# Supporting Information for
# "Response of Marine-Terminating Glaciers to Forcing: Time Scales, Sensitivities, Instabilities and Stochastic Dynamics"

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## Contents



## 1 Two-stage model equations

For reference in the derivations below, we provide the two-stage model, which is derived in the main text

$$\frac{dH}{dt} = P - \frac{Q_g}{L} - \frac{H}{h_g L}\left(Q - Q_g\right) \tag{1}$$

$$\frac{dL}{dt} = \frac{1}{h_g}\left(Q - Q_g\right) \tag{2}$$

$$Q = \nu \frac{H^\alpha}{L^\gamma} \tag{3}$$

$$Q_g = \Omega h_g^\beta \tag{4}$$

$$h_g = -\lambda b(L) \tag{5}$$

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## 2 Fast and slow time scales

The two-stage model (equations 1-5) can be linearized into evolution equations for fluctuations ($H'$ and $L'$) about a stable equilibrium ($\bar{H}$ and $\bar{L}$)

$$\frac{dH'}{dt} = A_H(\bar{H},\bar{L})H' + A_L(\bar{H},\bar{L})L' \tag{6}$$

$$\frac{dL'}{dt} = B_H(\bar{H},\bar{L})H' + B_L(\bar{H},\bar{L})L', \tag{7}$$

where the feedbacks between thickness and grounding line position are

$$A_H(\bar{H},\bar{L}) = \Omega\bar{h}_g^{\beta-1}\bar{L}^{-1} - \nu\,(\alpha+1)\,\bar{H}^{\alpha}\bar{L}^{-(\gamma+1)}\bar{h}_g^{-1} \tag{8}$$

$$A_L(\bar{H},\bar{L}) = \Omega\bar{L}^{-2}\bar{h}_g^{\beta}\left[1+\beta\lambda\bar{b}_x\bar{L}\bar{h}_g^{-1} - \bar{H}\bar{h}_g^{-1} - (\beta-1)\lambda\bar{b}_x\bar{L}\bar{h}_g^{-2}\bar{H}\right] - \tag{9}$$

$$\nu\bar{H}^{\alpha}\bar{L}^{-(\gamma+2)}\left[\bar{H}\bar{h}_g^{-2}\bar{L}\lambda\bar{b}_x - (\gamma+1)\,\bar{H}\bar{h}_g^{-1}\right] \tag{10}$$

$$B_H(\bar{H},\bar{L}) = \nu\alpha\bar{h}_g^{-1}\bar{H}^{-1} \tag{11}$$

$$B_L(\bar{H},\bar{L}) = \nu\bar{H}^{\alpha}\bar{L}^{-\gamma}\left(\bar{h}_g^{-2}\lambda\bar{b}_x - \gamma\bar{h}_g^{-1}\bar{L}^{-1}\right) + (\beta-1)\Omega\bar{h}_g^{\beta-2}\lambda\bar{b}_x. \tag{12}$$

At stable equilibrium, the interior and grounding zone flux balance each another: $\bar{Q} = \bar{Q}_g$. Thus, we can simplify the above feedbacks to

$$A_H(\bar{H},\bar{L}) = -\bar{Q}_g\alpha\bar{h}_g^{-1}\bar{L}^{-1} \tag{13}$$

$$A_L(\bar{H},\bar{L}) = \bar{Q}_g\bar{L}^{-2}\left[1+\gamma\bar{H}\bar{h}_g^{-1} + \beta\lambda\bar{b}_x\bar{L}\bar{h}_g^{-1}\left(1-\bar{H}\bar{h}_g^{-1}\right)\right] \tag{14}$$

$$B_H(\bar{H},\bar{L}) = \bar{Q}_g\alpha\bar{H}^{-1}\bar{h}_g^{-1} \tag{15}$$

$$B_L(\bar{H},\bar{L}) = \bar{Q}_g\bar{h}_g^{-1}\left(\beta\lambda\bar{b}_x\bar{h}_g^{-1} - \gamma\bar{L}^{-1}\right). \tag{16}$$

Equations (6) and (7) can be combined into a second-order homogenous differential equation

$$\frac{d^2L'}{dt^2} - (A_H + B_L)\frac{dL'}{dt} + (A_H B_L - A_L B_H)L' = 0. \tag{17}$$

The solution to this differential equation is two damped exponential functions if the exponents are negative and real-valued. To find the exponents, we generally must solve the corresponding characteristic quadratic equation

$$r^2 - (A_H + B_L)r + (A_H B_L - A_L B_H) = 0. \tag{18}$$

We can take a critical shortcut by guessing (or checking numerically) that one root of this quadratic will be much larger than the other. As Figure 3 in the main text shows, for parameter values that are plausible for actual glaciers, one root is generally at least an order of magnitude larger than the other and both roots are real (i.e. there are no oscillatory solutions). Thus, we can solve for the roots of the characteristic equation using Vieta's formulas

$$r_1 = A_H + B_L \tag{19}$$

$$r_2 = \frac{A_H B_L - A_L B_H}{A_H + B_L}. \tag{20}$$

We can use these solutions to simplify and solve for the fast and slow time scales, which are equivalent to $r_1^{-1}$ and $r_2^{-1}$, respectively. As given in the main text, the two characteristic response time scales are

$$T_F = \frac{\bar{L}\bar{h}_g}{\bar{Q}\,(\alpha+\gamma)} - \frac{\bar{h}_g^2}{\bar{Q}_g \beta\lambda\bar{b}_x} \tag{21}$$

$$T_S = -\frac{\bar{H}\bar{h}_g\bar{L}^2}{\alpha T_F \bar{Q}}\left[\bar{Q} + \left(\frac{\beta\lambda b_x \bar{L}}{h_g}\right)\bar{Q}_g\right]^{-1}. \tag{22}$$

When note that $\bar{P}\bar{L} = \bar{Q} = \bar{Q}_g$ at a stable equilibrium, we can simplify to

$$T_F = \frac{\bar{h}_g}{\bar{P}}\left(\alpha + \gamma - \frac{\beta\lambda\bar{b}_x\bar{L}}{\bar{h}_g}\right)^{-1} \tag{23}$$

$$T_S = -\frac{\bar{H}\bar{h}_g}{\alpha T_F \bar{P}^2 S_T}. \tag{24}$$

## 3 Magnitude of glacier sensitivity to external forcing

We derive the sensitivity of marine-terminating glaciers to forcing by linearizing the two-stage model about the stable equibrium glacier state ($\bar{H}$, $\bar{L}$), and the time-averaged parameter values (e.g. $\bar{P}$). We start by decomposing $P$ (the spatially-averaged surface mass balance) into time-averaged and perturbed components

$$P = \bar{P} + P' \tag{25}$$

which leads to linear equations for glacier state that include a perturbation in surface mass balance

$$\frac{dH'}{dt} = A_H(\bar{H},\bar{L})H' + A_L(\bar{H},\bar{L})L' + P' \tag{26}$$

$$\frac{dL'}{dt} = B_H(\bar{H},\bar{L})H' + B_L(\bar{H},\bar{L})L', \tag{27}$$

which now includes the glacier response to perturbations in surface mass balance. Given a perturbation in mass balance, we would like to calculate the resulting change in glacier state that occurs after the glacier has transiently equilibrated and reached a new stable equilibrium. Or, in other words, we set $\frac{dH'}{dt} = 0$ and $\frac{dL'}{dt} = 0$ in equations (26) and (27)

$$0 = \left[-\bar{Q}_g \alpha \bar{h}_g^{-1} \bar{L}^{-1}\right] H' + \bar{Q}_g \bar{L}^{-2} \left[1 + \gamma \bar{H} \bar{h}_g^{-1} + \beta \lambda \bar{b}_x \bar{L} \bar{h}_g^{-1} \left(1 - \bar{H} \bar{h}_g^{-1}\right)\right] L' + P' \tag{28}$$

$$0 = \left[\bar{Q}_g \alpha \bar{H}^{-1} \bar{h}_g^{-1}\right] H' + \left[\bar{Q}_g \bar{h}_g^{-1} \left(\beta \lambda \bar{b}_x \bar{h}_g^{-1} - \gamma \bar{L}^{-1}\right)\right] L'. \tag{29}$$

We solves this linear system of equations for fractional changes in glacier state, relative to the stable equilibrium state:

$$\frac{H'}{\bar{H}} = \frac{1}{\alpha S_T} \left(\frac{\beta \lambda \bar{b}_x \bar{L}}{\bar{h}_g} - \gamma\right) \frac{P'}{\bar{P}} \tag{30}$$

$$\frac{L'}{\bar{L}} = -\frac{1}{S_T} \frac{P'}{\bar{P}}. \tag{31}$$

We can also derive the glacier sensitivity to changes in observable parameters that go into $\Omega$. Taking the form of grounding line flux for a glacier strongly buttressed by an ice shelf that primarily loses ice through calving,

$$\Omega = (n/2)^n (n+1)^{-(n+1)} \left[\rho_i g \left(1 - \lambda^{-1}\right)\right]^n A_g L_s^{-n} W_s^{n+1}, \tag{32}$$

We can take the same approach as above and linearize the two-stage model about a stable equilibrium and time-averaged ice shelf length

$$\frac{dH'}{dt} = A_H(\bar{H}, \bar{L}) H' + A_L(\bar{H}, \bar{L}) L' - n\bar{\phi} \bar{L}_s^{-(n+1)} \bar{h}_g^{\beta-1} \left(\frac{\bar{H}}{\bar{L}} - \frac{\bar{h}_g}{\bar{L}}\right) L_s' \tag{33}$$

$$\frac{dL'}{dt} = B_H(\bar{H}, \bar{L}) H' + B_L(\bar{H}, \bar{L}) L' + n\bar{\phi} \bar{L}_s^{-(n+1)} \bar{h}_g^{\beta-1} L_s', \tag{34}$$

where $\bar{\phi} = (n/2)^n (n+1)^{-(n+1)} \left[\rho_i g \left(1 - \lambda^{-1}\right)\right]^n A_g W_s^{n+1}$, or all the parameters in equation (32), except for the parameter being perturbed, $L_s$. We can now set the LHS to zero

$$0 = A_H(\bar{H}, \bar{L}) H' + A_L(\bar{H}, \bar{L}) L' - n\bar{\phi} \bar{L}_s^{-(n+1)} \bar{h}_g^{\beta-1} \left(\frac{\bar{H}}{\bar{L}} - \frac{\bar{h}_g}{\bar{L}}\right) L_s' \tag{35}$$

$$0 = B_H(\bar{H}, \bar{L}) H' + B_L(\bar{H}, \bar{L}) L' + n\bar{\phi} \bar{L}_s^{-(n+1)} \bar{h}_g^{\beta-1} L_s', \tag{36}$$

and solve for the the fractional glacier sensitivity to changes in the ice-shelf length ($L_s$)

$$\frac{H'}{\bar{H}} = -\frac{(\gamma+1)n}{\alpha S_T} \left(\frac{L_s'}{\bar{L}_s}\right) \tag{37}$$

$$\frac{L'}{\bar{L}} = -\frac{n}{S_T} \left(\frac{L_s'}{\bar{L}_s}\right). \tag{38}$$

We also consider an ice shelf that strongly buttresses a glacier and loses mass entirely through basal melting where

$$\Omega = (n+1)^{-\frac{1}{n+1}} \left[\rho_i g \left(1 - \lambda^{-1}\right)\right]^{\frac{n}{n+1}} A_g^{\frac{1}{n+1}} W_s \left(-\frac{\dot{b}}{2}\right)^{\frac{n}{n+1}}. \tag{39}$$

Linearizing the two-stage model about a stable equilibrium and time-averaged basal melt rate

$$\frac{dH'}{dt} = A_H(\bar{H},\bar{L})H' + A_L(\bar{H},\bar{L})L' + \left(\frac{n}{n+1}\right)\bar{\psi}\bar{\dot{b}}^{\frac{1}{n}}\bar{h}_g^{\beta-1}\left(\frac{\bar{H}}{\bar{L}} - \frac{\bar{h}_g}{\bar{L}}\right)\dot{b}' \tag{40}$$

$$\frac{dL'}{dt} = B_H(\bar{H},\bar{L})H' + B_L(\bar{H},\bar{L})L' + \left(\frac{n}{n+1}\right)\bar{\psi}\bar{\dot{b}}^{\frac{1}{n}}\bar{h}_g^{\beta-1}\dot{b}', \tag{41}$$

where $\bar{\psi} = (n+1)^{-\frac{1}{n+1}}\left[\rho_i g\left(1-\lambda^{-1}\right)\right]^{\frac{n}{n+1}} A_g^{\frac{1}{n+1}} W_s \left(-\frac{1}{2}\right)^{\frac{n}{n+1}}$, or all the parameters in equation (39), except for the parameter being perturbed, $\dot{b}$. We can now set the LHS to zero

$$0 = A_H(\bar{H},\bar{L})H' + A_L(\bar{H},\bar{L})L' + \left(\frac{n}{n+1}\right)\bar{\psi}\bar{\dot{b}}^{\frac{1}{n}}\bar{h}_g^{\beta-1}\left(\frac{\bar{H}}{\bar{L}} - \frac{\bar{h}_g}{\bar{L}}\right)\dot{b}' \tag{42}$$

$$0 = B_H(\bar{H},\bar{L})H' + B_L(\bar{H},\bar{L})L' + \left(\frac{n}{n+1}\right)\bar{\psi}\bar{\dot{b}}^{\frac{1}{n}}\bar{h}_g^{\beta-1}\dot{b}', \tag{43}$$

and solve for the the fractional glacier sensitivity to changes in the ice-shelf length ($\dot{b}$)

$$\frac{H'}{\bar{H}} = \frac{(\gamma+1)n}{\alpha(n+1)S_T}\left(\frac{\dot{b}'}{\bar{\dot{b}}}\right) \tag{44}$$

$$\frac{L'}{\bar{L}} = \frac{n}{(n+1)S_T}\left(\frac{\dot{b}'}{\bar{\dot{b}}}\right). \tag{45}$$

## 4 Transient glacier response to trend in external forcing

We define a linear trend in surface mass balance

$$P'(t) = \dot{P}t, \tag{46}$$

where $\dot{P}$ is the time rate of change of surface mass balance, and $t$ is the time (where the onset time of the trend occurs at $t = 0$). We also assume that the glacier begins at stable equilibrium

$$L'(t=0) = 0 \tag{47}$$

$$\left.\frac{dL'}{dt}\right|_{t=0} = 0. \tag{48}$$

The linearized two-stage model equations with a trend in surface mass balance (equations 26-27 where $P'$ is defined as in equation 46) can be combined to form a second-order nonhomogenous differential equation for grounding-line position

$$\frac{d^2L'}{dt^2} - (A_H + B_L)\frac{dL'}{dt} + (A_H B_L - A_L B_H)L' = B_H\dot{P}t, \tag{49}$$

which can be rewritten

$$\frac{d^2L'}{dt^2} = -T_F^{-1}\frac{dL'}{dt} - T_F^{-1}T_S^{-1}L' - T_F^{-1}T_S^{-1}L_P\dot{P}t, \tag{50}$$

where

$$L_P = \frac{d}{dP'}\left(-\frac{\bar{L}P'}{S_T\bar{P}}\right) = -\frac{\bar{L}}{S_T\bar{P}} \tag{51}$$

is the grounding-line sensitivity to perturbations in surface mass balance (which is also derived for various parameters in the last section). We can find a general time-dependent solution to this equation using the method of undetermined coefficients

$$L'(t) = C_F e^{-\frac{t}{T_F}} + C_S e^{-\frac{t}{T_S}} + B_H \dot{P} T_S^2 T_F (T_L + t), \tag{52}$$

Using the initial conditions, we then find a particular solution

$$L'(t) = \dot{P} L_P T_S \left[\frac{1}{2}(1-\tau)\, e^{-\frac{t}{T_F}} + \frac{1}{2}(1+\tau)\, e^{-\frac{t}{T_S}} - 1 + \frac{t}{T_S}\right], \tag{53}$$

where

$$\tau = \frac{T_S - 2T_F}{\left(T_S^2 - 4T_S T_F\right)^{\frac{1}{2}}} \tag{54}$$

is a dimensionless parameter that defines the relative importance of the fast and slow time scales to the magnitude of the response. This solution is valid only when $T_S > 4T_F$.

## 5 Expected glacier variability for noisy external forcing

The linearized two-stage model equations with included perturbations in a forcing parameter (equations 26-27) can be combined to form a second-order ordinary differential equation for grounding-line position

$$\frac{d^2 L'}{dt^2} = -T_F^{-1}\frac{dL'}{dt} - T_F^{-1}T_S^{-1}L' - T_F^{-1}T_S^{-1}L_P P', \tag{55}$$

with coefficients related to the slow and fast time scales, and where $P'$ represents noise in surface mass balance (this can be formulated for additive noise in other forcing parameters as well). Discretizing using the forward Euler method in time, this leads to a second-order autoregressive (AR(2)) model for the grounding-line position

$$L_t = \phi_1 L_{t-\Delta t} + \phi_2 L_{t-2\Delta t} - T_F^{-1}T_S^{-1}\Delta t^2 L_P P', \tag{56}$$

where the coefficients are given by

$$\phi_1 = 2 - T_F^{-1}\Delta t - T_F^{-1}T_S^{-1}\Delta t^2 \tag{57}$$

$$\phi_2 = -1 + T_F^{-1}\Delta t, \tag{58}$$

where $\Delta t$ is the discrete time step length at which the noisy process occurs (throughout this study, we take $\Delta t = 1$ year). Writing the two-stage model in the form of an autoregressive process allows us to leverage existing statistical characterizations of low-order autoregressive models. *Box et al.* [2015] gives the variance of a second-order autoregressive process ($\sigma_L^2$) forced by white additive noise (i.e. no interannual persistence or memory) as

$$\sigma_L^2 = \frac{1-\phi_2}{1+\phi_2}\frac{B_H^2\sigma_P^2}{(1-\phi_2)^2-\phi_1^2}, \tag{59}$$

where $\sigma_P^2$ is the variance of the white noise forcing process (surface mass balance in this case). This gives

$$\sigma_L^2 = \frac{2-T_F^{-1}\Delta t}{-T_F^{-1}\Delta t}\frac{B_H^2\sigma_P^2}{\left(2-T_F^{-1}\Delta t\right)^2-\left(2-T_F^{-1}\Delta t-T_F^{-1}T_S^{-1}\Delta t^2\right)^2}, \tag{60}$$

Expanding, and then assuming that the timescale of stochastic perturbations is small compared to the fast time scale of grounding-line response, $\Delta t << T_F$, we can simplify and derive the variance of the grounding-line position ($\sigma_L^2$)

$$\sigma_L^2 = \frac{T_S\Delta t}{2}\left[\frac{\alpha T_F\bar{P}\bar{L}}{\bar{H}\bar{h}_g}\right]^2\sigma_P^2. \tag{61}$$

## References

Box, G., G. Jenkins, G. Reinsel, and G. Ljung (2015), *Time Series Analysis: Forecasting and Control*, Wiley Series in Probability and Statistics, Wiley.