



# Universal eigenstate entanglement of chaotic local Hamiltonians

Yichen Huang (黄溢辰)

*Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, CA 91125, USA*

Received 22 June 2018; received in revised form 17 September 2018; accepted 18 September 2018

Available online 9 October 2018

Editor: Hubert Saleur

---

## Abstract

In systems governed by “chaotic” local Hamiltonians, we conjecture the universality of eigenstate entanglement (defined as the average entanglement entropy of all eigenstates) by proposing an exact formula for its dependence on the subsystem size. This formula is derived from an analytical argument based on a plausible assumption, and is supported by numerical simulations.

© 2018 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP<sup>3</sup>.

---

## 1. Introduction

Entanglement, a concept of quantum information theory, has been widely used in condensed matter and statistical physics to provide insights beyond those obtained via “conventional” quantities. For ground states of local Hamiltonians, it characterizes quantum criticality [24,52,36,9,10] and topological order [22,35,37,38,11,28]. The scaling of entanglement reflects physical properties (e.g., correlation decay [7,8,21] and dynamical localization [56,5,26]) and is quantitatively related to the classical simulability of quantum many-body systems [51,47,42,20,25].

Besides ground states, it is also important to understand the entanglement of excited eigenstates. Significant progress has been made for a variety of local Hamiltonians [14,46,15,32,6,30,33,55,12,53,16,19,29]. In many-body localized systems [41,1,50,31], one expects an “area law” [17], i.e., the eigenstate entanglement between a subsystem and its complement scales as the

---

*E-mail address:* [yichen.huang@microsoft.com](mailto:yichen.huang@microsoft.com).

boundary (area) rather than the volume of the subsystem [6,32,30]. In translationally invariant free-fermion systems, the average entanglement entropy of all eigenstates obeys a volume law with non-maximal coefficients due to the integrability of the model [53].

In this paper we consider “chaotic” quantum many-body systems. We are not able to specify the precise meaning of being chaotic, for there is no clear-cut definition of quantum chaos. Intuitively, this class of systems should include non-integrable models in which energy is the only local conserved quantity. For such systems, there are some widely accepted opinions [14,46,15,12,19]:

1. The entanglement entropy of an eigenstate for a subsystem smaller than half the system size is (to leading order) equal to the thermodynamic entropy of the subsystem at the same energy density.
2. The entanglement entropy of an eigenstate at the mean energy density (of the Hamiltonian) is indistinguishable from that of a random (pure) state.
3. The entanglement entropy of a generic eigenstate is indistinguishable from that of a random state.

We briefly explain the reasoning behind these opinions. The eigenstate thermalization hypothesis (ETH) states that for expectation values of local observables, a single eigenstate resembles a thermal state with the same energy density [13,49,44]. Opinion 1 is a variant of ETH for entropy. Opinion 2 follows from Opinion 1 and the fact that the entanglement entropy of a random state is nearly maximal [43]. Opinion 3 follows from Opinion 2 because a generic eigenstate is at the mean energy density (Lemma 3 in Ref. [27]).

These opinions concern the scaling of the entanglement entropy only to leading order. A more ambitious goal is to find the exact value of eigenstate entanglement. We conjecture that the average entanglement entropy of all eigenstates is universal (model-independent), and propose a formula for its dependence on the subsystem size. This formula is derived from an analytical argument based on an assumption that characterizes the chaoticity of the model. It is also supported by numerical simulations of a non-integrable spin chain.

The formula implies that by taking into account sub-leading corrections not captured in Opinion 3, a generic eigenstate is distinguishable from a random state in the sense of being less entangled. Indeed, this implication can be proved rigorously for any (not necessarily chaotic) local Hamiltonian. The proof also solves an open problem of Keating et al. [33].

The paper is organized as follows. Section 2 gives a brief review of random-state entanglement. Section 3 proves that for any (not necessarily chaotic) local Hamiltonian, the average entanglement entropy of all eigenstates is smaller than that of random states. Sections 4 and 5 provide an analytical argument and numerical evidence, respectively, for the universality of eigenstate entanglement in chaotic systems. The main text of this paper should be easy to read, for most of the technical details are deferred to Appendices A and B.

## 2. Entanglement of random states

We begin with a brief review of random-state entanglement. We use the natural logarithm throughout this paper.

**Definition 1** (*entanglement entropy*). The entanglement entropy of a bipartite pure state  $\rho_{AB} = |\psi\rangle\langle\psi|$  is defined as the von Neumann entropy

$$S(\rho_A) = -\text{tr}(\rho_A \ln \rho_A) \tag{1}$$

of the reduced density matrix  $\rho_A = \text{tr}_B \rho_{AB}$ . It is the Shannon entropy of  $\rho_A$ 's eigenvalues, which form a probability distribution as  $\rho_A \geq 0$  (positive semidefinite) and  $\text{tr} \rho_A = 1$  (normalization).

**Theorem 1** (conjectured and partially proved by Page [43]; proved in Refs. [18,45,48]). *Let  $\rho_{AB}$  be a bipartite pure state chosen uniformly at random with respect to the Haar measure. In average,*

$$S(\rho_A) = \sum_{k=d_B+1}^{d_A d_B} \frac{1}{k} - \frac{d_A - 1}{2d_B} = \ln d_A - \frac{d_A}{2d_B} + \frac{O(1)}{d_A d_B}, \tag{2}$$

where  $d_A \leq d_B$  are the local dimensions of subsystems A and B, respectively.

Let  $\gamma \approx 0.5772$  be the Euler–Mascheroni constant. The second step of Eq. (2) uses the formula

$$\sum_{k=1}^{d_B} \frac{1}{k} = \ln d_B + \gamma + \frac{1}{2d_B} + O(1/d_B^2). \tag{3}$$

The concentration bound proved in Ref. [23] using Levy’s lemma [40] shows that the deviation of  $S(\rho_A)$  (from the mean) for a typical state  $\rho_{AB}$  is exponentially small.

### 3. Rigorous bounds on eigenstate entanglement

This section proves a rigorous upper bound on the average entanglement entropy of all eigenstates. The bound holds for any (not necessarily chaotic) local Hamiltonian, and distinguishes the entanglement entropy of a generic eigenstate from that of a random state.

For ease of presentation, consider a chain of  $n$  spin-1/2’s governed by a local Hamiltonian

$$H = \sum_{i=1}^n H_i, \quad H_i := H'_i + H'_{i,i+1}, \tag{4}$$

where  $H'_i$  acts only on spin  $i$ , and  $H'_{i,i+1}$  represents the nearest-neighbor interaction between spins  $i$  and  $i + 1$ . We use periodic boundary conditions by identifying the indices  $i$  and  $(i \bmod n)$ . Suppose  $H'_i$  and  $H'_{i,i+1}$  are linear combinations of one- and two-local Pauli operators, respectively, so that  $\text{tr} H'_i = \text{tr} H'_{i,i+1} = 0$  (traceless) and  $\text{tr}(H_i H_{i'}) = 0$  for  $i \neq i'$ . We assume translational invariance and  $\|H_i\| = 1$  (unit operator norm). Let  $d := 2^n$  and  $\{|j\rangle\}_{j=1}^d$  be a complete set of translationally invariant eigenstates of  $H$  with corresponding eigenvalues  $\{E_j\}$ .

**Lemma 1.** *Consider the spin chain as a bipartite quantum system  $A \otimes B$ . Subsystem A consists of spins 1, 2, . . . , m. Assume without loss of generality that m is even and  $f := m/n \leq 1/2$ . Then,*

$$S(\rho_{j,A}) \leq m \ln 2 - f E_j^2 / (4n), \tag{5}$$

where  $\rho_{j,A} := \text{tr}_B |j\rangle\langle j|$  is the reduced density matrix of  $|j\rangle$  for A.

**Proof.** See Appendix A.  $\square$

We are now ready to prove the main result of this section:

**Theorem 2.** *In the setting of Lemma 1,*

$$\bar{S} := \frac{1}{d} \sum_{j=1}^d S(\rho_{j,A}) \leq m \ln 2 - f \langle H_1^2 \rangle / 4, \tag{6}$$

where  $\langle \dots \rangle := d^{-1} \text{tr} \dots$  denotes the expectation value of an operator at infinite temperature.

**Proof.** Theorem 2 follows from Lemma 1 and the observation that

$$\frac{1}{d} \sum_{j=1}^d E_j^2 = \langle H^2 \rangle = \sum_{i,i'=1}^n \langle H_i H_{i'} \rangle = \sum_{i=1}^n \langle H_i^2 \rangle = n \langle H_1^2 \rangle. \quad \square \tag{7}$$

Recall that Theorem 2 assumes  $\|H_i\| = 1$ . Without this assumption, (6) should be modified to

$$\bar{S} \leq m \ln 2 - f \langle H_1^2 \rangle / (4 \|H_1\|^2). \tag{8}$$

For  $2 \leq m = O(1)$ , Theorem 2 gives the upper bound

$$\bar{S} \leq m \ln 2 - \Theta(1/n). \tag{9}$$

A lower bound can be easily derived from Theorem 1 in Ref. [33]

$$\bar{S} \geq m \ln 2 - \Theta(1/n). \tag{10}$$

Therefore, both bounds are tight. This answers an open question in Section 6.1 of Ref. [33].

Without translational invariance (e.g., in weakly disordered systems), a similar result is obtained by averaging over all possible ways of “cutting out” a region of length  $m$ . Here,  $\|H_i\|$  may be site dependent but should be  $\Theta(1)$  for all  $i$ .

**Corollary 1.** *The average entanglement entropy  $\bar{S}$  of a random eigenstate for a random consecutive region of size  $m$  is upper bounded by  $m \ln 2 - \Theta(f)$ .*

**Proof.** First, we follow the proof of Lemma 1. Without translational invariance, (21) remains valid upon replacing  $\epsilon_{j,i}$  by  $\epsilon_{j,i}/\|H_i\| = \Theta(\epsilon_{j,i})$ . By the RMS-AM inequality and Eq. (7), we have

$$\begin{aligned} \bar{S} &\leq m \ln 2 - \frac{\Theta(f)}{4d} \sum_{j=1}^d \sum_{i=1}^n \epsilon_{j,i}^2 \leq m \ln 2 - \frac{\Theta(f)}{4dn} \sum_{j=1}^d E_j^2 = m \ln 2 - \frac{\Theta(f)}{4n} \sum_{i=1}^n \langle H_i^2 \rangle \\ &= m \ln 2 - \Theta(f). \quad \square \end{aligned} \tag{11}$$

It is straightforward to extend all the results of this section to higher spatial dimensions.

#### 4. Eigenstate entanglement of “chaotic” Hamiltonians

Suppose the Hamiltonian (4) is chaotic in a sense to be made precise below. This section provides an analytical argument for

**Conjecture 1** (*Universal eigenstate entanglement*). Consider the spin chain as a bipartite quantum system  $A \otimes B$ . Subsystem  $A$  consists of spins  $1, 2, \dots, m$ . For a fixed constant  $f := m/n \leq 1/2$ , the average entanglement entropy of all eigenstates is

$$\bar{S} = m \ln 2 + (\ln(1 - f))/2 - 2\delta_{f,1/2}/\pi \tag{12}$$

in the thermodynamic limit  $n \rightarrow +\infty$ , where  $\delta$  is the Kronecker delta.

We split the Hamiltonian (4) into three parts:  $H = H_A + H_\partial + H_B$ , where  $H_{A(B)}$  contains terms acting only on subsystem  $A(B)$ , and  $H_\partial = H'_{m,m+1} + H'_{n,1}$  consists of boundary terms. Let  $\{|j\rangle_A\}_{j=1}^{2^m}$  and  $\{|k\rangle_B\}_{k=1}^{2^{n-m}}$  be complete sets of eigenstates of  $H_A$  and  $H_B$  with corresponding eigenvalues  $\{\epsilon_j\}$  and  $\{\epsilon_k\}$ , respectively. Since  $H_A$  and  $H_B$  are decoupled from each other, product states  $\{|j\rangle_A|k\rangle_B\}$  form a complete set of eigenstates of  $H_A + H_B$  with eigenvalues  $\{\epsilon_j + \epsilon_k\}$ . Due to the presence of  $H_\partial$ , a (normalized) eigenstate  $|\psi\rangle$  of  $H$  with eigenvalue  $E$  is a superposition

$$|\psi\rangle = \sum_{j=1}^{2^m} \sum_{k=1}^{2^{n-m}} c_{jk} |j\rangle_A |k\rangle_B. \tag{13}$$

The locality of  $H_\partial$  implies a strong constraint stating that the population of  $|j\rangle_A|k\rangle_B$  is significant only when  $\epsilon_j + \epsilon_k$  is close to  $E$ .

**Lemma 2.** *There exist constants  $c, \Delta > 0$  such that*

$$\sum_{|\epsilon_j + \epsilon_k - E| \geq \Delta} |c_{jk}|^2 \leq ce^{-\Delta/\Delta}. \tag{14}$$

**Proof.** This is a direct consequence of Theorem 2.3 in Ref. [2].  $\square$

In chaotic systems, we expect

**Assumption 1.** The expansion (13) of a generic eigenstate  $|\psi\rangle$  is a random superposition subject to the constraint (14).

This assumption is consistent with, but goes beyond, the semiclassical approximation Eq. (16) of Ref. [16].

We now show that Assumption 1 implies Conjecture 1. Consider the following simplified setting. Let  $M_j$  be the set of computational basis states with  $j$  spins up and  $n - j$  spins down, and  $U_j \in \mathcal{U}(|M_j|) = \mathcal{U}(\binom{n}{j})$  be a Haar-random unitary on  $\text{span } M_j$ . Define  $M'_j = \{U_j|\phi\rangle : \forall |\phi\rangle \in M_j\}$  so that  $M := \bigcup_{j=0}^n M'_j$  is a complete set of eigenstates of the Hamiltonian

$$H = \sum_{i=1}^n \sigma_i^z. \tag{15}$$

The set  $M$  captures the essentials of Assumption 1. Every state in  $M$  satisfies

$$\sum_{|\epsilon_j + \epsilon_k - E| \geq 1} |c_{jk}|^2 = 0, \tag{16}$$

which is a hard version of the constraint (14). The random unitary  $U_j$  ensures that Eq. (13) is a random superposition. Thus, we establish Conjecture 1 by

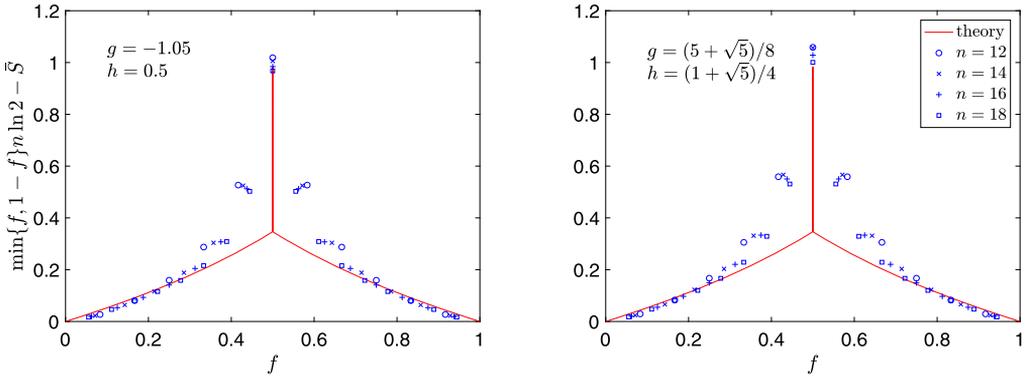


Fig. 1. Numerical check of Conjecture 1 for two sets of parameters  $(g, h)$ . The horizontal axes are the fraction  $f$  of spins in subsystem A. To be aesthetically pleasing, we allow  $0 < f < 1$  so that the plots are mirror symmetric with respect to  $f = 1/2$ . Blue symbols represent corrections obtained by subtracting the average entanglement entropy  $\bar{S}$  of all eigenstates from the leading-order term  $\min\{f, 1 - f\}n \ln 2$ . Different symbols correspond to different system sizes. Red curves are the theoretical prediction given by Eq. (12). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

**Proposition 1.** *The average entanglement entropy of all states in  $M$  is given by Eq. (12).*

### 5. Numerics

To provide numerical evidence for Conjecture 1, consider the spin-1/2 Hamiltonian [4,34]

$$H = \sum_{i=1}^n H_i, \quad H_i := \sigma_i^z \sigma_{i+1}^z + g \sigma_i^x + h \sigma_i^z, \tag{17}$$

where  $\sigma_i^x, \sigma_i^y, \sigma_i^z$  are the Pauli matrices at site  $i (\leq n)$ , and  $\sigma_{n+1}^z := \sigma_1^z$  (periodic boundary condition). For generic values of  $g, h$ , this model is non-integrable in the sense of Wigner–Dyson level statistics [4,34]. We compute the average entanglement entropy of all eigenstates by exact diagonalization in every symmetry sector.

Fig. 1 shows the numerical results, which semiquantitatively support Conjecture 1. Noticeable deviations from Eq. (12) are expected due to significant finite-size effects. However, the trend appears to be that the difference between theory and numerics decreases as the system size increases.

### Acknowledgments and notes

We would like to thank Fernando G.S.L. Brandão and Xiao-Liang Qi for interesting discussions and Anatoly Dymarsky for insightful comments. We acknowledge funding provided by the Institute for Quantum Information and Matter, an NSF Physics Frontiers Center (NSF Grant PHY-1733907). Additional funding support was provided by NSF DMR-1654340.

After this paper appeared on arXiv, we became aware of a simultaneous work [54] and a slightly later one [39].

**Appendix A. Proof of Lemma 1**

Let  $\epsilon_{j,i} := \langle j|H_i|j \rangle$  so that  $|\epsilon_{j,i}| \leq 1$ . Let  $\rho_{j,i}$  be the reduced density matrix of  $|j \rangle$  for spins  $i$  and  $i + 1$ . Let  $I_4$  be the identity matrix of order 4. Let  $\|X\|_1 := \text{tr} \sqrt{X^\dagger X}$  be the trace norm. Since  $H_i$  is traceless,  $|\epsilon_{j,i}|$  provides a lower bound on the deviation of  $\rho_{j,i}$  from the maximally mixed state:

$$|\epsilon_{j,i}| = |\text{tr}(\rho_{j,i} H_i)| = |\text{tr}((\rho_{j,i} - I_4/4) H_i)| \leq \|\rho_{j,i} - I_4/4\|_1 \|H_i\| = \|\rho_{j,i} - I_4/4\|_1 = \sum_{k=1}^4 |\lambda_k - 1/4|, \tag{18}$$

where  $\lambda_k$ 's are the eigenvalues of  $\rho_{j,i}$ . An upper bound on  $S(\rho_{j,i})$  is

$$\max \left\{ -\sum_{k=1}^4 p_k \ln p_k \right\}; \quad \text{s. t.} \quad \sum_{k=1}^4 p_k = 1, \quad \sum_{k=1}^4 |p_k - 1/4| \geq \epsilon_{j,i}. \tag{19}$$

Since the Shannon entropy is concave, it suffices to consider

- $p_1 = p_2 = 1/4 + \epsilon_{j,i}/4, p_3 = p_4 = 1/4 - \epsilon_{j,i}/4;$
- (if  $\epsilon_{j,i} \geq -1/2$ )  $p_1 = 1/4 + \epsilon_{j,i}/2, p_2 = p_3 = p_4 = 1/4 - \epsilon_{j,i}/6;$
- (if  $\epsilon_{j,i} \leq 1/2$ )  $p_1 = 1/4 - \epsilon_{j,i}/2, p_2 = p_3 = p_4 = 1/4 + \epsilon_{j,i}/6.$

For  $|\epsilon_{j,i}| \ll 1$ , by Taylor expansion we can prove

$$S(\rho_{j,i}) \leq 2 \ln 2 - \epsilon_{j,i}^2/2. \tag{20}$$

We have checked numerically that this inequality remains valid for any  $|\epsilon_{j,i}| \leq 1$ . Therefore,

$$S(\rho_{j,A}) \leq \sum_{k=0}^{m/2-1} S(\rho_{j,2k+1}) \leq m \ln 2 - \frac{1}{2} \sum_{k=0}^{m/2-1} \epsilon_{j,2k+1}^2 \tag{21}$$

due to the subadditivity [3] of the von Neumann entropy. We complete the proof using  $\epsilon_{j,i} = E_j/n$ .

**Appendix B. Proof of Proposition 1**

Assume without loss of generality that  $n$  is even. Let  $L_j$  ( $R_j$ ) be the set of computational basis states of subsystem  $A$  ( $B$ ) with  $j$  spins up and  $m - j$  ( $n - m - j$ ) spins down so that

$$|L_j| = \binom{m}{j}, \quad |R_j| = \binom{n-m}{j}, \quad \text{and} \quad M_j = \bigcup_{k=\max\{0, m-n+j\}}^{\min\{m, j\}} L_k \times R_{j-k}. \tag{22}$$

Thus, any (normalized) state  $|\psi\rangle$  in  $M'_j$  can be decomposed as

$$|\psi\rangle = \sum_{k=\max\{0, m-n+j\}}^{\min\{m, j\}} c_k |\phi_k\rangle, \tag{23}$$

where  $|\phi_k\rangle$  is a normalized state in  $\text{span } L_k \otimes \text{span } R_{j-k}$ . Let  $\rho_A$  and  $\sigma_{k,A}$  be the reduced density matrices of  $|\psi\rangle$  and  $|\phi_k\rangle$  for  $A$ , respectively. It is easy to see

$$\rho_A = \bigoplus_{k=\max\{0, m-n+j\}}^{\min\{m, j\}} |c_k|^2 \sigma_{k,A} \implies S(\rho_A) = \sum_{k=\max\{0, m-n+j\}}^{\min\{m, j\}} |c_k|^2 S(\sigma_{k,A}) - |c_k|^2 \ln |c_k|^2. \tag{24}$$

Since  $|\psi\rangle$  is a random state in  $\text{span } M_j$ , each  $|\phi_k\rangle$  is a (Haar-)random state in  $\text{span } L_k \otimes \text{span } R_{j-k}$ . Theorem 1 implies that in average,

$$S(\sigma_{k,A}) = \ln \min\{|L_k|, |R_{j-k}|\} - \min\{|L_k|, |R_{j-k}|\} / (2 \max\{|L_k|, |R_{j-k}|\}). \tag{25}$$

In average, the population  $|c_k|^2$  is proportional to the dimension of  $\text{span } L_k \otimes \text{span } R_{j-k}$ :

$$|c_k|^2 = |L_k| |R_{j-k}| / |M_j|. \tag{26}$$

The deviation of  $|c_k|^2$  (from the mean) for a typical state  $|\psi\rangle \in \text{span } M_j$  is exponentially small. In the thermodynamic limit,  $j, k$  can be promoted to continuous real variables so that  $|M_j|, |L_k|$  follow normal distributions with means  $n/2, fn/2$  and variances  $n/4, fn/4$ , respectively. Let

$$J := j/\sqrt{n} - \sqrt{n}/2, \quad K := k/\sqrt{n} - f\sqrt{n}/2. \tag{27}$$

We have

$$|L_k| = \sqrt{2}d^f e^{-2K^2/f} / \sqrt{f\pi n}, \quad |R_{j-k}| = \sqrt{2}d^{1-f} e^{-2(J-K)^2/(1-f)} / \sqrt{(1-f)\pi n}, \tag{28}$$

$$|M_j| = \sqrt{2}d e^{-2J^2} / \sqrt{\pi n}, \quad |c_k|^2 = \sqrt{2}e^{2J^2-2K^2/f-2(J-K)^2/(1-f)} / \sqrt{f(1-f)\pi n}. \tag{29}$$

For any fixed constant  $f < 1/2$ , it is almost always the case that  $|L_k| \ll |R_{j-k}|$ . Hence,

$$S(\sigma_{k,A}) = (m + 1/2) \ln 2 - (\ln(f\pi n))/2 - 2K^2/f. \tag{30}$$

Substituting Eqs. (29) and (30) into Eq. (24),

$$\begin{aligned} S_j := S(\rho_A) &= \int_{-\infty}^{+\infty} |c_k|^2 (S(\sigma_{k,A}) - \ln |c_k|^2) dk \\ &= (m + 1/2) \ln 2 - (\ln(f\pi n))/2 + (f - 4fJ^2 - 1)/2 \\ &\quad + (1 + \ln(f - f^2) + \ln(\pi n/2))/2 = m \ln 2 + f(1 - 4J^2)/2 + (\ln(1 - f))/2. \end{aligned} \tag{31}$$

Averaging over all states in  $M$ ,

$$\begin{aligned} \bar{S} &= \frac{1}{d} \sum_{j=0}^n |M_j| S_j \approx m \ln 2 + (\ln(1 - f))/2 + \frac{f}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2J^2} (1 - 4J^2) dJ \\ &= m \ln 2 + (\ln(1 - f))/2. \end{aligned} \tag{32}$$

For  $f = 1/2$ , assume without of generality that  $j \leq n/2$  and  $k \leq j/2$  (i.e.,  $J \leq 0$  and  $K \leq J/2$ ) so that  $|L_k| \leq |R_{j-k}|$ . Hence,

$$S(\sigma_{k,A}) = (n/2 + 1) \ln 2 - (\ln(\pi n))/2 - 4K^2 - e^{4J^2-8JK} / 2. \tag{33}$$

Let

$$\operatorname{erfc} x := \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt \quad (34)$$

be the complementary error function. Substituting Eqs. (29) and (33) into Eq. (24),

$$\begin{aligned} S_j &:= S(\rho_A) = 2 \int_{-\infty}^{j/2} |c_k|^2 S(\sigma_{k,A}) dk - \int_{-\infty}^{+\infty} |c_k|^2 \ln |c_k|^2 dk \\ &= \left(\frac{n}{2} + 1\right) \ln 2 - \frac{\ln(\pi n)}{2} - \frac{1}{4} + J \sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2J^2} \operatorname{erfc}(-\sqrt{2}J)}{2} + \frac{1 + \ln(\pi n/8)}{2} \\ &= \frac{n-1}{2} \ln 2 + \frac{1}{4} + J \sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2J^2} \operatorname{erfc}(-\sqrt{2}J)}{2}. \end{aligned} \quad (35)$$

This is the average entanglement entropy of a random state in span  $M_j$  for  $j \leq n/2$ . For  $j > n/2$ , Eq. (35) remains valid upon replacing  $J$  by  $-J$ . Averaging over all states in  $M$ ,

$$\begin{aligned} \bar{S} &= \frac{1}{d} \sum_{j=0}^n |M_j| S_j \\ &\approx \frac{n-1}{2} \ln 2 + \sqrt{\frac{8}{\pi}} \int_{-\infty}^0 e^{-2J^2} \left( \frac{1}{4} + J \sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2J^2} \operatorname{erfc}(-\sqrt{2}J)}{2} \right) dJ \\ &= (n-1)(\ln 2)/2 - 2/\pi. \end{aligned} \quad (36)$$

Equation (12) follows from Eqs. (32) and (36).

## References

- [1] E. Altman, R. Vosk, Universal dynamics and renormalization in many-body-localized systems, *Annu. Rev. Condens. Matter Phys.* 6 (1) (2015) 383–409.
- [2] I. Arad, T. Kuwahara, Z. Landau, Connecting global and local energy distributions in quantum spin models on a lattice, *J. Stat. Mech. Theory Exp.* 2016 (3) (2016) 033301.
- [3] H. Araki, E.H. Lieb, Entropy inequalities, *Commun. Math. Phys.* 18 (2) (1970) 160–170.
- [4] M.C. Bañuls, J.I. Cirac, M.B. Hastings, Strong and weak thermalization of infinite nonintegrable quantum systems, *Phys. Rev. Lett.* 106 (5) (2011) 050405.
- [5] J.H. Bardarson, F. Pollmann, J.E. Moore, Unbounded growth of entanglement in models of many-body localization, *Phys. Rev. Lett.* 109 (1) (2012) 017202.
- [6] B. Bauer, C. Nayak, Area laws in a many-body localized state and its implications for topological order, *J. Stat. Mech. Theory Exp.* 2013 (09) (2013) P09005.
- [7] F.G.S.L. Brandao, M. Horodecki, An area law for entanglement from exponential decay of correlations, *Nat. Phys.* 9 (11) (2013) 721–726.
- [8] F.G.S.L. Brandao, M. Horodecki, Exponential decay of correlations implies area law, *Commun. Math. Phys.* 333 (2) (2015) 761–798.
- [9] P. Calabrese, J. Cardy, Entanglement entropy and quantum field theory, *J. Stat. Mech. Theory Exp.* 2004 (06) (2004) P06002.
- [10] P. Calabrese, J. Cardy, Entanglement entropy and conformal field theory, *J. Phys. A, Math. Theor.* 42 (50) (2009) 504005.
- [11] X. Chen, Z.-C. Gu, X.-G. Wen, Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order, *Phys. Rev. B* 82 (15) (2010) 155138.
- [12] L. D'Alessio, Y. Kafri, A. Polkovnikov, M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, *Adv. Phys.* 65 (3) (2016) 239–362.

- [13] J.M. Deutsch, Quantum statistical mechanics in a closed system, *Phys. Rev. A* 43 (4) (1991) 2046–2049.
- [14] J.M. Deutsch, Thermodynamic entropy of a many-body energy eigenstate, *New J. Phys.* 12 (7) (2010) 075021.
- [15] J.M. Deutsch, H. Li, A. Sharma, Microscopic origin of thermodynamic entropy in isolated systems, *Phys. Rev. E* 87 (4) (2013) 042135.
- [16] A. Dymarsky, N. Lashkari, H. Liu, Subsystem eigenstate thermalization hypothesis, *Phys. Rev. E* 97 (1) (2018) 012140.
- [17] J. Eisert, M. Cramer, M.B. Plenio, Colloquium: area laws for the entanglement entropy, *Rev. Mod. Phys.* 82 (1) (2010) 277–306.
- [18] S.K. Foong, S. Kanno, Proof of Page’s conjecture on the average entropy of a subsystem, *Phys. Rev. Lett.* 72 (8) (1994) 1148–1151.
- [19] J.R. Garrison, T. Grover, Does a single eigenstate encode the full Hamiltonian?, *Phys. Rev. X* 8 (2) (2018) 021026.
- [20] S. Gharibian, Y. Huang, Z. Landau, S.W. Shin, Quantum Hamiltonian complexity, *Found. Trends Theor. Comput. Sci.* 10 (3) (2015) 159–282.
- [21] D. Gosset, Y. Huang, Correlation length versus gap in frustration-free systems, *Phys. Rev. Lett.* 116 (9) (2016) 097202.
- [22] A. Hamma, R. Ionicioiu, P. Zanardi, Ground state entanglement and geometric entropy in the Kitaev model, *Phys. Lett. A* 337 (1–2) (2005) 22–28.
- [23] P. Hayden, D.W. Leung, A. Winter, Aspects of generic entanglement, *Commun. Math. Phys.* 265 (1) (2006) 95–117.
- [24] C. Holzhey, F. Larsen, F. Wilczek, Geometric and renormalized entropy in conformal field theory, *Nucl. Phys. B* 424 (3) (1994) 443–467.
- [25] Y. Huang, Classical Simulation of Quantum Many-Body Systems, PhD thesis, University of California, Berkeley, 2015.
- [26] Y. Huang, Entanglement dynamics in critical random quantum Ising chain with perturbations, *Ann. Phys.* 380 (2017) 224–227.
- [27] Y. Huang, F.G.S.L. Brandao, Y.-L. Zhang, Finite-size scaling of out-of-time-ordered correlators at late times, arXiv:1705.07597, 2017.
- [28] Y. Huang, X. Chen, Quantum circuit complexity of one-dimensional topological phases, *Phys. Rev. B* 91 (19) (2015) 195143.
- [29] Y. Huang, Y. Gu, Eigenstate entanglement in the Sachdev–Ye–Kitaev model, arXiv:1709.09160, 2017.
- [30] Y. Huang, J.E. Moore, Excited-state entanglement and thermal mutual information in random spin chains, *Phys. Rev. B* 90 (22) (2014) 220202.
- [31] Y. Huang, Y.-L. Zhang, X. Chen, Out-of-time-ordered correlators in many-body localized systems, *Ann. Phys.* 529 (7) (2017) 1600318.
- [32] D.A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, S.L. Sondhi, Localization-protected quantum order, *Phys. Rev. B* 88 (1) (2013) 014206.
- [33] J.P. Keating, N. Linden, H.J. Wells, Spectra and eigenstates of spin chain Hamiltonians, *Commun. Math. Phys.* 338 (1) (2015) 81–102.
- [34] H. Kim, D.A. Huse, Ballistic spreading of entanglement in a diffusive nonintegrable system, *Phys. Rev. Lett.* 111 (12) (2013) 127205.
- [35] A. Kitaev, J. Preskill, Topological entanglement entropy, *Phys. Rev. Lett.* 96 (11) (2006) 110404.
- [36] J.I. Latorre, E. Rico, G. Vidal, Ground state entanglement in quantum spin chains, *Quantum Inf. Comput.* 4 (1) (2004) 48–92.
- [37] M. Levin, X.-G. Wen, Detecting topological order in a ground state wave function, *Phys. Rev. Lett.* 96 (11) (2006) 110405.
- [38] H. Li, F.D.M. Haldane, Entanglement spectrum as a generalization of entanglement entropy: identification of topological order in non-abelian fractional quantum Hall effect states, *Phys. Rev. Lett.* 101 (1) (2008) 010504.
- [39] T.-C. Lu, T. Grover, Renyi entropy of chaotic eigenstates, arXiv:1709.08784, 2017.
- [40] V.D. Milman, G. Schechtman, Asymptotic Theory of Finite Dimensional Normed Spaces, *Lecture Notes in Mathematics*, vol. 1200, Springer, 1986.
- [41] R. Nandkishore, D.A. Huse, Many-body localization and thermalization in quantum statistical mechanics, *Annu. Rev. Condens. Matter Phys.* 6 (1) (2015) 15–38.
- [42] T.J. Osborne, Hamiltonian complexity, *Rep. Prog. Phys.* 75 (2) (2012) 022001.
- [43] D.N. Page, Average entropy of a subsystem, *Phys. Rev. Lett.* 71 (9) (1993) 1291–1294.
- [44] M. Rigol, V. Dunjko, M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, *Nature* 452 (7) (2008) 854–858.
- [45] J. Sánchez-Ruiz, Simple proof of Page’s conjecture on the average entropy of a subsystem, *Phys. Rev. E* 52 (5) (1995) 5653–5655.

- [46] L.F. Santos, A. Polkovnikov, M. Rigol, Weak and strong typicality in quantum systems, *Phys. Rev. E* 86 (1) (2012) 010102.
- [47] N. Schuch, M.M. Wolf, F. Verstraete, J.I. Cirac, Entropy scaling and simulability by matrix product states, *Phys. Rev. Lett.* 100 (3) (2008) 030504.
- [48] S. Sen, Average entropy of a quantum subsystem, *Phys. Rev. Lett.* 77 (1) (1996) 1–3.
- [49] M. Srednicki, Chaos and quantum thermalization, *Phys. Rev. E* 50 (2) (1994) 888–901.
- [50] R. Vasseur, J.E. Moore, Nonequilibrium quantum dynamics and transport: from integrability to many-body localization, *J. Stat. Mech. Theory Exp.* 2016 (6) (2016) 064010.
- [51] F. Verstraete, J.I. Cirac, Matrix product states represent ground states faithfully, *Phys. Rev. B* 73 (9) (2006) 094423.
- [52] G. Vidal, J.I. Latorre, E. Rico, A. Kitaev, Entanglement in quantum critical phenomena, *Phys. Rev. Lett.* 90 (22) (2003) 227902.
- [53] L. Vidmar, L. Hackl, E. Bianchi, M. Rigol, Entanglement entropy of eigenstates of quadratic fermionic Hamiltonians, *Phys. Rev. Lett.* 119 (2) (2017) 020601.
- [54] L. Vidmar, M. Rigol, Entanglement entropy of eigenstates of quantum chaotic Hamiltonians, *Phys. Rev. Lett.* 119 (22) (2017) 220603.
- [55] Z.-C. Yang, C. Chamon, A. Hama, E.R. Mucciolo, Two-component structure in the entanglement spectrum of highly excited states, *Phys. Rev. Lett.* 115 (26) (2015) 267206.
- [56] M. Znidaric, T. Prosen, P. Prelovsek, Many-body localization in the Heisenberg  $XXZ$  magnet in a random field, *Phys. Rev. B* 77 (6) (2008) 064426.