $+i0$ prescription will be implicit.

We are interested in the scattering amplitude for a process where \mathbf{p} and \mathbf{p}' are the incoming and outgoing three-momenta in the center-of-mass frame, and E_A and E_B are the energies of the incoming particles,

$$E_{A,B} = \sqrt{\mathbf{p}^2 + m_{A,B}^2} = \sqrt{\mathbf{p}'^2 + m_{A,B}^2}. \quad (6)$$

We define the total energy and the reduced energy ratio,

$$E = E_A + E_B \quad \text{and} \quad \xi = \frac{E_A E_B}{(E_A + E_B)^2}. \quad (7)$$

Note that $0 \leq \xi \leq 1/4$ and, moreover, ξ and E are dependent variables since E_A and E_B are related through Eq. (6). We also define the momentum transfer $\mathbf{q} = \mathbf{p} - \mathbf{p}' \propto J^{-1}$, with classical scaling dictated by Eq. (3).

The EFT amplitude can either be organized in terms of the κ expansion or in terms of loop orders, so

$$M_{\text{EFT}} = \sum_{i=1}^{\infty} M_{\text{EFT}}^{(i)} = \sum_{L=0}^{\infty} M_{\text{EFT}}^{L\text{loop}}, \quad (8)$$

where $M_{\text{EFT}}^{(i)}$ is at i th order in κ and arises from Feynman diagrams at $i - 1$ loops and below.

Since pair creation of matter particles is kinematically forbidden in the NR limit, the amplitude at L loops is comprised purely of iterated bubbles, so

$$M_{\text{EFT}}^{L\text{-loop}} = \dots \quad (9)$$

For convenience, we merge each pair of matter lines into an effective ‘‘two-body propagator,’’

$$\begin{aligned} \Delta(\mathbf{k}) &= i \int \frac{dk_0}{2\pi} \frac{1}{k_0 - \sqrt{\mathbf{k}^2 + m_A^2}} \frac{1}{E - k_0 - \sqrt{\mathbf{k}^2 + m_B^2}} \\ &= \frac{1}{E - \sqrt{\mathbf{k}^2 + m_A^2} - \sqrt{\mathbf{k}^2 + m_B^2}}, \end{aligned} \quad (10)$$

where the second line is obtained by closing the contour in k_0 either upwards or downwards in the complex plane. The contribution at L loops is then

$$\begin{aligned} M_{\text{EFT}}^{L\text{loop}} &= - \int_{\mathbf{k}_1 \dots \mathbf{k}_L} V(\mathbf{p}, \mathbf{k}_1) \Delta(\mathbf{k}_1) \dots \Delta(\mathbf{k}_L) V(\mathbf{k}_L, \mathbf{p}') \\ &= - \int_{\mathbf{k}_1 \dots \mathbf{k}_L} \frac{\mathcal{N}_{\text{EFT}}^{L\text{loop}}}{X_1^2 X_2^2 \dots X_{L+1}^2 Y_1 Y_2 \dots Y_L}, \end{aligned} \quad (11)$$

where in the second line we have substituted the internal loop momenta \mathbf{k}_n for equivalent variables,

$$X_n = |\mathbf{k}_{n-1} - \mathbf{k}_n| \quad \text{and} \quad Y_n = \mathbf{k}_n^2 - \mathbf{p}^2, \quad (12)$$

describing the momentum transfer at each vertex and the off shellness of each pair of matter propagators, respectively. Here $\mathbf{k}_0 = \mathbf{p}$ and $\mathbf{k}_{L+1} = \mathbf{p}'$ and $\mathcal{N}_{\text{EFT}}^{L\text{loop}}$ is a regular function of X_n and Y_n obtained from the Laurent expansion of the first line of Eq. (11) in those variables.

The variables in Eq. (12) have several advantages. First, since $X_n \propto J^{-1}$, $Y_n \propto J^{-1} + J^{-2}$ and $\int_k \propto J^{-3}$ in accordance with Eq. (3), we can trivially extract the classical contribution by expanding $\mathcal{N}_{\text{EFT}}^{L\text{loop}}$ in the limit of small Y_n , keeping terms through order $\mathcal{O}(Y_n^L)$ and all orders in velocity. Second, Eq. (11) manifests all singularities from matter particles as simple poles in Y_n . These singularities correspond to lower order iterated contributions that are infrared divergent. As we will see, since these iterations must exactly cancel against similar terms in the full theory, they play no role in the determination of the c_i . On the other hand, terms that are regular in Y_n produce rational functions of the kinematic variables after integration and do affect c_i .

Full theory.—Scalar integral decomposition: We now decompose all full theory amplitudes into a basis of scalar functions of the external four-momenta. Here, $p_1 = (E_A, \mathbf{p})$ and $p_2 = (E_B, -\mathbf{p})$ are the incoming four-momenta while $p_3 = (E_A, \mathbf{p}')$ and $p_4 = (E_B, -\mathbf{p}')$ are outgoing. Like before, we decompose the full theory amplitude,

$$M = \sum_{i=1}^{\infty} M^{(i)}, \quad (13)$$

where $M^{(i)}$ is the contribution at i th order in κ , which arises purely at $i - 1$ loops in the full theory. Here we define M to have NR normalization, so it is proportional to the usual relativistic amplitude $\tilde{M} = 4E_A E_B M$. At tree level, the relativistic tree amplitude is

$$\kappa^{-1} \tilde{M}^{(1)} = d_{\perp} \mathcal{I}_{\perp}, \quad (14)$$

where d_{\perp} is a function of the external kinematics and the scalar tree function is defined as

$$\mathcal{I}_{\perp} = \frac{1}{(p_1 - p_3)^2} = -\frac{1}{\mathbf{q}^2}. \quad (15)$$

Similarly, the one-loop amplitude is

$$\kappa^{-2} \tilde{M}^{(2)} = d_{\square} \mathcal{I}_{\square} + d_{\nabla} \mathcal{I}_{\nabla} + d_{\Delta} \mathcal{I}_{\Delta} + \dots, \quad (16)$$

where the ellipsis denotes rational, bubble, and crossed-box contributions which do not contribute classically. The scalar basis integrals are

$$\begin{aligned} \mathcal{I}_{\square} &= \int_k \frac{1}{(p_1 - k)^2} \frac{1}{(k - p_3)^2} \frac{1}{k^2 - m_A^2} \\ &\quad \times \frac{1}{(p_1 + p_2 - k)^2 - m_B^2}, \\ \mathcal{I}_{\nabla, \Delta} &= \int_k \frac{1}{(p_1 - k)^2} \frac{1}{(k - p_3)^2} \frac{1}{k^2 - m_{A,B}^2}, \end{aligned} \quad (17)$$

where $\int_k = \int [d^4 k / (2\pi)^4]$. The scalar coefficients d_{\square} , d_{∇} , and d_{Δ} are rational functions of the external kinematics.

Reduction to three-dimensional integrals.—To compare the EFT and full theory amplitudes at the integrand level, we reduce the four-dimensional integrals in Eq. (17) to three-dimensional integrals expressed in terms of X_n and Y_n . Our approach is similar to the method of regions [19,20] except with an alternative prescription for contour integrals. While the relativistic one-loop integrals in Eq. (17) have been computed previously, the procedure outlined here is formulated with the expressed purpose of scaling mechanically to higher loop orders.

In terms of the variables defined in Eq. (12), the triangle integral in Eq. (17) is

$$\mathcal{I}_{\nabla} = \int \frac{1}{k_0^2 - X_1^2 + i0} \frac{1}{k_0^2 - X_2^2 + i0} \frac{1}{k_0^2 + 2E_A k_0 - Y_1 + i0}, \quad (18)$$

where we parametrize $k = (E_A + k_0, \mathbf{k}_1)$ so that k_0 describes deviations from an instantaneous potential. The

classical potential is generated by off-shell mediators in the potential region with $|k_0| \ll X_{1,2}$. Thus, we consider the contribution to the integral in Eq. (18) from a contour on the real k_0 axis along this interval.

We can evaluate this by pushing the contour either upwards or downwards, provided one includes nonzero contributions from the upper or lower arc. Including relative signs from contour orientation, these arc contributions are equal and opposite and their difference is the residue at infinity, which is in general nonzero. Thus, an equivalent but more convenient prescription is to take the average result from pushing the contour upwards and downwards, i.e., half the sum of all residues enclosed by the full circle, including signs from orientation. Crucially, this region does *not* contain any poles from mediator propagators, since $|k_0| \ll X_{1,2}$. Furthermore, while the matter propagator contains both a particle and antiparticle pole, at most one can lie in the potential region. So, e.g., in Eq. (18) we would only include the pole at $k_0 = \sqrt{E_A^2 + Y_1} - E_A - i0$ because it vanishes in the instantaneous limit, $Y_1 = 0$. The upshot is that the energy integral from the potential region effectively yields

$$\int \frac{dk_0}{2\pi} (\cdot) = \frac{i}{2} \left(\sum_{k_* \in \mathbb{H}^+} \text{Res}(\cdot) - \sum_{k_* \in \mathbb{H}^-} \text{Res}(\cdot) \right), \quad (19)$$

where the sum runs over residues k_* from the potential region matter poles in the upper or lower half planes, \mathbb{H}^\pm . This prescription is equivalent to applying key identities used in showing the exponentiation of the eikonal amplitude [21,22].

After performing the k_0 integration we expand the remaining three-dimensional integrand in the NR limit of large $m_{A,B}$. We then extract the classical contribution according to the J power counting discussed below Eq. (12). For example, the triangle integrals in Eq. (17) depend only on the four-momenta p_1 and $p_1 - p_3$, whose on-shell inner products are functions only of $|\mathbf{q}|$ and $m_{A,B}$. Hence, the NR expansion is a power series in $|\mathbf{q}|/m_{A,B}$ and since $\mathbf{q} \propto J^{-1}$ it is obvious that the classical term coincides with the leading term in the large mass expansion. Similarly, for the box integral we expand in large $E_{A,B}$.

In summary, the scalar integrals can be written as

$$\begin{aligned} \mathcal{I}_\square &= \frac{i}{2E} \int_k \frac{1}{X_1^2 X_2^2 Y_1} + \dots, \\ \mathcal{I}_{\nabla, \Delta} &= -\frac{i}{4m_{A,B}} \int_k \frac{1}{X_1^2 X_2^2} + \dots, \end{aligned} \quad (20)$$

where the ellipses denote contributions which are higher order in J^{-1} and thus quantum, and $\int_k (1/X_1^2 X_2^2) = (1/8|\mathbf{q}|)$ by standard integral formulas [23]. Including the coupling constant, we find that $\kappa^2 \mathcal{I}_\square \propto J^4$ and $\kappa^2 \mathcal{I}_{\nabla, \Delta} \propto J^3$, so the triangle is classical but the box is actually superclassical

since it encodes iterations of the tree-level potential that will cancel with similar terms in the EFT.

Matching calculation.—The potential coefficients c_i are obtained by matching the EFT and full theory amplitudes order by order in κ , so $M^{(i)} - M_{\text{EFT}}^{(i)} = 0$. This procedure is greatly simplified by expressing this difference of amplitudes at the integrand level, since terms with poles in Y_n , which evaluate to infrared nonanalyticities are canceled without performing complicated integrations. This cancellation occurs because the EFT and full theory have identical cut structure at low energies, as mandated by the starting assumption that the theories describe the same infrared dynamics. This holds at all loops, provided all relevant momentum regions have been included in the EFT.

That such a subtraction can be done at the integrand level should not be obvious because loop momenta in distinct diagrams generally have ambiguous relative orientation since there is no intrinsic origin in loop momentum space. Crucially, in our case the integrands can be aligned by matching their Y_n poles. The remaining terms then trivially integrate to rational functions of the external kinematics.

At leading and next-to-leading order in κ we find

$$\begin{aligned} M^{(1)} - M_{\text{EFT}}^{(1)} &= \frac{\kappa}{q^2} \left[c_1(\mathbf{p}^2) - \frac{d_\perp}{4E^2 \xi} \right], \\ M^{(2)} - M_{\text{EFT}}^{(2)} &= \kappa^2 \left[\frac{c_2(\mathbf{p}^2)}{|\mathbf{q}|} + \int_k \frac{\mathcal{N}^{\text{1loop}} - \mathcal{N}_{\text{EFT}}^{\text{1loop}}}{X_1^2 X_2^2 Y_1} \right], \end{aligned} \quad (21)$$

where the EFT and full theory integrand numerators are

$$\begin{aligned} \mathcal{N}_{\text{EFT}}^{\text{1loop}} &= \left[2E\xi + Y_1 \left(\frac{1-3\xi}{2E\xi} + E\xi \partial_{p^2} \right) \right] c_1^2(\mathbf{p}^2), \\ \mathcal{N}^{\text{1loop}} &= \frac{d_\square}{8E^3 \xi} - \frac{Y_1}{16E^2 \xi} \left(\frac{d_\nabla}{m_A} + \frac{d_\Delta}{m_B} \right). \end{aligned} \quad (22)$$

Since the left-hand sides of Eq. (21) are zero, we can solve explicitly for c_1 and c_2 . We find the following solutions, which apply to all orders in velocity:

$$\begin{aligned} c_1(\mathbf{p}^2) &= \frac{d_\perp}{4E^2 \xi} \quad \text{and} \\ c_2(\mathbf{p}^2) &= \frac{1}{128E^2 \xi} \left[\frac{(-1 + \xi + 2E^2 \xi^2 \partial_{p^2}) d_\perp^2}{2E^3 \xi^2} + \frac{d_\nabla}{m_A} + \frac{d_\Delta}{m_B} \right]. \end{aligned} \quad (23)$$

Note that c_2 is simply a rational function of \mathbf{p}^2 since, as discussed above, terms in the integral that have poles in Y_1 that would yield infrared logarithms cancel exactly at the integrand level. In particular, the $\mathcal{O}(Y_1^0)$ term in the difference of numerators is

$$\mathcal{N}^{\text{1loop}} - \mathcal{N}_{\text{EFT}}^{\text{1loop}} = \frac{1}{8E^3 \xi} [d_\square - d_\perp^2] + \mathcal{O}(Y_1), \quad (24)$$

which implies that $d_{\square} = d_{\perp}^2$. Indeed, this relation is obvious from the point of view of unitarity in the full theory, since the coefficient of the scalar box integral is given by the product of tree amplitudes.

Gravity results.—We have computed the classical potential at leading and next-to-leading order, c_1 and c_2 in Eq. (23), expressed in terms of the scalar functions d_{\perp} , d_{∇} , and d_{Δ} that are the natural outputs of a generalized unitarity calculation. For GR, both the full amplitude calculation [24] as well as the unitarity calculation have been completed [4,8,9,11], yielding

$$\begin{aligned} d_{\perp} &= 4[m_A^2 m_B^2 - 2(p_1 \cdot p_2)^2], \\ d_{\nabla, \Delta} &= 12m_{A,B}^2 [m_A^2 m_B^2 - 5(p_1 \cdot p_2)^2], \end{aligned} \quad (25)$$

where $p_1 \cdot p_2 = E_A E_B + \mathbf{p}^2$ and we have kept only classical contributions. Note that these quantities can also be constructed with the aid of color-kinematics duality [25–27], provided one can eliminate unphysical dilaton and axion modes (see also Refs. [28–33] for the related classical double copy). Inserting these into Eq. (23) yields

$$\begin{aligned} c_1(\mathbf{p}^2) &= \frac{1}{E^2 \xi} [m_A^2 m_B^2 - 2(p_1 \cdot p_2)^2], \\ c_2(\mathbf{p}^2) &= \frac{1}{32E^2 \xi} [2E(\xi - 1)c_1^2(\mathbf{p}^2) - 16E(p_1 \cdot p_2)c_1(\mathbf{p}^2) \\ &\quad + 3(m_A + m_B)(m_A^2 m_B^2 - 5(p_1 \cdot p_2)^2)], \end{aligned} \quad (26)$$

which specify the potential at 2PM, i.e., $\mathcal{O}(G^2)$ and to all orders in velocity. In position space, our classical potential in the center-of-mass frame is

$$V(\mathbf{p}, \mathbf{r}) = \frac{Gc_1(\mathbf{p}^2)}{|\mathbf{r}|} + \frac{8G^2 c_2(\mathbf{p}^2)}{r^2} + \dots, \quad (27)$$

where \mathbf{r} is the distance vector between the black holes. It would be interesting to connect our results to those of Damour [7,10], who has used effective one-body methods to compute the 2PM potential from scattering angles (for calculations of the 1PM potential see Refs. [34,35]).

Since these expressions are valid at all orders in velocity they can be compared against state of the art calculations that extend up to 4PN order. A nontrivial complication is that our potential in Eq. (27) differs from those in Refs. [15,16] by a gauge transformation. While this gauge transformation can, in principle, be constructed [36], this requires considerable effort. Here we employ a new approach: we instead compute the on-shell scattering amplitude M_{EFT} for the two potentials under comparison. Since the amplitude encodes all the relevant dynamics and is gauge invariant, they will match provided the potentials are gauge equivalent. Calculating M_{EFT} for our potential and comparing it to M_{EFT} computed from the known

potentials given in Eq. (223) of Ref. [37] and Eq. (8.41) of Ref. [16], we obtain exact agreement including all terms through $\mathcal{O}(G^2)$. We have also checked that in the limit $m_A/m_B \gg 1$ our result agrees with the potential for a test body orbiting a Schwarzschild black hole to $\mathcal{O}(G^2)$ and all orders in velocity [38].

Given that the on-shell amplitude is unique and the classical potential is not, it may seem strange that our construction extracts a unique expression for the latter from the former. However, recall that our starting point in Eq. (4) does not include terms that can be eliminated by equations of motion, which in itself is the choice of gauge [39]. Such terms do not affect on-shell amplitudes, but enter in loops and change the resulting potential coefficients in a way that is a pure gauge transformation.

Conclusions.—We have derived a systematic map between the classical potential for spinless particles and their corresponding on-shell scattering amplitudes. Our main result, summarized in Eq. (23), relates the classical potential coefficients c , describing effects at leading and next-to-leading order in the coupling constant and all orders in velocity to the scalar coefficients d , which are the natural output of a unitarity calculation of the on-shell scattering amplitude in a general QFT. For the special case of GR, we have verified agreement of our results, summarized in Eq. (26) and Eq. (27), to all known formulas in the literature, which extend up to 4PN order.

Our current results may have utility for future work on gravitational waves. Since Eq. (23) includes all orders in velocity, it will serve as a check of higher order PN calculations. Furthermore, it is applicable to any QFT, so it allows for modifications of GR involving new light states or higher dimension operators.

This Letter leaves several promising avenues for future work. Foremost is the extension to next-to-next-to-leading order in the coupling constant, which for gravity is 3PM order. While our methods are by construction scalable to higher orders, new subtleties may emerge, e.g., from the logarithms of momentum in the 3PM potential. Moreover, we will eventually encounter infrared divergences due to overlap from radiation modes at 4PN [40] as well as ultraviolet divergences which formally enter at 3PN and affect physical observables at 5PN when finite size effects become relevant. Application of our methods to particles with spin and for gravitational wave emission should also be interesting.

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