BORG–MARCHENKO-TYPE UNIQUENESS RESULTS FOR CMV OPERATORS

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Dedicated with great pleasure to Pavel Exner on the occasion of his 60th birthday

Abstract. We prove local and global versions of Borg–Marchenko-type uniqueness theorems for half-lattice and full-lattice CMV operators (CMV for Cantero, Moral, and Velázquez [15]). While our half-lattice results are formulated in terms of Weyl–Titchmarsh functions, our full-lattice results involve the diagonal and main off-diagonal Green's functions.

1. Introduction

To set the stage, we briefly review the history of Borg–Marchenko-type uniqueness theorems. Apparently, it all started in connection with Schrödinger operators on half-lines, and so we turn to that case first.

Let $H_j = -\frac{d^2}{dx^2} + V_j$, $V_j \in L^1([0,R];dx)$ for all $R > 0$, V_j real-valued, $j = 1,2$, be two self-adjoint operators in $L^2([0,\infty);dx)$ which, just for simplicity, have a Dirichlet boundary condition at $x = 0$ (and possibly a self-adjoint boundary condition at infinity). Let $m_j(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, be the Weyl-Titchmarsh m-functions associated with H_j, $j = 1,2$. Then the celebrated Borg–Marchenko uniqueness theorem, in this particular context, reads as follows:

Theorem 1.1. *Suppose*

$$m_1(z) = m_2(z), \quad z \in \mathbb{C}\backslash\mathbb{R}, \quad \textit{then } V_1(x) = V_2(x) \ \textit{ for a.e. } x \in [0,\infty). \tag{1.1}$$

This result was published by Marchenko [47] in 1950. Marchenko's extensive treatise on spectral theory of one-dimensional Schrödinger operators [48], repeating the proof of his uniqueness theorem, then appeared in 1952, which also marked the appearance of Borg's proof of the uniqueness theorem [11] (apparently, based on his lecture at the 11th Scandinavian Congress of Mathematicians held at Trondheim, Norway in 1949).

As pointed out by Levitan [43] in the Notes to Chapter 2, Borg and Marchenko were actually preceded by Tikhonov [68] in 1949, who proved a special case of Theorem 1.1 in connection with the string equation (and hence under certain additional hypotheses on V_j). Since Weyl–Titchmarsh functions $m(z)$ are uniquely related to the spectral measure $d\rho$ of a self-adjoint (Dirichlet) Schrödinger operator $H = -\frac{d^2}{dx^2} + V$ in $L^2([0,\infty))$ by the standard Herglotz representation

$$m(z) = \mathrm{Re}(m(i)) + \int_{\mathbb{R}} d\rho(\lambda)[(\lambda - z)^{-1} - \lambda(1+\lambda^2)^{-1}], \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{1.2}$$

Theorem 1.2 is equivalent to the following statement: Denote by $d\rho_j$ the spectral measures of H_j, $j = 1,2$. Then

$$d\rho_1 = d\rho_2 \ \text{ implies } V_1 = V_2 \text{ a.e. on } [0,\infty). \tag{1.3}$$

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In fact, Marchenko's proof takes the spectral measures $d\rho_j$ as the point of departure while Borg focuses on the Weyl–Titchmarsh functions m_j.

We emphasize at this point that Borg and Marchenko also treat the general case of non-Dirichlet boundary conditions at $x = 0$ (in which equality of the two m-functions also identifies the two boundary conditions); moreover, Marchenko also simultaneously discussed the half-line and the finite interval case. For brevity we chose to illustrate the simplest possible case only.

To the best of our knowledge, the only alternative approaches to Theorem 1.1 are based on the Gelfand–Levitan solution [20] of the inverse spectral problem published in 1951 (see also Levitan and Gasymov [44]) and alternative variants due to M. Krein [41], [42]. For over 45 years, Theorem 1.1 stood the test of time and resisted any improvements. Finally, in 1998, Simon [59] proved the following spectacular result, a local Borg–Marchenko theorem (see part (i) below) and a significant improvement of the original Borg–Marchenko theorem (see part (ii) below):

Theorem 1.2.
(i) Let $a > 0$, $0 < \varepsilon < \pi/2$ and suppose that

$$|m_1(z) - m_2(z)| \underset{|z|\to\infty}{=} O(e^{-2\mathrm{Im}(z^{1/2})a}) \tag{1.4}$$

along the ray $\arg(z) = \pi - \varepsilon$. *Then*

$$V_1(x) = V_2(x) \ \text{ for a.e. } x \in [0, a]. \tag{1.5}$$

(ii) Let $0 < \varepsilon < \pi/2$ and suppose that for all $a > 0$,

$$|m_1(z) - m_2(z)| \underset{|z|\to\infty}{=} O(e^{-2\mathrm{Im}(z^{1/2})a}) \tag{1.6}$$

along the ray $\arg(z) = \pi - \varepsilon$. *Then*

$$V_1(x) = V_2(x) \ \text{ for a.e. } x \in [0, \infty). \tag{1.7}$$

The ray $\arg(z) = \pi - \varepsilon$, $0 < \varepsilon < \pi/2$ chosen in Theorem 1.2 is of no particular importance. A limit taken along any non-self-intersecting curve $\mathcal{C}$ going to infinity in the sector $\arg(z) \in ((\pi/2)+\varepsilon, \pi-\varepsilon)$ is permissible. For simplicity we only discussed the Dirichlet boundary condition $u(0) = 0$ thus far. However, everything extends to the case of general boundary condition $u'(0) + hu(0) = 0$, $h \in \mathbb{R}$. Moreover, the case of a finite interval problem on $[0, b]$, $b \in (0, \infty)$, instead of the half-line $[0, \infty)$ in Theorem 1.2 (i), with $0 < a < b$, and a self-adjoint boundary condition at $x = b$ of the type $u'(b) + h_b u(b) = 0$, $h_b \in \mathbb{R}$, can be handled as well. All of this is treated in detail in [34].

Remarkably enough, the local Borg–Marchenko theorem proven by Simon [59] was just a by-product of his new approach to inverse spectral theory for half-line Schrödinger operators. Actually, Simon's original result in [59] was obtained under a bit weaker conditions on V; the result as stated in Theorem 1.2 is taken from [34] (see also [33]). While the original proof of the local Borg–Marchenko theorem in [59] relied on the full power of a new formalism in inverse spectral theory, a short and fairly elementary proof of Theorem 1.2 was presented in [34]. Without going into further details at this point, we also mention that [34] contains the analog of the local Borg–Marchenko uniqueness result, Theorem 1.2 for Schrödinger operators on the real line. In addition, the case of half-line Jacobi operators and half-line matrix-valued Schrödinger operators was dealt with in [34].

We should also mention some work of Ramm [54], [55], who provided a proof of Theorem 1.2 (i) under the additional assumption that V_j are short-range potentials satisfying $V_j \in L^1([0, \infty); (1 + |x|)dx)$, $j = 1, 2$. A very short proof of Theorem 1.2, close in spirit to Borg's original paper [11], was subsequently found by Bennewitz [7]. Still other proofs were presented in [38] and [39]. Various local and global uniqueness results for matrix-valued Schrödinger, Dirac-type, and Jacobi operators were considered in [16], [19], [32], [56], [57]. A local Borg–Marchenko theorem for complex-valued potentials has been proved in [13]; the case of semi-infinite Jacobi operators with complex-valued coefficients was studied in [73]. This circle of ideas has been reviewed in [28].

After this review of Borg–Marchenko-type uniqueness results for Schrödinger operators, we now turn to the principal object of our interest in this paper, the so-called CMV operators. CMV operators are a special class of unitary semi-infinite five-diagonal matrices. But for simplicity, we confine ourselves in this introduction to a discussion of CMV operators on $\mathbb{Z}$, that is, doubly infinite CMV operators. Denoting by $\mathbb{D}$ the open unit disk in $\mathbb{C}$, let α be a sequence of complex numbers in $\mathbb{D}$, $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}} \subset \mathbb{D}$. The unitary CMV operator U on $\ell^2(\mathbb{Z})$ then can be written as a special five-diagonal doubly infinite matrix in the standard basis of $\ell^2(\mathbb{Z})$ according to [62, Sects. 4.5, 10.5]) as in (2.8). For the corresponding half-lattice CMV operators U_{+,k_0} in $\ell^2([k_0,\infty)\cap\mathbb{Z})$ we refer to (2.15)–(2.17).

The actual history of CMV operators is more involved: The corresponding unitary semi-infinite five-diagonal matrices were first introduced in 1991 by Bunse–Gerstner and Elsner [14], and subsequently treated in detail by Watkins [72] in 1993 (cf. the recent discussion in Simon [64]). They were subsequently rediscovered by Cantero, Moral, and Velázquez (CMV) in [15]. In [62, Sects. 4.5, 10.5], Simon introduced the corresponding notion of unitary doubly infinite five-diagonal matrices and coined the term "extended" CMV matrices. For simplicity, we will just speak of CMV operators whether or not they are half-lattice or full-lattice operators. We also note that in a context different from orthogonal polynomials on the unit circle, Bourget, Howland, and Joye [12] introduced a family of doubly infinite matrices with three sets of parameters which, for special choices of the parameters, reduces to two-sided CMV matrices on $\mathbb{Z}$. Moreover, it is possible to connect unitary block Jacobi matrices to the trigonometric moment problem (and hence to CMV matrices) as discussed by Berezansky and Dudkin [9], [10].

The relevance of this unitary operator U on $\ell^2(\mathbb{Z})$, more precisely, the relevance of the corresponding half-lattice CMV operator $U_{+,0}$ in $\ell^2(\mathbb{N}_0)$ is derived from its intimate relationship with the trigonometric moment problem and hence with finite measures on the unit circle $\partial\mathbb{D}$. (Here $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.) This will be reviewed in some detail in Section 2 but we also refer to the monumental two-volume treatise by Simon [62] (see also [61] and [63]) and the exhaustive bibliography therein. For classical results on orthogonal polynomials on the unit circle we refer, for instance, to [6], [25]–[27], [40], [65]–[67], [70], [71]. More recent references relevant to the spectral theoretic content of this paper are [18], [22]–[24], [35]–[37], [46], [53], and [60]. The full-lattice CMV operator U on $\mathbb{Z}$ is closely related to an important, and only recently intensively studied, completely integrable version of the defocusing nonlinear Schrödinger equation (continuous in time but discrete in space), a special case of the Ablowitz–Ladik system. Relevant references in this context are, for instance, [1]–[5], [21], [29]–[31], [45], [49]–[52], [58], [69], and the literature cited therein.

Next, we briefly summarize some of the principal results proven in this paper. For brevity we just focus on CMV operators on $\mathbb{Z}$. We use the following notation for the diagonal and for the neighboring off-diagonal entries of the Green's function of U (i.e., the discrete integral kernel of $(U-zI)^{-1}$),

$$g(z,k) = (U-Iz)^{-1}(k,k), \quad h(z,k) = \begin{cases} (U-Iz)^{-1}(k-1,k), & k \text{ odd},\\ (U-Iz)^{-1}(k,k-1), & k \text{ even}, \end{cases} \quad k\in\mathbb{Z},\ z\in\mathbb{D}. \tag{1.8}$$

Then the following uniqueness results for CMV operators U on $\mathbb{Z}$ will be proven in Section 3:

Theorem 1.3. *Assume $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}} \subset \mathbb{D}$ and let $k_0\in\mathbb{Z}$. Then any of the following two sets of data*

(i) $g(z,k_0)$ and $h(z,k_0)$ for all z in a sufficiently small neighborhood of the origin under the assumption $h(0,k_0)\neq 0$;

(*ii*) *$g(z,k_0-1)$ and $g(z,k_0)$ for all z in a sufficiently small neighborhood of the origin and α_{k_0} under the assumption $\alpha_{k_0}\neq 0$;*

uniquely determine the Verblunsky coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$, and hence the full-lattice CMV operator U.

In the following local uniqueness result, $g^{(j)}$ and $h^{(j)}$ denote the corresponding quantities (1.8) and (1.9) associated with the Verblunsky coefficients $\alpha^{(j)}$, $j=1,2$.

Theorem 1.4. *Assume $\alpha^{(\ell)}=\{\alpha_k^{(\ell)}\}_{k\in\mathbb{Z}}\subset\mathbb{D}$, $\ell=1,2$, and let $k_0\in\mathbb{Z}$, $N\in\mathbb{N}$. Then for the full-lattice problems associated with $\alpha^{(1)}$ and $\alpha^{(2)}$, the following local uniqueness results hold:*

(*i*) *If either $h^{(1)}(0,k_0)$ or $h^{(2)}(0,k_0)$ is nonzero and*

$$\begin{aligned} &|g^{(1)}(z,k_0)-g^{(2)}(z,k_0)|+|h^{(1)}(z,k_0)-h^{(2)}(z,k_0)| \underset{z\to 0}{=} o(z^N), \\ &\text{then } \alpha_k^{(1)}=\alpha_k^{(2)} \text{ for } k_0-N\leq k\leq k_0+N+1. \end{aligned} \tag{1.9}$$

(*ii*) *If $\alpha_{k_0}^{(1)}=\alpha_{k_0}^{(2)}\neq 0$ and*

$$\begin{aligned} &|g^{(1)}(z,k_0-1)-g^{(2)}(z,k_0-1)|+|g^{(1)}(z,k_0)-g^{(2)}(z,k_0)| \underset{z\to 0}{=} o(z^N), \\ &\text{then } \alpha_k^{(1)}=\alpha_k^{(2)} \text{ for } k_0-N-1\leq k\leq k_0+N+1. \end{aligned} \tag{1.10}$$

Finally, a brief description of the content of each section in this paper: In Section 2 we review the basic Weyl–Titchmarsh theory for CMV operators, discussed in great detail in [35], as this plays a fundamental role in our principal Section 3. In Section 3 we first provide an alternative proof of the known Borg–Marchenko-type uniqueness results for half-lattice CMV operators recorded, for instance, in Simon in [62, Thm. 1.5.5] (cf. our Theorems 3.1 and 3.2). Then we turn to the case of full-lattice CMV operators and prove our principal new results in Theorems 3.3 and 3.4 (summarized as Theorems 1.3 and 1.4 above). In particular, we note that our discussion of CMV operators on the full-lattice will be undertaken in the spirit of [32], where (local and global) uniqueness theorems for full-line (resp., full-lattice) problems are formulated in terms of diagonal Green's functions $g(z,x_0)$ and their x-derivatives $g'(z,x_0)$ at some fixed $x_0\in\mathbb{R}$, for Schrödinger and Dirac-type operators on $\mathbb{R}$ and similarly for Jacobi operators on $\mathbb{Z}$.

An extension of the results of this paper to matrix-valued Verblunsky coefficients appeared in [17].

2. A Summary of Weyl–Titchmarsh Theory for CMV Operators on Half-Lattices and on $\mathbb{Z}$

We start by introducing some of the basic notations used throughout this paper. Detailed proofs of all facts in this preparatory section (and a lot of additional material) can be found in [35].

In the following, let $\ell^2(\mathbb{Z})$ be the usual Hilbert space of all square summable complex-valued sequences with scalar product $(\cdot,\cdot)_{\ell^2(\mathbb{Z})}$ linear in the second argument. The *standard basis* in $\ell^2(\mathbb{Z})$ is denoted by

$$\{\delta_k\}_{k\in\mathbb{Z}}, \quad \delta_k=(\dots,0,\dots,0,\underbrace{1}_{k},0,\dots,0,\dots)^\top,\ k\in\mathbb{Z}. \tag{2.1}$$

For $m\in\mathbb{N}$ and $J\subseteq\mathbb{R}$ an interval, we will identify $\oplus_{j=1}^m\ell^2(J\cap\mathbb{Z})$ and $\ell^2(J\cap\mathbb{Z})\otimes\mathbb{C}^m$ and then use the simplified notation $\ell^2(J\cap\mathbb{Z})^m$. For simplicity, the identity operators on $\ell^2(J\cap\mathbb{Z})$ and $\ell^2(J\cap\mathbb{Z})^m$ are abbreviated by I and I_m, respectively, without separately indicating its dependence on J.

By a *Laurent polynomial* we denote a finite linear combination of terms z^k, $k\in\mathbb{Z}$, with complex-valued coefficients.

Throughout this section we make the following basic assumption:

Hypothesis 2.1. *Let α be a sequence of complex numbers such that*

$$\alpha = \{\alpha_k\}_{k\in\mathbb{Z}} \subset \mathbb{D}. \tag{2.2}$$

Given a sequence α satisfying (2.2), we define the sequence of positive real numbers $\{\rho_k\}_{k\in\mathbb{Z}}$ and two sequences of complex numbers with positive real parts $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ by

$$\rho_k = \sqrt{1-|\alpha_k|^2}, \quad a_k = 1+\alpha_k, \quad b_k = 1-\alpha_k, \quad k\in\mathbb{Z}. \tag{2.3}$$

Following Simon [62], we call α_k the Verblunsky coefficients in honor of Verblunsky's pioneering work in the theory of orthogonal polynomials on the unit circle [70], [71].

Next, we also introduce a sequence of 2×2 unitary matrices Θ_k by

$$\Theta_k = \begin{pmatrix} -\alpha_k & \rho_k \\ \rho_k & \overline{\alpha_k} \end{pmatrix}, \quad k\in\mathbb{Z}, \tag{2.4}$$

and two unitary operators V and W on $\ell^2(\mathbb{Z})$ by their matrix representations in the standard basis of $\ell^2(\mathbb{Z})$ as follows,

$$V = \begin{pmatrix} \ddots & & & \\ & \Theta_{2k-2} & & 0 \\ & & \Theta_{2k} & \\ 0 & & & \ddots \end{pmatrix}, \quad W = \begin{pmatrix} \ddots & & & \\ & \Theta_{2k-1} & & 0 \\ & & \Theta_{2k+1} & \\ 0 & & & \ddots \end{pmatrix}, \tag{2.5}$$

where

$$\begin{pmatrix} V_{2k-1,2k-1} & V_{2k-1,2k} \\ V_{2k,2k-1} & V_{2k,2k} \end{pmatrix} = \Theta_{2k}, \quad \begin{pmatrix} W_{2k,2k} & W_{2k,2k+1} \\ W_{2k+1,2k} & W_{2k+1,2k+1} \end{pmatrix} = \Theta_{2k+1}, \quad k\in\mathbb{Z}. \tag{2.6}$$

Moreover, we introduce the unitary operator U on $\ell^2(\mathbb{Z})$ by

$$U = VW, \tag{2.7}$$

or in matrix form, in the standard basis of $\ell^2(\mathbb{Z})$, by

$$U = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & 0 & -\alpha_0\rho_{-1} & -\overline{\alpha_{-1}}\alpha_0 & -\alpha_1\rho_0 & \rho_0\rho_1 & & & 0 \\ & & \rho_{-1}\rho_0 & \overline{\alpha_{-1}}\rho_0 & -\overline{\alpha_0}\alpha_1 & \overline{\alpha_0}\rho_1 & 0 & & \\ & & & 0 & -\alpha_2\rho_1 & -\overline{\alpha_1}\alpha_2 & -\alpha_3\rho_2 & \rho_2\rho_3 & \\ & 0 & & & \rho_1\rho_2 & \overline{\alpha_1}\rho_2 & -\overline{\alpha_2}\alpha_3 & \overline{\alpha_2}\rho_3 & 0 \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.8}$$

$$\begin{aligned} &= \rho^-\rho\,\delta_{\text{even}}\,S^{--} + (\overline{\alpha}^-\rho\,\delta_{\text{even}} - \alpha^+\rho\,\delta_{\text{odd}})S^- - \overline{\alpha}\alpha^+ + (\overline{\alpha}\rho^+\,\delta_{\text{even}} - \alpha^{++}\rho^+\,\delta_{\text{odd}})S^+ \\ &\quad + \rho^+\rho^{++}\,\delta_{\text{odd}}\,S^{++}, \end{aligned} \tag{2.9}$$

where δ_{even} and δ_{odd} denote the characteristic functions of the even and odd integers,

$$\delta_{\text{even}} = \chi_{2\mathbb{Z}}, \quad \delta_{\text{odd}} = 1-\delta_{\text{even}} = \chi_{2\mathbb{Z}+1}. \tag{2.10}$$

Here the diagonal entries in the infinite matrix (2.8) are given by $U_{k,k} = -\overline{\alpha_k}\alpha_{k+1}$, $k\in\mathbb{Z}$.

As explained in the introduction, in the recent literature on orthogonal polynomials on the unit circle, such operators U are frequently called CMV operators.

Next we recall some of the principal results of [35] needed in this paper.

Lemma 2.2. *Let $z \in \mathbb{C}\backslash\{0\}$ and $\{u(z,k)\}_{k\in\mathbb{Z}}, \{v(z,k)\}_{k\in\mathbb{Z}}$ be sequences of complex functions. Then the following items (i)–(iii) are equivalent:*

$$(i) \quad (Uu(z,\cdot))(k) = zu(z,k), \quad (Wu(z,\cdot)) = zv(z,k), \quad k \in \mathbb{Z}. \tag{2.11}$$

$$(ii) \quad (Wu(z,\cdot))(k) = zv(z,k), \quad (Vv(z,\cdot))(k) = u(z,k), \quad k \in \mathbb{Z}. \tag{2.12}$$

$$(iii) \quad \begin{pmatrix} u(z,k) \\ v(z,k) \end{pmatrix} = T(z,k) \begin{pmatrix} u(z,k-1) \\ v(z,k-1) \end{pmatrix}, \quad k \in \mathbb{Z}, \tag{2.13}$$

where the transfer matrices $T(z,k)$, $z \in \mathbb{C}\backslash\{0\}$, $k \in \mathbb{Z}$, are given by

$$T(z,k) = \begin{cases} \frac{1}{\rho_k} \begin{pmatrix} \alpha_k & z \\ 1/z & \overline{\alpha_k} \end{pmatrix}, & k \text{ odd}, \\ \frac{1}{\rho_k} \begin{pmatrix} \overline{\alpha_k} & 1 \\ 1 & \alpha_k \end{pmatrix}, & k \text{ even}. \end{cases} \tag{2.14}$$

If one sets $\alpha_{k_0} = e^{is}$, $s \in [0, 2\pi)$, for some reference point $k_0 \in \mathbb{Z}$, then the operator U splits into a direct sum of two half-lattice operators $U^{(s)}_{-,k_0-1}$ and $U^{(s)}_{+,k_0}$ acting on $\ell^2((-\infty, k_0-1] \cap \mathbb{Z})$ and on $\ell^2([k_0, \infty) \cap \mathbb{Z})$, respectively. Explicitly, one obtains

$$\begin{aligned} &U = U^{(s)}_{-,k_0-1} \oplus U^{(s)}_{+,k_0} \text{ in } \ell^2((-\infty, k_0-1] \cap \mathbb{Z}) \oplus \ell^2([k_0,\infty) \cap \mathbb{Z}) \\ &\text{if } \alpha_{k_0} = e^{is}, \ s \in [0, 2\pi). \end{aligned} \tag{2.15}$$

(Strictly, speaking, setting $\alpha_{k_0} = e^{is}$, $s \in [0, 2\pi)$, for some reference point $k_0 \in \mathbb{Z}$ contradicts our basic Hypothesis 2.1. However, as long as the exception to Hypothesis 2.1 refers to only one site, we will safely ignore this inconsistency in favor of the notational simplicity it provides by avoiding the introduction of a properly modified hypothesis on $\{\alpha_k\}_{k\in\mathbb{Z}}$.) Similarly, one obtains $W^{(s)}_{-,k_0-1}$, $V^{(s)}_{-,k_0-1}$ and $W^{(s)}_{+,k_0}$, $V^{(s)}_{+,k_0}$ such that

$$U^{(s)}_{\pm,k_0} = V^{(s)}_{\pm,k_0} W^{(s)}_{\pm,k_0}. \tag{2.16}$$

For simplicity we will abbreviate

$$U_{\pm,k_0} = U^{(s=0)}_{\pm,k_0} = V^{(s=0)}_{\pm,k_0} W^{(s=0)}_{\pm,k_0} = V_{\pm,k_0} W_{\pm,k_0}. \tag{2.17}$$

Lemma 2.3. *Let $k_0 \in \mathbb{Z}$, $z \in \mathbb{C}\backslash\{0\}$, and $\{\hat p_+(z,k,k_0)\}_{k\geq k_0}$, $\{\hat r_+(z,k,k_0)\}_{k\geq k_0}$ be two sequences of complex functions. Then the following items (i)–(iii) are equivalent:*

$$(i) \quad (U_{+,k_0}\hat p_+(z,\cdot,k_0))(k) = z\hat p_+(z,k,k_0), \quad (W_{+,k_0}\hat p_+(z,\cdot,k_0))(k) = z\hat r_+(z,k,k_0), \quad k \geq k_0. \tag{2.18}$$

$$(ii) \quad (W_{+,k_0}\hat p_+(z,\cdot,k_0))(k) = z\hat r_+(z,k,k_0), \quad (V_{+,k_0}\hat r_+(z,\cdot,k_0))(k) = \hat p_+(z,k,k_0), \quad k \geq k_0. \tag{2.19}$$

$$(iii) \quad \begin{pmatrix} \hat p_+(z,k,k_0) \\ \hat r_+(z,k,k_0) \end{pmatrix} = T(z,k) \begin{pmatrix} \hat p_+(z,k-1,k_0) \\ \hat r_+(z,k-1,k_0) \end{pmatrix}, \quad k > k_0, \quad \textit{with initial condition}$$

$$\hat p_+(z,k_0,k_0) = \begin{cases} z\hat r_+(z,k_0,k_0), & k_0 \text{ odd}, \\ \hat r_+(z,k_0,k_0), & k_0 \text{ even}. \end{cases} \tag{2.20}$$

Next, consider sequences $\{\hat p_-(z,k,k_0)\}_{k\leq k_0}$, $\{\hat r_-(z,k,k_0)\}_{k\leq k_0}$. Then the following items (iv)–(vi) are equivalent:

$$(iv) \quad (U_{-,k_0}\hat p_-(z,\cdot,k_0))(k) = z\hat p_-(z,k,k_0), \quad (W_{-,k_0}\hat p_-(z,\cdot,k_0))(k) = z\hat r_-(z,k,k_0), \quad k \leq k_0. \tag{2.21}$$

(v) $\quad (W_{-,k_0}\hat p_-(z,\cdot,k_0))(k) = z\hat r_-(z,k,k_0), \quad (V_{-,k_0}\hat r_-(z,\cdot,k_0))(k) = \hat p_-(z,k,k_0), \quad k \le k_0.$ (2.22)

(vi) $\quad \begin{pmatrix}\hat p_-(z,k-1),k_0\\ \hat r_-(z,k-1,k_0)\end{pmatrix} = T(z,k)^{-1}\begin{pmatrix}\hat p_-(z,k,k_0)\\ \hat r_-(z,k,k_0)\end{pmatrix}, \quad k \le k_0,$ *with initial condition*

$$\hat p_-(z,k_0,k_0) = \begin{cases}\hat r_-(z,k_0,k_0), & k_0 \text{ odd},\\ -z\hat r_-(z,k_0,k_0), & k_0 \text{ even}.\end{cases} \tag{2.23}$$

In the following, we denote by $\left(\begin{smallmatrix}p_\pm(z,k,k_0)\\ r_\pm(z,k,k_0)\end{smallmatrix}\right)_{k\in\mathbb{Z}}$ and $\left(\begin{smallmatrix}q_\pm(z,k,k_0)\\ s_\pm(z,k,k_0)\end{smallmatrix}\right)_{k\in\mathbb{Z}}$, $z \in \mathbb{C}\backslash\{0\}$, four linearly independent solutions of (2.13) with the following initial conditions:

$$\begin{pmatrix}p_+(z,k_0,k_0)\\ r_+(z,k_0,k_0)\end{pmatrix} = \begin{cases}\binom{z}{1}, & k_0 \text{ odd},\\ \binom{1}{1}, & k_0 \text{ even},\end{cases} \quad \begin{pmatrix}q_+(z,k_0,k_0)\\ s_+(z,k_0,k_0)\end{pmatrix} = \begin{cases}\binom{z}{-1}, & k_0 \text{ odd},\\ \binom{-1}{1}, & k_0 \text{ even}.\end{cases} \tag{2.24}$$

$$\begin{pmatrix}p_-(z,k_0,k_0)\\ r_-(z,k_0,k_0)\end{pmatrix} = \begin{cases}\binom{1}{-1}, & k_0 \text{ odd},\\ \binom{-z}{1}, & k_0 \text{ even},\end{cases} \quad \begin{pmatrix}q_-(z,k_0,k_0)\\ s_-(z,k_0,k_0)\end{pmatrix} = \begin{cases}\binom{1}{1}, & k_0 \text{ odd},\\ \binom{z}{1}, & k_0 \text{ even}.\end{cases} \tag{2.25}$$

Then it follows that $p_\pm(z,k,k_0)$, $q_\pm(z,k,k_0)$, $r_\pm(z,k,k_0)$, and $s_\pm(z,k,k_0)$, $k,k_0 \in \mathbb{Z}$, are Laurent polynomials in z.

Lemma 2.4. *Let $k_0 \in \mathbb{Z}$. Then the sets $\{p_\pm(\cdot,k,k_0)\}_{k\gtreqless k_0}$ and $\{r_\pm(\cdot,k,k_0)\}_{k\gtreqless k_0}$ form complete orthonormal systems of Laurent polynomials in $L^2(\partial\mathbb{D};d\mu_\pm(\cdot,k_0))$, where*

$$d\mu_\pm(\zeta,k_0) = d(\delta_{k_0},E_{U_{\pm,k_0}}(\zeta)\delta_{k_0})_{\ell^2(\mathbb{Z}\cap[k_0,\pm\infty))}, \quad \zeta \in \partial\mathbb{D}, \tag{2.26}$$

and $dE_{U_{\pm,k_0}}(\cdot)$ denote the operator-valued spectral measures of the operators $U_{\pm,k_0}$,

$$U_{\pm,k_0} = \oint_{\partial\mathbb{D}} dE_{U_{\pm,k_0}}(\zeta)\,\zeta. \tag{2.27}$$

Moreover, the half-lattice CMV operators $U_{\pm,k_0}$ are unitarily equivalent to the operators of multiplication by the function id (where $id(\zeta)=\zeta$, $\zeta\in\partial\mathbb{D}$) on $L^2(\partial\mathbb{D};d\mu_\pm(\cdot,k_0))$. In particular,

$$\sigma(U_{\pm,k_0}) = \operatorname{supp}(d\mu_\pm(\cdot,k_0)) \tag{2.28}$$

and the spectrum of $U_{\pm,k_0}$ is simple.

We note that the measures $d\mu_\pm(\cdot,k_0)$, $k_0 \in \mathbb{Z}$, are nonnegative and supported on infinite sets.

Corollary 2.5. *Let $k_0 \in \mathbb{Z}$.*
The Laurent polynomials $\{p_+(\cdot,k,k_0)\}_{k\ge k_0}$ can be constructed by Gram–Schmidt orthogonalizing

$$\begin{cases}\zeta,\ 1,\ \zeta^2,\ \zeta^{-1},\ \zeta^3,\ \zeta^{-2},\dots, & k_0 \text{ odd},\\ 1,\ \zeta,\ \zeta^{-1},\ \zeta^2,\ \zeta^{-2},\zeta^3,\ \dots, & k_0 \text{ even}\end{cases} \tag{2.29}$$

in $L^2(\partial\mathbb{D};d\mu_+(\cdot,k_0))$.
The Laurent polynomials $\{r_+(\cdot,k,k_0)\}_{k\ge k_0}$ can be constructed by Gram–Schmidt orthogonalizing

$$\begin{cases}1,\ \zeta,\ \zeta^{-1},\ \zeta^2,\ \zeta^{-2},\zeta^3,\ \dots, & k_0 \text{ odd},\\ 1,\ \zeta^{-1},\ \zeta,\ \zeta^{-2},\ \zeta^2,\zeta^{-3},\ \dots, & k_0 \text{ even}\end{cases} \tag{2.30}$$

in $L^2(\partial\mathbb{D};d\mu_+(\cdot,k_0))$.
The Laurent polynomials $\{p_-(\cdot,k,k_0)\}_{k\le k_0}$ can be constructed by Gram–Schmidt orthogonalizing

$$\begin{cases}1,\ -\zeta,\ \zeta^{-1},\ -\zeta^2,\ \zeta^{-2},-\zeta^3,\ \dots, & k_0 \text{ odd},\\ -\zeta,\ 1,\ -\zeta^2,\ \zeta^{-1},\ -\zeta^3,\ \zeta^{-2},\dots, & k_0 \text{ even}\end{cases} \tag{2.31}$$

in $L^2(\partial\mathbb{D}; d\mu_-(\cdot,k_0))$.
The Laurent polynomials $\{r_-(\cdot,k,k_0)\}_{k\le k_0}$ *can be constructed by Gram–Schmidt orthogonalizing*

$$\begin{cases} -1,\,\zeta^{-1},\,-\zeta,\,\zeta^{-2},\,-\zeta^2,\zeta^{-3},\,\dots, & k_0 \text{ odd,}\\ 1,\,-\zeta,\,\zeta^{-1},\,-\zeta^2,\,\zeta^{-2},-\zeta^3,\,\dots, & k_0 \text{ even}\end{cases} \tag{2.32}$$

in $L^2(\partial\mathbb{D}; d\mu_-(\cdot,k_0))$.

Theorem 2.6. *Let* $k_0\in\mathbb{Z}$ *and* $d\mu_\pm(\cdot,k_0)$ *be nonnegative finite measures on* $\partial\mathbb{D}$ *which are supported on infinite sets and normalized by*

$$\oint_{\partial\mathbb{D}} d\mu_\pm(\zeta,k_0)=1. \tag{2.33}$$

Then $d\mu_\pm(\cdot,k_0)$ *are necessarily the spectral measures for some half-lattice CMV operators* $U_{\pm,k_0}$ *with coefficients* $\{\alpha_k\}_{k\ge k_0+1}$*, respectively* $\{\alpha_k\}_{k\le k_0}$*, defined as follows,*

$$\alpha_k=-\begin{cases}\big(p_+(\cdot,k-1,k_0),M_{\pm,k_0}(id)r_+(\cdot,k-1,k_0)\big)_{L^2(\partial\mathbb{D};d\mu_+(\cdot,k_0))}, & k \text{ odd,}\\ \big(r_+(\cdot,k-1,k_0),p_+(\cdot,k-1,k_0)\big)_{L^2(\partial\mathbb{D};d\mu_+(\cdot,k_0))}, & k \text{ even}\end{cases} \tag{2.34}$$

for all $k\ge k_0+1$ *and*

$$\alpha_k=-\begin{cases}\big(p_-(\cdot,k-1,k_0),M_{\pm,k_0}(id)r_-(\cdot,k-1,k_0)\big)_{L^2(\partial\mathbb{D};d\mu_-(\cdot,k_0))}, & k \text{ odd,}\\ \big(r_-(\cdot,k-1,k_0),p_-(\cdot,k-1,k_0)\big)_{L^2(\partial\mathbb{D};d\mu_-(\cdot,k_0))}, & k \text{ even}\end{cases} \tag{2.35}$$

for all $k\le k_0$*. Here* $\{p_+(\cdot,k,k_0),r_+(\cdot,k,k_0)\}_{k\ge k_0}$ *and* $\{p_-(\cdot,k,k_0),r_-(\cdot,k,k_0)\}_{k\le k_0}$ *denote the Laurent orthonormal polynomials constructed in Corollary* 2.5*.*

Next, we introduce the functions $m_\pm(z,k_0)$ by

$$\begin{aligned} m_\pm(z,k_0)&=\pm(\delta_{k_0},(U_{\pm,k_0}+zI)(U_{\pm,k_0}-zI)^{-1}\delta_{k_0})_{\ell^2(\mathbb{Z}\cap[k_0,\pm\infty))}\\ &=\pm\oint_{\partial\mathbb{D}} d\mu_\pm(\zeta,k_0)\,\frac{\zeta+z}{\zeta-z},\quad z\in\mathbb{C}\backslash\partial\mathbb{D},\end{aligned} \tag{2.36}$$

with

$$m_\pm(0,k_0)=\pm\oint_{\partial\mathbb{D}} d\mu_\pm(\zeta,k_0)=\pm1. \tag{2.37}$$

Theorem 2.7. *Let* $k_0\in\mathbb{Z}$*. Then there exist unique functions* $M_\pm(\cdot,k_0)$ *such that*

$$\begin{pmatrix}u_\pm(z,\cdot,k_0)\\ v_\pm(z,\cdot,k_0)\end{pmatrix}=\begin{pmatrix}q_+(z,\cdot,k_0)\\ s_+(z,\cdot,k_0)\end{pmatrix}+M_\pm(z,k_0)\begin{pmatrix}p_+(z,\cdot,k_0)\\ r_+(z,\cdot,k_0)\end{pmatrix}\in\ell^2([k_0,\pm\infty)\cap\mathbb{Z})^2,\quad z\in\mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\}). \tag{2.38}$$

We will call $u_\pm(z,\cdot,k_0)$ (resp., $v_\pm(z,\cdot,k_0)$) *Weyl–Titchmarsh solutions* of U (resp., $U^\top$). Similarly, we will call $m_\pm(z,k_0)$ as well as $M_\pm(z,k_0)$ the *half-lattice Weyl–Titchmarsh m-functions* associated with $U_{\pm,k_0}$. (See also [60] for a comparison of various alternative notions of Weyl–Titchmarsh m-functions for U_{+,k_0}.)

One verifies that

$$M_+(z,k_0)=m_+(z,k_0),\quad z\in\mathbb{C}\backslash\partial\mathbb{D}, \tag{2.39}$$

$$M_+(0,k_0)=1, \tag{2.40}$$

$$M_-(z,k_0)=\frac{\mathrm{Re}(a_{k_0})+i\mathrm{Im}(b_{k_0})m_-(z,k_0-1)}{i\mathrm{Im}(a_{k_0})+\mathrm{Re}(b_{k_0})m_-(z,k_0-1)}=\frac{(1-z)m_-(z,k_0)+(1+z)}{(1+z)m_-(z,k_0)+(1-z)}$$

$$= \frac{(m_-(z,k_0)+1) - z(m_-(z,k_0)-1)}{(m_-(z,k_0)+1) + z(m_-(z,k_0)-1)}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}, \tag{2.41}$$

$$M_-(0,k_0) = \frac{\alpha_{k_0}+1}{\alpha_{k_0}-1}, \tag{2.42}$$

$$\begin{aligned} m_-(z,k_0) &= \frac{\mathrm{Re}(a_{k_0+1}) - i\mathrm{Im}(a_{k_0+1})M_-(z,k_0+1)}{\mathrm{Re}(b_{k_0+1})M_-(z,k_0) - i\mathrm{Im}(b_{k_0+1})} = \frac{(1+z)-(1-z)M_-(z,k_0)}{(1+z)M_-(z,k_0)-(1-z)} \\ &= \frac{z(M_-(z,k_0)+1) - (M_-(z,k_0)-1)}{z(M_-(z,k_0)+1) + (M_-(z,k_0)-1)}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}. \end{aligned} \tag{2.43}$$

In particular, one infers that $M_\pm$ are analytic at $z=0$.

Next, we introduce the functions $\Phi_\pm(\cdot,k)$, $k \in \mathbb{Z}$, by

$$\Phi_\pm(z,k) = \frac{M_\pm(z,k)-1}{M_\pm(z,k)+1}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}. \tag{2.44}$$

One then verifies,

$$M_\pm(z,k) = \frac{1+\Phi_\pm(z,k)}{1-\Phi_\pm(z,k)}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}, \tag{2.45}$$

$$m_-(z,k) = \frac{z-\Phi_-(z,k)}{z+\Phi_-(z,k)}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}. \tag{2.46}$$

Finally, we turn to the resolvent of U:

Lemma 2.8. *Let $z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\})$ and fix $k_0 \in \mathbb{Z}$. Then the resolvent $(U-zI)^{-1}$ of the unitary CMV operator U on $\ell^2(\mathbb{Z})$ is given in terms of its matrix representation in the standard basis of $\ell^2(\mathbb{Z})$ by*

$$\begin{aligned} (U-zI)^{-1}(k,k') &= \frac{-1}{2z[M_+(z,k_0)-M_-(z,k_0)]} \\ &\quad \times \begin{cases} \widetilde{u}_-(z,k,k_0)v_+(z,k',k_0), & k<k' \text{ and } k=k' \text{ odd}, \\ v_-(z,k',k_0)\widetilde{u}_+(z,k,k_0), & k'<k \text{ and } k=k' \text{ even}, \end{cases} \quad k,k' \in \mathbb{Z}, \end{aligned} \tag{2.47}$$

where

$$\widetilde{u}_\pm(z,k,k_0) = \begin{cases} u_\pm(z,k,k_0)/z, & k_0 \text{ odd}, \\ u_\pm(z,k,k_0), & k_0 \text{ even}. \end{cases} \tag{2.48}$$

Moreover, since $0 \in \mathbb{C}\backslash\sigma(U)$, (2.47) analytically extends to $z=0$. In addition, the following formulas hold,

$$(U-zI)^{-1}(k,k) = \frac{[1-M_+(z,k)][1+M_-(z,k)]}{2z[M_+(z,k)-M_-(z,k)]}, \tag{2.49}$$

$$(U-zI)^{-1}(k-1,k-1) = \frac{[\overline{a_k}-\overline{b_k}M_+(z,k)][a_k+b_kM_-(z,k)]}{2z\rho_k^2[M_+(z,k)-M_-(z,k)]}, \tag{2.50}$$

$$(U-zI)^{-1}(k-1,k) = -\frac{\begin{cases} [1-M_+(z,k)][\overline{a_k}-\overline{b_k}M_-(z,k)], & k \text{ odd}, \\ [1+M_+(z,k)][a_k+b_kM_-(z,k)], & k \text{ even}, \end{cases}}{2z\rho_k[M_+(z,k)-M_-(z,k)]} \tag{2.51}$$

$$(U-zI)^{-1}(k,k-1) = -\frac{\begin{cases} [1+M_+(z,k)][a_k+b_kM_-(z,k)], & k \text{ odd}, \\ [1-M_+(z,k)][\overline{a_k}-\overline{b_k}M_-(z,k)], & k \text{ even}. \end{cases}}{2z\rho_k[M_+(z,k)-M_-(z,k)]} \tag{2.52}$$

3. Borg–Marchenko-type Uniqueness Results for CMV Operators

In this section we prove (local) Borg–Marchenko-type uniqueness results for CMV operators with scalar-valued Verblunsky coefficients on the full lattice $\ell^2(\mathbb{Z})$ and on half-lattices $\ell^2([k_0, \pm\infty) \cap \mathbb{Z})$. The principal results in the full lattice case, Theorems 3.3 and 3.4 are new. We freely use the notation established in Section 2.

We start with uniqueness results for CMV operators on half-lattices. While these results are known and have recently been recorded by Simon in [62, Thm. 1.5.5], we present the proofs for the convenience of the reader as the half-lattice results are crucial ingredients for our new full lattice results.

Theorem 3.1. *Assume Hypothesis 2.1 and let $k_0 \in \mathbb{Z}$, $N \in \mathbb{N}$. Then, for the right half-lattice problem, the following sets of data (i)–(v) are equivalent:*

$$(i) \quad \{\alpha_{k_0+k}\}_{k=1}^N. \tag{3.1}$$

$$(ii) \quad \left\{ \oint_{\partial\mathbb{D}} \zeta^k \, d\mu_+(\zeta, k_0) \right\}_{k=1}^N. \tag{3.2}$$

(iii) $\{m_{+,k}(k_0)\}_{k=1}^N$, *where $m_{+,k}(k_0)$, $k \geq 0$, are the Taylor coefficients of $m_+(z, k_0)$ at $z = 0$, that is,*

$$m_+(z, k_0) = \sum_{k=0}^{\infty} m_{+,k}(k_0) z^k, \ z \in \mathbb{D}. \tag{3.3}$$

(iv) $\{M_{+,k}(k_0)\}_{k=1}^N$, *where $M_{+,k}(k_0)$, $k \geq 0$, are the Taylor coefficients of $M_+(z, k_0)$ at $z = 0$, that is,*

$$M_+(z, k_0) = \sum_{k=0}^{\infty} M_{+,k}(k_0) z^k, \ z \in \mathbb{D}. \tag{3.4}$$

(v) $\{\phi_{+,k}(k_0)\}_{k=1}^N$, *where $\phi_{+,k}(k_0)$, $k \geq 0$, are the Taylor coefficients of $\Phi_+(z, k_0)$ at $z = 0$, that is,*

$$\Phi_+(z, k_0) = \sum_{k=0}^{\infty} \phi_{+,k}(k_0) z^k, \ z \in \mathbb{D}. \tag{3.5}$$

Similarly, for the left half-lattice problem, the following sets of data (vi)–(x) are equivalent:

$$(vi) \quad \{\alpha_{k_0-k}\}_{k=0}^{N-1}. \tag{3.6}$$

$$(vii) \quad \left\{ \oint_{\partial\mathbb{D}} \zeta^k \, d\mu_-(\zeta, k_0) \right\}_{k=1}^N. \tag{3.7}$$

$(viii)$ $\{m_{-,k}(k_0)\}_{k=1}^N$, *where $m_{-,k}(k_0)$, $k \geq 0$, are the Taylor coefficients of $m_-(z, k_0)$ at $z = 0$, that is,*

$$m_-(z, k_0) = \sum_{k=0}^{\infty} m_{-,k}(k_0) z^k. \tag{3.8}$$

(ix) $\{M_{-,k}(k_0)\}_{k=0}^{N-1}$, *where $M_{-,k}(k_0)$, $k \geq 0$, are the Taylor coefficients of $M_-(z, k_0)$ at $z = 0$, that is,*

$$M_-(z, k_0) = \sum_{k=0}^{\infty} M_{-,k}(k_0) z^k. \tag{3.9}$$

(x) $\{\varphi_{-,k}(k_0)\}_{k=0}^{N-1}$, *where $\varphi_{-,k}(k_0)$, $k \geq 0$, are the Taylor coefficients of $\Phi_-(z, k_0)^{-1}$ at $z = 0$, that is,*

$$\Phi_-(z, k_0)^{-1} = \sum_{k=0}^{\infty} \varphi_{-,k}(k_0) z^k. \tag{3.10}$$

Proof. The crucial equivalence of items (i) and (ii) can be found in Simon [62, Thm. 1.5.5] (where a more general result is proven). For the convenience of the reader we present an alternative proof below.
$(i) \Rightarrow (ii)$ and $(vi) \Rightarrow (vii)$: First, utilizing relations (2.20) and (2.23) with the initial conditions (2.24) and (2.25), one constructs $\{p_\pm(z, k_0 \pm k, k_0)\}_{k=1}^N$ and $\{r_\pm(z, k_0 \pm k, k_0)\}_{k=1}^N$. We note that

the polynomials

$$\begin{cases} z^{-1}p_+(z,k_0+k,k_0),\ r_-(z,k_0-k,k_0), & k_0 \text{ odd},\\ r_+(z,k_0+k,k_0),\ z^{-1}p_-(z,k_0-k,k_0), & k_0 \text{ even}, \end{cases} \tag{3.11}$$

are linear combinations of

$$\begin{cases} 1, z^{-1}, z, z^{-2}, z^2, \dots, z^{(k-1)/2}, z^{-(k+1)/2}, & k \text{ odd},\\ 1, z^{-1}, z, z^{-2}, z^2, \dots, z^{-k/2}, z^{k/2}, & k \text{ even}, \end{cases} \tag{3.12}$$

and

$$\begin{cases} r_+(z,k_0+k,k_0),\ p_-(z,k_0-k,k_0), & k_0 \text{ odd},\\ p_+(z,k_0+k,k_0),\ r_-(z,k_0-k,k_0), & k_0 \text{ even}, \end{cases} \tag{3.13}$$

are linear combinations of

$$\begin{cases} 1, z, z^{-1}, z^2, z^{-2}, \dots, z^{-(k-1)/2}, z^{(k+1)/2}, & k \text{ odd},\\ 1, z, z^{-1}, z^2, z^{-2}, \dots, z^{k/2}, z^{-k/2}, & k \text{ even}. \end{cases} \tag{3.14}$$

Moreover, the last elements of the sequences in (3.12) and (3.14) represent the leading-order terms of the polynomials in (3.11) and (3.13), respectively, and the corresponding leading-order coefficients are nonzero.

Next, assume k_0 and k to be odd. Then utilizing (3.13) and (3.14) one finds constants $c_{\pm,j}$ and $d_{\pm,j}$, $0 \le j \le k$, such that

$$z^{-(k-1)/2} = \sum_{j=0}^{k} c_{+,j}\, r_+(z,k_0+j,k_0), \quad z^{(k+1)/2} = \sum_{j=0}^{k} d_{+,j}\, r_+(z,k_0+j,k_0), \tag{3.15}$$

$$z^{-(k-1)/2} = \sum_{j=0}^{k} c_{-,j}\, p_-(z,k_0-j,k_0), \quad z^{(k+1)/2} = \sum_{j=0}^{k} d_{-,j}\, p_-(z,k_0-j,k_0), \tag{3.16}$$

and, using Lemma 2.4, computes

$$\oint_{\partial\mathbb{D}} \zeta^k d\mu_\pm(\zeta,k_0) = \oint_{\partial\mathbb{D}} \overline{\zeta^{-(k-1)/2}}\zeta^{(k+1)/2} d\mu_\pm(\zeta,k_0) = \sum_{j=0}^{k} \overline{c_{\pm,j}}\, d_{\pm,j}. \tag{3.17}$$

The remaining cases of k_0 and k follow similarly.

$(ii) \Rightarrow (i)$ and $(vii) \Rightarrow (vi)$: Since the measures $d\mu_\pm(\cdot,k_0)$ are real-valued and normalized, one has

$$\oint_{\partial\mathbb{D}} \zeta^{-k} d\mu_\pm(\zeta,k_0) = \overline{\oint_{\partial\mathbb{D}} \zeta^k d\mu_\pm(\zeta,k_0)} \quad \text{and} \quad \oint_{\partial\mathbb{D}} d\mu_\pm(\zeta,k_0) = 1, \tag{3.18}$$

that is, the knowledge of positive moments imply the knowledge of negative ones. Applying Corollary 2.5 and Theorem 2.6 one constructs orthonormal polynomials $\{p_\pm(\zeta,k_0\pm k,k_0)\}_{k=1}^N$ and $\{r_\pm(\zeta,k_0\pm k,k_0)\}_{k=1}^N$ and subsequently the Verblunsky coefficients in (i) and (vi) using formulas (2.34) and (2.35).

$(ii) \Leftrightarrow (iii)$ and $(vii) \Leftrightarrow (viii)$: These equivalences follow directly from (2.36),

$$m_\pm(z,k_0) = \pm\oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\, d\mu_\pm(\zeta,k_0) = \pm 1 \pm 2\sum_{k=1}^{\infty} z^k \overline{\oint_{\partial\mathbb{D}} \zeta^k d\mu_\pm(\zeta,k_0)}, \quad z \in \mathbb{D}. \tag{3.19}$$

$(iii) \Leftrightarrow (iv)$: This follows from (2.39).

$(iv) \Leftrightarrow (v)$: This follows from (2.40), (2.44), (2.45), and the fact that $|\Phi_+(z,k_0)| < 1$, $z \in \mathbb{D}$,

$$M_+(z,k_0) = \frac{1+\Phi_+(z,k_0)}{1-\Phi_+(z,k_0)} \underset{z\to 0}{=} [1+\Phi_+(z,k_0)] \sum_{k=0}^{\infty} \Phi_+(z,k_0)^k, \tag{3.20}$$

$$\Phi_+(z,k_0) = \frac{2^{-1}[M_+(z,k_0)-1]}{2^{-1}[M_+(z,k_0)-1]+1} \underset{z\to 0}{=} -\sum_{k=1}^{\infty} 2^{-k}[1-M_+(z,k_0)]^k. \tag{3.21}$$

$(ix) \Leftrightarrow (x)$: This follows from (2.42), (2.44), (2.45), and the fact that $|\Phi_-(z,k_0)^{-1}| < 1$, $z \in \mathbb{D}$,

$$M_-(z,k_0) = \frac{1/\Phi_-(z,k_0)+1}{1/\Phi_-(z,k_0)-1} \underset{z\to 0}{=} -[1/\Phi_-(z,k_0)+1] \sum_{k=0}^{\infty} \Phi_-(z,k_0)^{-k}, \tag{3.22}$$

$$\begin{aligned} 1/\Phi_-(z,k_0) &= \frac{M_-(z,k_0)+1}{M_-(z,k_0)-M_-(0,k_0)+M_-(0,k_0)-1} \\ &= \frac{[M_-(z,k_0)+1][M_-(0,k_0)-1]^{-1}}{[M_-(z,k_0)-M_-(0,k_0)][M_-(0,k_0)-1]^{-1}+1} \\ &\underset{z\to 0}{=} \frac{M_-(z,k_0)+1}{M_-(0,k_0)-1} \sum_{k=0}^{\infty} \left(\frac{M_-(z,k_0)-M_-(0,k_0)}{1-M_-(0,k_0)}\right)^k. \end{aligned} \tag{3.23}$$

$(viii) \Leftrightarrow (x)$: This follows from (2.37), (2.46), and the fact that $|\Phi_-(z,k_0)^{-1}| < 1$, $z \in \mathbb{D}$,

$$m_-(z,k_0) = \frac{z/\Phi_-(z,k_0)-1}{z/\Phi_-(z,k_0)+1} \underset{z\to 0}{=} [z/\Phi_-(z,k_0)-1] \sum_{k=0}^{\infty} \big(-z/\Phi_-(z,k_0)\big)^k, \tag{3.24}$$

$$z/\Phi_-(z,k_0) = \frac{1+m_-(z,k_0)}{1-m_-(z,k_0)} \underset{z\to 0}{=} \frac{2^{-1}[1+m_-(z,k_0)]}{1-2^{-1}[1+m_-(z,k_0)]} = \sum_{k=1}^{\infty} \left(\frac{1+m_-(z,k_0)}{2}\right)^k. \tag{3.25}$$

□

We restate Theorem 3.1 as follows:

Theorem 3.2. *Assume Hypothesis 2.1 for two sequences $\alpha^{(1)}$, $\alpha^{(2)}$ and let $k_0 \in \mathbb{Z}$, $N \in \mathbb{N}$. Then for the right half-lattice problems associated with $\alpha^{(1)}$ and $\alpha^{(2)}$ the following items (i)–(iv) are equivalent:*

$$(i) \quad \alpha_k^{(1)} = \alpha_k^{(2)}, \quad k_0+1 \le k \le k_0+N. \tag{3.26}$$

$$(ii) \quad m_+^{(1)}(z,k_0) - m_+^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^N). \tag{3.27}$$

$$(iii) \quad M_+^{(1)}(z,k_0) - M_+^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^N). \tag{3.28}$$

$$(iv) \quad \Phi_+^{(1)}(z,k_0) - \Phi_+^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^N). \tag{3.29}$$

Similarly, for the left half-lattice problems associated with $\alpha^{(1)}$ and $\alpha^{(2)}$, the following items (v)–$(viii)$ are equivalent:

$$(v) \quad \alpha_k^{(1)} = \alpha_k^{(2)}, \quad k_0-N+1 \le k \le k_0. \tag{3.30}$$

$$(vi) \quad m_-^{(1)}(z,k_0) - m_-^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^N). \tag{3.31}$$

$$(vii) \quad M_-^{(1)}(z,k_0) - M_-^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^{N-1}). \tag{3.32}$$

$$(viii) \quad 1/\Phi_-^{(1)}(z,k_0) - 1/\Phi_-^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^{N-1}). \tag{3.33}$$

Proof. This follows immediately from Theorem 3.1. □

Finally, we turn to CMV operators on $\mathbb{Z}$.

To start, we introduce the following notation for the diagonal and for the neighboring off-diagonal entries of the Green's function of U (i.e., the discrete integral kernel of $(U-zI)^{-1}$),

$$g(z,k) = (U-Iz)^{-1}(k,k), \tag{3.34}$$

$$h(z,k) = \begin{cases} (U-Iz)^{-1}(k-1,k), & k \text{ odd}, \\ (U-Iz)^{-1}(k,k-1), & k \text{ even}, \end{cases} \quad k\in\mathbb{Z},\ z\in\mathbb{D}. \tag{3.35}$$

Then the following uniqueness results hold for the full-lattice CMV operator U:

Theorem 3.3. *Assume Hypothesis* 2.1 *and let* $k_0\in\mathbb{Z}$. *Then any of the following two sets of data*

(*i*) $g(z,k_0)$ *and* $h(z,k_0)$ *for all* z *in some open* (*nonempty*) *neighborhood of the origin under the assumption* $h(0,k_0)\neq 0$;

(*ii*) $g(z,k_0-1)$ *and* $g(z,k_0)$ *for all* z *in some open* (*nonempty*) *neighborhood of the origin and* α_{k_0} *under the assumption* $\alpha_{k_0}\neq 0$;

uniquely determines the Verblunsky coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$, *and hence the full-lattice CMV operator* U.

Proof. Case (i). First, note that it follows from (2.8) that

$$g(0,k_0) = (U^{-1})_{k_0,k_0} = (U^*)_{k_0,k_0} = \overline{U_{k_0,k_0}} = -\alpha_{k_0}\overline{\alpha_{k_0+1}}, \tag{3.36}$$

$$h(0,k_0) = \begin{cases} (U^{-1})_{k_0-1,k_0} = \overline{U_{k_0,k_0-1}} = -\overline{\alpha_{k_0+1}}\rho_{k_0}, & k_0 \text{ odd}, \\ (U^{-1})_{k_0,k_0-1} = \overline{U_{k_0-1,k_0}} = -\overline{\alpha_{k_0+1}}\rho_{k_0}, & k_0 \text{ even}. \end{cases} \tag{3.37}$$

Since by hypothesis $h(0,k_0)\neq 0$, one can solve the above equalities for α_{k_0},

$$\frac{g(0,k_0)}{h(0,k_0)} = \alpha_{k_0}/\rho_{k_0}, \quad |\alpha_{k_0}|^2 = \frac{|g(0,k_0)|^2}{|g(0,k_0)|^2+|h(0,k_0)|^2}, \quad \rho_{k_0} = \frac{|h(0,k_0)|}{\sqrt{|g(0,k_0)|^2+|h(0,k_0)|^2}}, \tag{3.38}$$

and hence,

$$\alpha_{k_0} = \frac{g(0,k_0)|h(0,k_0)|}{h(0,k_0)\sqrt{|g(0,k_0)|^2+|h(0,k_0)|^2}}. \tag{3.39}$$

Recalling (2.3), one has $a_{k_0}=1+\alpha_{k_0}$ and $b_{k_0}=1-\alpha_{k_0}$.

Next, utilizing (2.49), (2.51), and (2.52), one computes,

$$\frac{g(z,k_0)}{h(z,k_0)} = \frac{\rho_{k_0}[1+M_-(z,k_0)]}{\overline{b_{k_0}}M_-(z,k_0)-\overline{a_{k_0}}}, \quad z\in\mathbb{D}. \tag{3.40}$$

Solving for $M_-(z,k_0)$, one then obtains

$$M_-(z,k_0) = \frac{2g(z,k_0)}{\overline{b_{k_0}}g(z,k_0)-\rho_{k_0}h(z,k_0)} - 1, \quad z\in\mathbb{D}. \tag{3.41}$$

Here, the denominator may have only a discrete set of zeros (i.e., without accumulation points in $\mathbb{D}$), corresponding to removable singularities of the fraction. Otherwise, the denominator is identically zero in $\mathbb{D}$ since the functions g and h are analytic in $\mathbb{D}$. This in turn implies that the numerator is identically zero in $\mathbb{D}$, and hence, h is identically zero in $\mathbb{D}$, contradicting our assumption $h(0,k_0)\neq 0$.

Next, having obtained $M_-(z,k_0)$, one solves

$$h(z,k_0) = -\frac{[1-M_+(z,k_0)][\overline{a_k}-\overline{b_k}M_-(z,k_0)]}{2z\rho_k[M_+(z,k_0)-M_-(z,k_0)]}, \quad z\in\mathbb{D} \tag{3.42}$$

for $M_+(z,k_0)$ and obtains,

$$\begin{aligned} M_+(z,k_0) &= \frac{[\overline{a_{k_0}} - \overline{b_{k_0}} M_-(z,k_0)] - 2z\rho_{k_0} h(z,k_0) M_-(z,k_0)}{[\overline{a_{k_0}} - \overline{b_{k_0}} M_-(z,k_0)] - 2z\rho_{k_0} h(z,k_0)} \\ &= \frac{2(1+zg(z,k_0))}{1+z[\overline{b_{k_0}} g(z,k_0) - \rho_{k_0} h(z,k_0)]} - 1, \quad z \in \mathbb{D}. \end{aligned} \tag{3.43}$$

Here, the denominator may have only a discrete set of zeros, corresponding to removable singularities of the fraction (3.43). Otherwise, one concludes again that the denominator is identically zero in $\mathbb{D}$, contradicting the fact that g and h are analytic in $\mathbb{D}$.

Finally, Theorem 3.1 (parts (i), (iv) and (vi), (ix)) implies that $M_\pm(z,k_0)$, $z \in \mathbb{D}$, uniquely determine the Verblunsky coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$.

Case (ii). First, using (2.44) and

$$1 + zg(z,k_0) = \frac{[1+M_+(z,k_0)][1-M_-(z,k_0)]}{2[M_+(z,k_0) - M_-(z,k_0)]}, \tag{3.44}$$

which follows from (2.49), one rewrites (2.49) and (2.50) as

$$zg(z,k_0) = [\Phi_+(z,k_0)/\Phi_-(z,k_0)][1+zg(z,k_0)]. \tag{3.45}$$

$$\begin{aligned} z\rho_{k_0}^2 g(z,k_0-1) &= \frac{\big[[1-M_+(z,k_0)] + \overline{\alpha_{k_0}}[1+M_+(z,k_0)]\big]\big[[1+M_-(z,k_0)] + \alpha_{k_0}[1-M_-(z,k_0)]\big]}{2[M_+(z,k_0) - M_-(z,k_0)]} \\ &= [\overline{\alpha_{k_0}} - \Phi_+(z,k_0)][\alpha_{k_0} - 1/\Phi_-(z,k_0)][1+zg(z,k_0)], \end{aligned} \tag{3.46}$$

Next, introducing the functions $A(z,k_0)$ and $B(z,k_0)$ by

$$A(z,k_0) = 1 + zg(z,k_0), \quad B(z,k_0) = z\rho_{k_0}^2 g(z,k_0-1) - zg(z,k_0) - |\alpha_{k_0}|^2 A(z,k_0), \tag{3.47}$$

one obtains from (3.46) and (3.45) that $\Phi_+(z,k_0)$ satisfies the following quadratic equation

$$\alpha_{k_0} A(z,k_0)\Phi_+(z,k_0)^2 + B(z,k_0)\Phi_+(z,k_0) + \overline{\alpha_{k_0}} zg(z,k_0) = 0. \tag{3.48}$$

In addition, it follows from (2.44) that $\Phi_+(0,k_0) = 0$ since by (2.40) $M_+(0,k_0) = 1$. Hence, $\Phi_+(z,k_0)$ can be uniquely determined for sufficiently small $|z|$ from (3.48),

$$\Phi_+(z,k_0) = \frac{-B(z,k_0) - \sqrt{B(z,k_0)^2 - 4|\alpha_{k_0}|^2 A(z,k_0) zg(z,k_0)}}{2\alpha_{k_0} A(z,k_0)}. \tag{3.49}$$

Utilizing (3.46), one also finds $1/\Phi_-(z,k_0)$ for sufficiently small $|z|$,

$$1/\Phi_-(z,k_0) = \alpha_{k_0} - \frac{z\rho_{k_0}^2 g(z,k_0-1)}{[1+zg(z,k_0)][\overline{\alpha_{k_0}} - \Phi_+(z,k_0)]}. \tag{3.50}$$

Finally, Theorem 3.1 (parts (i), (v) and (vi), (x)) implies again that for $|z|$ sufficiently small, $\Phi_\pm(z,k_0)^{\pm 1}$ uniquely determine the Verblunsky coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$. $\square$

In the following result, $g^{(j)}$ and $h^{(j)}$ denote the corresponding quantities (3.34) and (3.35) associated with the Verblunsky coefficients $\alpha^{(j)}$, $j = 1, 2$.

Theorem 3.4. *Assume Hypothesis* 2.1 *for two sequences* $\alpha^{(1)}$, $\alpha^{(2)}$ *and let* $k_0 \in \mathbb{Z}$, $N \in \mathbb{N}$. *Then for the full-lattice problems associated with* $\alpha^{(1)}$ *and* $\alpha^{(2)}$, *the following local uniqueness results hold:*

(i) *If either* $h^{(1)}(0,k_0)$ *or* $h^{(2)}(0,k_0)$ *is nonzero and*

$$\begin{aligned} &|g^{(1)}(z,k_0) - g^{(2)}(z,k_0)| + |h^{(1)}(z,k_0) - h^{(2)}(z,k_0)| \underset{z\to 0}{=} o(z^N), \\ &\text{then } \alpha_k^{(1)} = \alpha_k^{(2)} \text{ for } k_0 - N \le k \le k_0 + N + 1. \end{aligned} \tag{3.51}$$

(*ii*) *If* $\alpha_{k_0}^{(1)} = \alpha_{k_0}^{(2)} \neq 0$ *and*

$$\begin{aligned} &|g^{(1)}(z,k_0-1) - g^{(2)}(z,k_0-1)| + |g^{(1)}(z,k_0) - g^{(2)}(z,k_0)| \underset{z\to 0}{=} o(z^N), \\ &\textit{then } \ \alpha_k^{(1)} = \alpha_k^{(2)} \quad \textit{for} \quad k_0 - N - 1 \leq k \leq k_0 + N + 1. \end{aligned} \tag{3.52}$$

Proof. Case (*i*). The result follows from Theorem 3.2 (parts (*i*), (*iii*) and (*v*), (*vii*)) upon verifying that (3.41), (3.43), and (3.51) imply

$$M_+^{(1)}(z,k_0) - M_+^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^{N+1}), \tag{3.53}$$

$$M_-^{(1)}(z,k_0) - M_-^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^{N}). \tag{3.54}$$

The latter asymptotic behavior follows from the fact that the denominator in (3.51) is non-zero as $z \to 0$. Indeed, using (3.36) and (3.37) one computes

$$\overline{b_{k_0}} g(0,k_0) - \rho_{k_0} h(0,k_0) = (\overline{\alpha_{k_0}} - 1)\alpha_{k_0}\overline{\alpha_{k_0+1}} + \rho_{k_0}^2 \overline{\alpha_{k_0+1}} = (\alpha_{k_0} - 1) h(0,k_0)/\rho_{k_0} \neq 0. \tag{3.55}$$

Case (*ii*). The result follows from Theorem 3.2 (parts (*i*), (*iv*) and (*v*), (*viii*)) upon verifying that (3.49), (3.50), and (3.52) imply

$$|\Phi_+^{(1)}(z,k_0) - \Phi_+^{(2)}(z,k_0)| + |1/\Phi_-^{(1)}(z,k_0) - 1/\Phi_-^{(2)}(z,k_0)| \underset{z\to 0}{=} o(z^{N+1}). \tag{3.56}$$

□

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