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Response of the Bilinear Hysteretic System to Stationary Random Excitation

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Time-average statistics of the response of the bilinear hysteretic system to an excitation with approximately white-power spectral density and approximately Gaussian probability distribution are determined, using electronic-analog techniques. Results are presented for the mean-squared amplitude, the power spectral density, and the probability distribution of the response. The applicability of the Krylov–Bogoliubov method of equivalent linearization to this problem is investigated by comparing predicted and experimentally measured values of the mean-squared level of response.

INTRODUCTION

In recent years, there has been considerable interest in the application of approximate analytical techniques to problems of determining the response of nonlinear systems to stationary random excitation.[1–4] These analytical techniques have been used with apparent success on a variety of systems, but, in general, there has been little information that could be used to gauge the accuracy of these methods in a given application.

One of the systems that has been studied in some detail by these approximate analytical techniques is the so-called bilinear hysteretic system. This system is often used as an approximation to the yielding behavior of both materials and structures; as such, it has considerable importance from an engineering point of view. Furthermore, the problem of determining the response of such a system to a nondeterministic excitation has obvious importance in light of the types of loading that many modern structures are required to withstand. Whether the actual excitation in a given situation will approximate a stationary random signal is problematic, but certainly substantial knowledge about the general behavior of such systems can be obtained by a consideration of the stationary problem. From a more practical standpoint, this is the only problem that has thus far been treated in any detail by analytical techniques.

It is the purpose of the present paper to investigate the response of the bilinear hysteretic oscillator to stationary random excitation with a white-power spectral density and a Gaussian probability distribution, using electronic-analog techniques, and then to compare these results where possible with the results of one of the more widely used approximate analytical techniques. It is felt that this leads both to a better understanding of the behavior of a system of some practical importance and also to a clearer understanding of the applicability of certain approximate analytical approaches to nondeterministic response problems.

I. THE SYSTEM

The equation of motion for a single-degree-of-freedom bilinear hysteretic oscillator may be written as

$$\ddot{x}+2\beta_0\omega_0\dot{x}+\omega_0^2\varphi(x)=n(t)/m, \tag{1}$$

where m denotes the mass, ω_0 is the small-amplitude undamped natural circular frequency, β_0 is the small-amplitude fraction of critical damping, and $\varphi(x)$ is a bilinear hysteretic restoring force as shown in Fig. 1. Note that $\varphi(x)$ is chosen to have a unit slope for small amplitudes and a second slope α. For the present investigation, the excitation $n(t)$ is taken to be a stationary random function with a uniform power spectral density S_0 and a Gaussian probability distribution.

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[2] T. K. Caughey, "Equivalent Linearization Techniques," J. Acoust. Soc. Am. **35**, 1706–1711 (1963).

[3] R. H. Lyon, M. Heckl, and C. B. Hazelgrove, "Response of Hard-Spring Oscillator to Narrow-Band Excitation," J. Acoust. Soc. Am. **33**, 1404–1411 (1961).

[4] D. Karnapp and T. D. Scharton, "Plastic Deformation in Random Vibration," J. Acoust. Soc. Am. **39**, 1154–1161 (1966).

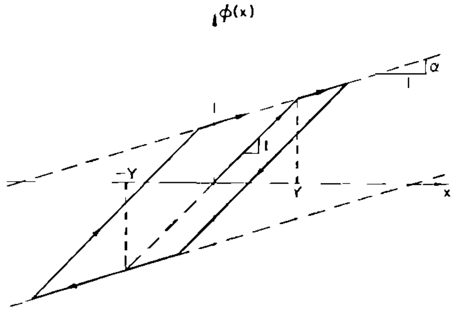


FIG. 1. Bilinear hysteretic restoring force.

A schematic of the electronic analog used to model Eq. 1 is shown in Fig. 2. The parameters of the analog were chosen in such a manner that the small-amplitude natural frequency of the system was approximately 500 cps and the damping factor could be varied between zero and arbitrarily large values. The elastoplastic function $\varphi_p(x)$ was generated by means of a solid-state flip–flop and holding circuit. The details of this circuit and a discussion of the specific equipment used in the analog are contained in Appendix A.

Photographs of oscilloscope traces of the elastoplastic function $\varphi_p(x)$ vs x for several frequencies of periodic motion are reproduced in Fig. 3. These photographs show that the function was essentially independent of frequency over the range from 5 to 1000 cps and was, in fact, a good approximation to the ideal elastoplastic function. The curvature of the sides of the hysteresis loop at very low frequencies is due to leakage currents, and the overshoot of $\varphi_p(x)$ at initiation of yield at high frequency is due to finite switching times associated with the circuit. An analysis of the errors introduced by these two effects indicates that they are negligible for the present study. Further error investigation indicates that the circuit described here should represent the system of Eq. 1 to within about 2%.

On the basis of a detailed analysis of the combined errors in the analog circuit and in the response measurements, it is felt that the over-all accuracy of the statistical results is within 4%–5%, except for certain limiting cases of probability distribution. These exceptional cases are discussed later in context.

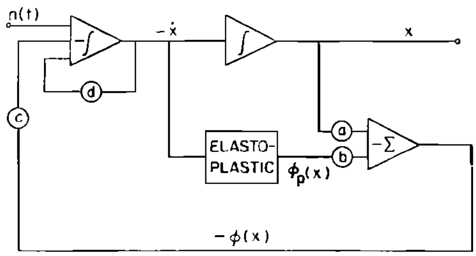


FIG. 2. Analog circuit for hysteretic oscillator.

II. SYSTEM RESPONSE

A. Mean-Squared Level of Response

The mean level is the simplest measure of a random signal, but it tells nothing of the variability of the signal. In fact, in the present study, the mean levels of both excitation and response were zero. One simple measure of how much a signal of zero mean value varies from that mean value is the mean-square level of the signal. In the present study, the rms level of displacement σ_x and the rms level of velocity $\sigma_{\dot{x}}$ were measured for the response of the bilinear system. These two measures of the response were both time independent. However, this fact in itself does not necessarily imply that the response was strongly stationary, since the probability distribution was, in general, non-Gaussian.

Figures 4 and 5 give the results of the measurements of rms response levels for systems with $\alpha=1/2$ and $1/21$ and with several values of the damping factor β_0. The factor $(S_0\omega_0)^{\frac{1}{2}}$ has been selected as a measure of the excitation level, since it is meaningless to talk about the rms level of a signal with white-power spectral density. The abscissas of the graphs may therefore be thought of as representing the ratio of the excitation level to the yield force level $m\omega_0^2Y$. The ordinates have been made nondimensional by dividing the rms displacement response by the factor $(S_0\omega_0)^{\frac{1}{2}}/m\omega_0^2$, and by dividing the rms velocity response by $(S_0\omega_0)^{\frac{1}{2}}/m\omega_0$. In this form, the response of a linear system would appear as a straight horizontal line on the graphs.

In some mechanical systems where yielding corresponds to straining of ductile members, the ratio σ_x/Y may be an important factor in failure criteria, since it is a measure of how much yielding is taking

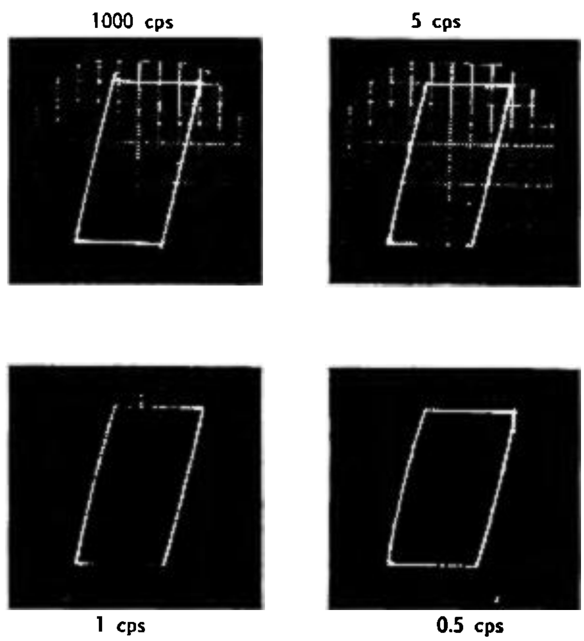
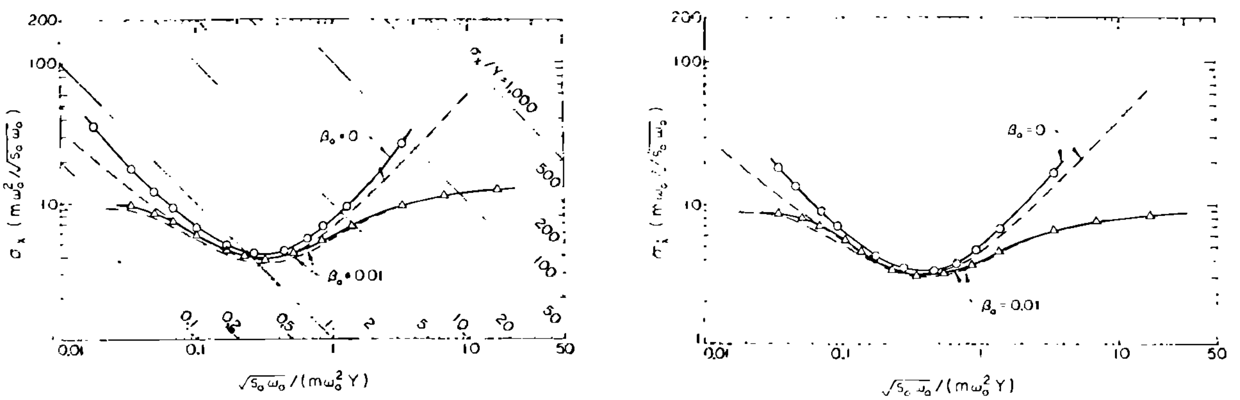


FIG. 3. Oscillator traces: $\varphi_p(x)$ vs x.



(a) (b)

FIG. 4. Root-mean-square response levels for system with $\alpha=\frac{1}{2}$. (a) Displacement response. (b) Velocity response. ——: Analog. - - - -: Theory.

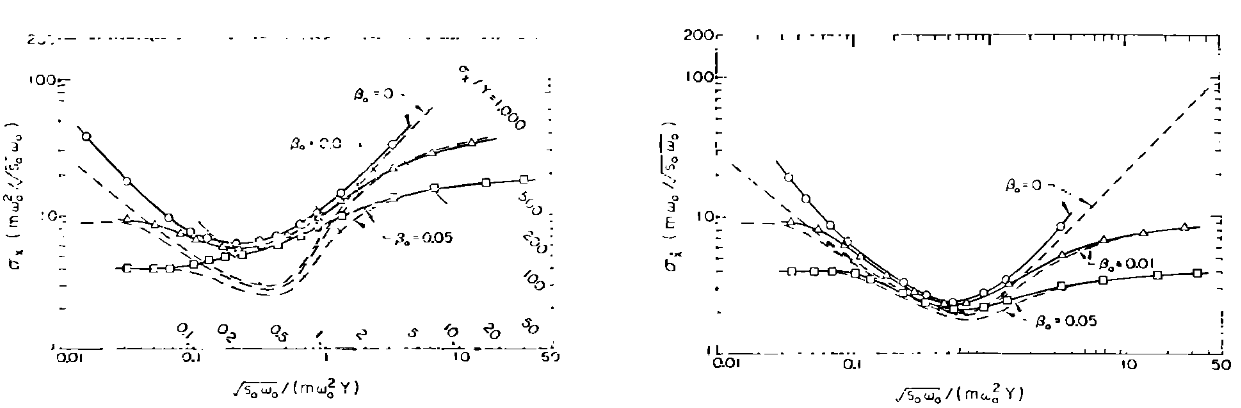
place. Grid lines for σ_x/Y are presented in Figs. 4(a) and 5(a) for convenient reference. The dashed lines in Figs. 4 and 5 represent the results of an approximate analytic solution and are discussed in detail later.

The elastoplastic system has often been chosen for study by investigators of nonlinear systems because of its simplicity and because many physical systems supposedly act in an approximately elastoplastic manner. The elastoplastic system cannot be used in the study of stationary response, however, since its response to stationary excitation is not a stationary signal, but rather wanders endlessly. The system with $\alpha=1/21$ was chosen to represent a nearly elastoplastic system while eliminating the problem of wander. The system with $\alpha=1/2$ was chosen to represent a system with a moderate nonlinearity.

B. Power Spectral Density

Representative results for the power spectral density of the response S_x are given for the system with $\alpha=1/2$ and $\beta_0=0$ in Fig. 6(a) and for the system with $\alpha=1/21$ and $\beta_0=0$ in Fig. 6(b). The parameter $m\omega_0^3[S_x(\omega)/S]^{\frac{1}{2}}$ was chosen to represent the power spectral density of the response of the bilinear systems in a manner that allows easy comparison with a linear system. For a linear oscillator, this parameter is identical to the well-known harmonic-excitation transfer function. However, this is generally not true for a nonlinear system, since separate solutions to a nonlinear problem do not superpose to form another solution.

The forms of the spectral density curves in Fig. 6(b) show that the response of a severely nonlinear hysteretic oscillator is not, in general, contained in a narrow frequency band. Figure 7 further illustrates this fact by showing plots of displacement response versus time. For comparison, Curve (a) shows the response of a linear system with approximately 1% of critical viscous damping. The displacement of this system has a clearly defined principal frequency and a slowly varying amplitude, as is typical of a narrow-band signal. Curve (b), for $\alpha=1/21$ and $\sigma_x/Y=1.6$, has a fairly well-defined high-frequency component that looks similar to Curve (a), but it also contains a large low-frequency component that gives a wandering effect to the result.



(a) (b)

FIG. 5. Root-mean-square response levels for system with $\alpha=1/21$. (a) Displacement response. (b) Velocity response. ——: Analog. - - - -: Theory.

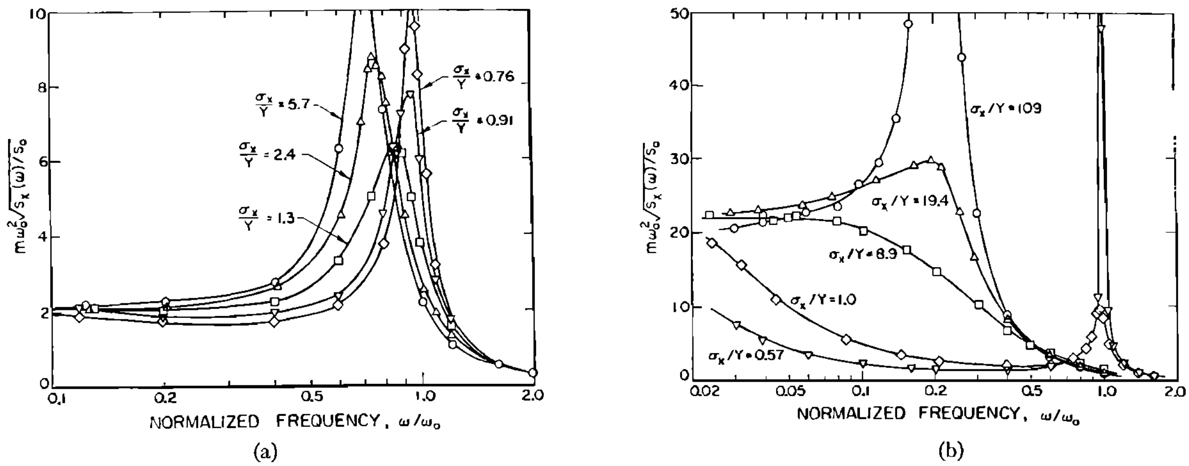


FIG. 6. Response power spectral density for (a) $\beta_0=0$, $\alpha=\frac{1}{2}$ and (b) $\alpha=1/21$.

The response of a system with $\alpha=1/21$ and $\sigma_x/Y=5.7$, as shown in Curve (c), shows almost no similarity to a narrow-band signal.

The magnitudes of the yield level are shown for Curves (b) and (c) of Fig. 7. At any time in the history of the motion, the bilinear system can execute oscillations of amplitude less than the yield level while acting in a purely elastic manner. The high-frequency component in Curve (b) clearly shows such elastic oscillations taking place at the natural frequency of the linear system. Curve (c) also shows some indication of elastic oscillations at the linear system natural frequency, but, because of the low yield level, the amplitude of such oscillations is very small as compared to the lower-frequency components present in the displacement.

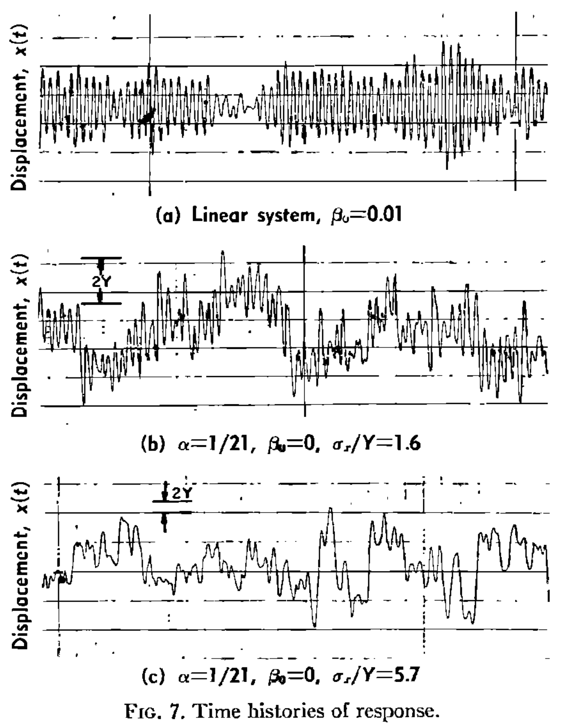


FIG. 7. Time histories of response.

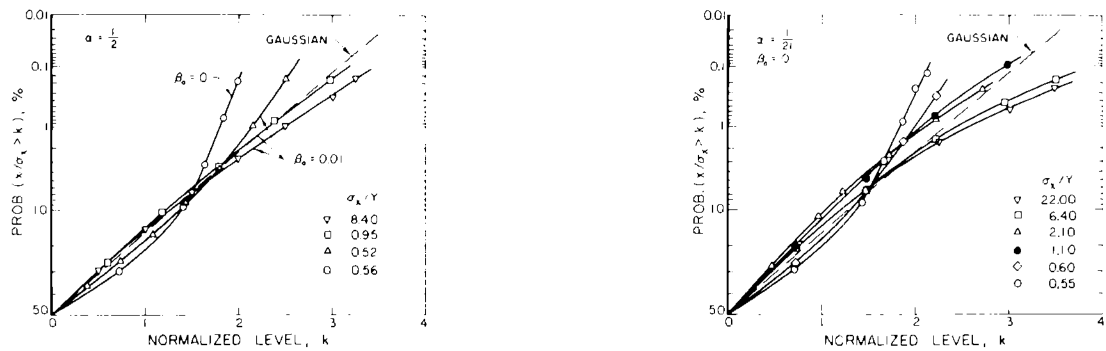
C. Probability-Distribution Measurements

The probability that a stationary signal exceeds a given level is simply the fraction of the time that the signal exceeds that level. Figure 8 shows the results of probability-distribution measurements for (a) the moderately nonlinear oscillator with $\alpha=1/2$ and for (b) the nearly elastoplastic oscillator with $\alpha=1/21$. The scales in these graphs are chosen so that the probability distribution of a Gaussian signal plots as a straight line and so that only distributions for positive x are shown, since those for negative x are identical. It should be noted that data points falling below the Gaussian distribution line for some arbitrary level k correspond to situations where the system response spends more time beyond that level than does a Gaussian signal—and the converse is true for points above the Gaussian distribution line.

III. DISCUSSION OF ANALOG RESULTS

Since the characteristics of the response of a linear system are well known, it is natural to discuss the response of a nonlinear system in terms of how it differs from that of a linear system. When the level of excitation of a linear system is raised or lowered, the levels of all measures of response are raised or lowered in direct proportion. Thus, ratios of mean-squared level of response to level of excitation, or of power spectral density of response to power spectral density of excitation, are independent of the level of excitation for a linear system. Similarly, for a linear system response, the probability distribution normalized by the rms level of the response is independent of the level of excitation.

It is immediately apparent from Figs. 4 and 5 that the mean-squared level of response for the bilinear hysteretic system differs markedly from that of a linear system. In particular, for most of the cases considered



(a) (b)

FIG. 8. Response probability distribution for (a) $\alpha=\frac{1}{2}$ and (b) $\alpha=1/21$, $\beta_0=0$.

here, the rms displacement-response ratio exhibits a definite minimum for some value of excitation yield ratio. Furthermore, in all cases, the rms displacement-response ratio is greater for large excitation levels than it is for small excitation levels. The rms velocity-response ratio also has a minimum for all cases considered, but the low- and high-excitation asymptotes are the same. Hence, yielding may either increase or decrease the rms displacement-response ratio but always acts to decrease (or leave unchanged) the rms velocity-response ratio. This result is not too surprising since the softening-spring effect of the nonlinearity always tends to increase the displacement response while the energy dissipation due to yielding tends to decrease this response. For large σ_x/Y, the softening-spring effect dominates, and the over-all response is increased. On the other hand, the velocity response reflects only the damping effect and so is always decreased by yielding.

From Figs. 4(a) and 5(a), it can be seen that, for small viscous damping in the two systems investigated, the rms displacement-response ratio is minimized when the excitation/yield ratio is such that $1<\sigma_x/Y<2$. For the system with $\alpha=1/2$ and $\beta_0=0$ and 0.01, this minimum response is approximately the same as that of a linear system with resonance at ω_0 and having about 5% of critical viscous damping. For the system with $\alpha=1/21$ and $\beta_0=0$ and 0.01, the minimum corresponds to a linear system with about 2% viscous damping.

Figures 4(a) and 5(a) also show that the region of reduced displacement response is much less for the nearly elastoplastic system than for the less severely nonlinear system. For the system with $\alpha=1/2$ and $\beta_0=0.01$, the rms displacement-response ratio is less than or equal to its low excitation value so long as $\sigma_x/Y<30$; however, for the system with $\alpha=1/21$ and $\beta_0=0.01$, this is only true when $\sigma_x/Y<7.5$. When $\alpha=1/21$ and $\beta_0=0.05$, the energy dissipation due to yielding is ineffective in reducing the over-all response ratio, which increases monotonically from its low excitation value.

From Figs. 4(b) and 5(b), one sees that the rms velocity-response ratio has a minimum for all cases considered. This minimum occurs generally within the range $1<\sigma_{\dot{x}}/\omega_0 Y<2$. For the system with $\alpha=1/2$, the minimum corresponds to that of a linear system with approximately 7% viscous damping, whereas, for $\alpha=1/21$, this figure is more like 15%. In every case, the rms velocity-response ratio is reduced by yielding, as one would expect.

As a final observation, it may be noted that the system with $\alpha=1/21$ results in considerably more reduction in the rms velocity-response ratio than does $\alpha=1/2$, whereas the less severely nonlinear system is more effective in reducing the rms displacement response. This again is due to the trade off between the effects of the softening spring and the hysteretic energy dissipation.

Figure 6 shows how the power spectral density is influenced by yielding. The primary effect in this case is the shifting of the frequency at which peak-power spectral density occurs from a value of ω_0 for small σ_x/Y to a value of $\alpha^{\frac{1}{2}}\omega_0$ for large σ_x/Y. For $\alpha=1/2$, this shift is relatively small, and the power spectral density remains noticeably peaked for all values of σ_x/Y. Hence, the system retains its narrow-band character, even with the introduction of yielding. However, for $\alpha=1/21$, the frequency shift is large; for values of σ_x/Y between 4 and 9, there was no peak at all. This gives rise to the broad-band response noted earlier in conjunction with Fig. 7.

When σ_x/Y is very large, both Figs. 6(a) and 6(b) show that the response is quite similar to that of a linear system with a natural frequency $\alpha^{\frac{1}{2}}\omega_0$. Similarly, for $\sigma_x/Y<1$, the power spectral density resembles that of a linear system with a natural frequency ω_0. However, close examination reveals that, in the latter case, the low-frequency spectrum approaches that of a system with resonance at $\alpha^{\frac{1}{2}}\omega_0$ instead of ω_0. This is especially noticeable in Fig. 6(b). All of the curves in Figs. 6(a) and 6(b) are bounded.

Figure 8 gives a further indication of the manner in which the response of the bilinear system differs from that of a linear system. Two significant trends may be noted from these Figures. The first is that, for large σ_x/Y, the response has a noticeably greater probability of being at large displacements, as compared to the rms level, than does a linear Gaussian system. For example, with $\alpha=1/21$ and $\sigma_x/Y=6.4$, the bilinear-system response spends approximately 0.4% of the time beyond $3\sigma_x$, whereas a Gaussian signal spends only about a third as much time beyond $3\sigma_x$.

The second important trend is that, for σ_x/Y small, there is a substantially smaller probability of the system response being at large amplitudes than for a Gaussian distribution. This effect can become quite pronounced, as indicated in Fig. 8(b) for $\alpha=1/21$ and $\sigma_x/Y=0.55$. In this case, the response of the bilinear system spends only about 0.3% of the time beyond $2\sigma_x$, whereas a Gaussian signal would spend about seven times as much time beyond that level. A similar situation exists for the system with $\alpha=1/2$. This marked tendency toward "amplitude limiting" in the bilinear system with no viscous damping is associated with the abrupt initiation of the hysteretic energy dissipation when the response exceeds Y. When viscous damping is introduced into the system, this effect becomes less significant as seen in the Fig. 8.

IV. COMPARISON WITH ANALYTIC SOLUTION

One common approximate analytical method for treating nonlinear problems is the Krylov–Bogoliubov method of equivalent linearization. In this method, two parameters, ω_{eq} and β_{eq}, are chosen so as to minimize the mean-squared difference between Eq. 1 and the linear equation

$$\ddot{x}+2\beta_{eq}\omega_{eq}\dot{x}+\omega_{eq}^2x=n(t)/m. \qquad (2)$$

Caughey has discussed the application of this method to problems of random vibration[2] and has used the technique for the particular problem of a bilinear hysteretic oscillator with "small" nonlinearity.[5] The assumptions made in obtaining ω_{eq} and β_{eq} are that (1) the response of the nonlinear system is assumed to be contained within a narrow frequency band; and (2) the probability density of the amplitude of this narrow-band response is assumed to be the Rayleigh distribution.

These assumptions lead to expressions for ω_{eq} and β_{eq} of the form

$$\left(\frac{\omega_{eq}}{\omega_0}\right)^2=1-\left[\frac{8(1-\alpha)}{\pi}\right]\int_1^\infty \times(z^{-3}+\lambda^{-1}z^{-1})(z-1)^{\frac{1}{2}}e^{-z^2/\lambda}dz \qquad (3)$$

and

$$\beta_{eq}=\beta_0(\omega_0/\omega_{eq}) +(\omega_0/\omega_{eq})^2(1-\alpha)(\pi\lambda)^{-\frac{1}{2}}\operatorname{erfc}(\lambda^{-\frac{1}{2}}), \qquad (4)$$

where

$$\lambda=2\sigma_x^2/Y^2. \qquad (5)$$

In general, Eq. 3 must be evaluated numerically. However, for the case of λ large, the asymptotic expansion below proves helpful:

$$(\omega_{eq}/\omega_0)^2=\alpha+[8(1-\alpha)/\pi](0.6043\lambda^{-\frac{3}{4}}-0.2451\lambda^{-5/4}-0.1295\lambda^{-7/4}), \quad \text{for } \lambda\gg1. \qquad (6)$$

After finding ω_{eq} and β_{eq}, one obtains rms levels of response of the "equivalent" linear system from

$$\sigma_{\dot{x}}^2=\omega_{eq}^2\sigma_x^2=\pi S_0/4m^2\beta_{eq}\omega_{eq}. \qquad (7)$$

The analog-computer investigations reported in Sec. III revealed that the power spectral density of the response of the nonlinear hysteretic system often is not contained within a narrow frequency band. This is particularly true for the nearly elastoplastic system. Further, determinations of probability distribution showed that the response was not, in general, Gaussian. Thus, it is very unlikely that the above technique for determining ω_{eq} and β_{eq} would result in minimizing the difference between Eqs. 1 and 2 for such systems. However, in some cases, the effects of the difference may be negligible even if ω_{eq} and β_{eq} are not chosen so as to minimize exactly the mean-squared value of the difference.

The response levels predicted by the Krylov–Bogaliubov method are indicated by dashed curves in Figs. 4–7. For the system with $\alpha=1/2$ and $\beta_0=0.01$, the predictions of σ_x and $\sigma_{\dot{x}}$ agree within about 10% with the analog-computer measurements for all values of yield level. The greatest discrepancy is when the excitation/yield ratio is in the range from 0.05 to 0.1. Figure 8(b) shows that the probability distribution of x was noticeably non-Gaussian in this range. For example, the probability $(x>2.5\sigma_x)$ for $(S_0\omega_0)^{\frac{1}{2}}/m\omega_0^2Y \simeq 0.08$ was about 0.18% as compared to 0.60% for a Gaussian signal. The fact that the predicted values of σ_x and $\sigma_{\dot{x}}$ did not err by more than 10% for this case illustrates the point made above that violation of Assumption (1) or (2) does not necessarily result in a large error in predicted level of response. For $\alpha=1/2$ and $\beta_0=0$, the predicted values of σ_x and $\sigma_{\dot{x}}$ agree with the analog-computer results within about 15%, when the excitation/yield ratio is in the range from 0.1 to 1.7. This range corresponds to σ_x/Y varying from about 20 down to 0.6. The noticeable error of the prediction for higher yield levels may be due to the severe amplitude limiting indicated in Fig. 8(a) for this system.

It appears that, for systems with $\alpha=1/2$, the above equivalent-linearization technique can be used to predict both displacement and velocity response within about 15%, except when β_0 is less than 0.01 and the excitation/yield ratio is outside the range of 0.1 to 1.7. For most physical systems, values of σ_x/Y as great as

[5] T. K. Caughey, "Random Excitation of a System with Bilinear Hysteresis," J. Appl. Mech. 27, 649–652 (1960).

20 could not normally be tolerated without failure. Thus, for such systems, the only practical limitation of the region of applicability of the equivalent-linearization technique for $\alpha=1/2$ is that of very small damping and low excitation.

It should perhaps be emphasized that the case of $\alpha=1/2$ actually represents a sizeable nonlinearity; that the approximate analysis gives such good results for this case indicates that the small nonlinearity assumption of the theory is not overly restrictive for this class of problem. For $\alpha>1/2$, one would anticipate obtaining even better results.

The predicted response is much less satisfactory for the nearly elastoplastic system with $\alpha=1/21$ than for $\alpha=1/2$. The only instances in which the theory accurately predicts the effect of the nonlinearity on σ_x, as shown in Fig. 5(a), is when σ_x/Y is greater than about 30. The predicted results also agree with the analog-computer results when σ_x/Y is less than 0.3 for the systems having viscous damping; in this case, the nonlinearity has no significant effect on the response.

When the excitation/yield ratio of the system with $\alpha=1/21$ is such that σ_x/Y is between 1 and 2, the small-nonlinearity theory predicts values of σ_x that are only about 50% of the values determined from the analog computer. Note that, for $\beta_0=0.05$, the theory fails to predict even the general character of the effect of yielding on σ_x. In particular, for $\sigma_x/Y=1$, the theory predicts a 38% decrease in σ_x as compared to a system with near-zero excitation, whereas the analog-computer studies revealed a 22% increase in σ_x due to yielding.

The equivalent-linearization technique is somewhat more accurate for predicting the velocity response than the displacement response of the nearly elastoplastic system. For $\beta_0\geq 0.01$, the predicted $\sigma_{\dot{x}}$ is at least 80% of that determined from the analog computer for all excitation/yield ratios; for the system with no viscous damping, the predicted $\sigma_{\dot{x}}$ is at least 75% of the proper value, except for very low excitation.

For nearly elastoplastic systems, it appears that the equivalent-linearization technique based on assumptions of small nonlinearity may be useful for obtaining a rough estimate of the effect of yielding on $\sigma_{\dot{x}}$. However, the technique yields useful information about the effect of yielding on σ_x only when σ_x/Y is greater than about 30.

V. SUMMARY AND CONCLUSIONS

The results of the analog investigation and the comparison with an approximate analytic solution may be summarized as follows.

- For bilinear hysteretic systems with low viscous damping, the ratios of the mean-squared levels of displacement and velocity response to the excitation level appear to have a definite minimum for ratios of σ_x/Y and $/\sigma\cdot\omega_0 Y$ between 1 and 2. More generally, yielding may either increase or decrease the rms displacement-response ratio but always acts to decrease the rms velocity-response ratio.

- The primary effect of yielding on the response-power spectral density is to cause a shift in peak frequency with changing excitation level. In some cases, this shift is accompanied by a significant broadening of the response peak or even elimination of the peak.

- The probability distribution of the bilinear hysteretic system response is strongly influenced by the level of excitation and is, in general, noticeably non-Gaussian. For low-level excitation and no viscous damping, the system exhibits a type of amplitude-limiting behavior.

- The Krylov–Bogoliubov approximation technique appears to give quite acceptable results for the rms response for systems with small to moderate nonlinearity ($\alpha\geq 1/2$) and small finite viscous damping. However, care must be exercised in attempting to apply the technique to more nearly elastoplastic systems.

ACKNOWLEDGMENT

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Appendix A. Details of Analog Investigation

The basic component of the electronic analog used in this investigation was a K7-A10 manifold of Philbrick model USA-3 universal stabilized amplifiers. A transistor switching circuit was used to produce the elastoplastic component of the restoring force; Fig. A-1 illustrates this circuit. Amplifier No. 4 is the integrator, and v_0 is the yield voltage. Amplifiers Nos. 1,–3 and transistors Q_1 and Q_3 compare $\varphi_{\mathrm{p}}(x)/R_4C_2$with v_0 and switch the base voltages of transistors Q_2 and Q_4 to produce yielding. When $|\varphi_{\mathrm{p}}(x)/R_4C_2|<v_0$, the bases of Q_2 and Q_4 are maintained at $+v_Z$ and $-v_Z$, respectively, where v_Z is the Zener breakdown voltage of the 1N476 diodes and $v_0<v_Z$. Thus, both Q_2 and Q_4 are "turned off" and amplifier No. 4 is a conventional integrator in this instance. When the output voltage of No. 4 reaches the level v_0, the base voltage of Q_2 is switched to the level v_0. This allows current to flow from emitter to collector of Q_2, and thus prevents further increase in $\varphi_{\mathrm{p}}(x)/R_4C_2$. The current through Q_2 stops, and integration on No. 4 proceeds when $\dot{x}$ becomes negative. Transistor Q_4 is similarly switched to bound $\varphi_{\mathrm{p}}(x)/R_4C_2$ at the level $-v_0$.

A General Radio random-noise generator, type 1390-B, was used to furnish the random exciting signal. The output from this equipment is an approximation to a white, Gaussian signal. In the mode of operation

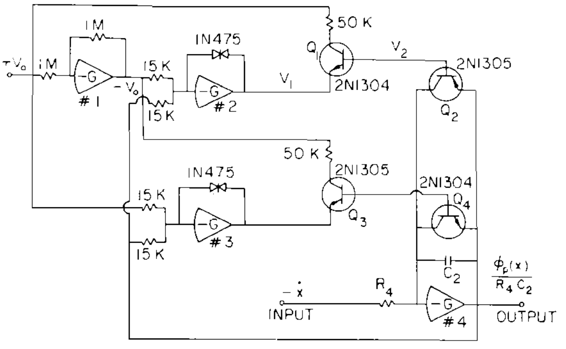

FIG. A-1. Elastoplastic function-generator circuit.

that was used, the power spectral density is essentially independent of frequency from about 20 000 cps down to near 100 cps. Below 100 cps, the power spectral density decreases gradually until at 10 cps, it is down 1–2 dB. The portion of the random signal contained within the frequency band from 0 to 1000 cps is essentially Gaussian. The higher-frequency components have a somewhat unsymmetric probability distribution, but this can be neglected since the system considered in this study did not respond significantly at frequencies above 1000 cps [see Figs. 6(a) and 6(b)].

In this study, rms levels were measured by using a Brüel and Kjær random-noise voltmeter, model 2417, which has an averaging time that is adjustable from 0.3 to 100 sec. Because of the well-known beat effect resulting from summing signals of nearly the same frequency, the amplitude of a narrow-band random signal tends to vary with a much lower frequency than the center frequency of the process. This low-frequency variation necessitates using relatively long averaging times in order to measure the true rms level. It can be shown that, for a narrow-band random signal with mean zero and a bandwidth of $2b$ rad/sec between half-power points, the normalized standard error in measuring mean-squared level is approximately $(bT)^{-\frac{1}{2}}$ when an averaging time of T sec is used. On the basis, the voltmeter used with $T=100$ was capable of determining the rms response levels reported in Figs. 4 and 5 within an accuracy of about 2%, except that, for the highest point of Curve A in both Figs. 4(a) and 5(a) $[(S_0\omega_0)^{\frac{1}{2}}/m\omega_0^2Y\approx 0.017]$, the accuracy is limited to about 3.5%.

Measurement of spectral density was accomplished by using a Radiometer wave analyzer, model FRA2. Experimental determination of the nominal 2-cps band of the wave analyzer gave an effective bandwidth of 4.53 cps. This value is sufficiently small to allow an accuracy of about 0.5% for all the experimental points presented in Figs. 6(a) and (b). The voltmeter described above was used to measure the rms level of the signal passing through the wave analyzer. The expected error in this measurement is about 1.5%.

Probability distributions were obtained by using a Quan-Tech Laboratories amplitude-distribution analyzer, model 317. An analysis of the probable error introduced by using a finite sampling time in determining the probability distribution of a stationary signal shows that the normalized error can be expected to be approximately inversely proportional to the square root of the product of the sampling time multiplied by the probability being determined. In this study, it was necessary to use an external filter and voltmeter to measure the mean output of the Quan-Tech Schmitt trigger. An RC filter having a time constant of 25 sec was used, and this allowed determination of a probability as small as 0.002 within an accuracy of about 10%.

A more detailed discussion of the entire analog system is contained in Ref. 6.

[6] L. D. Lutes, "Stationary Random Response of Bilinear Hysteretic Systems," PhD thesis, Calif. Inst. Technol., Pasadena, Calif., 1967.