Fast, High-Order Solution of Surface Scattering Problems

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Introduction. The calculation of scattering from surfaces of acoustically large objects remains one of the most important and challenging problems in computational science. Roughly, these problems present difficulties as they require accurate descriptions and manipulation of highly oscillatory functions. Scattering problems involving one dimensional integrals have been efficiently treated by means of high order integrators (including the exponentially accurate trapezoidal rule and other high-order schemes [3, 5]), which reduce dramatically the complexity necessary to meet a given accuracy requirement. Problems of scattering by two-dimensional surfaces require much more delicate treatments [2, 5, 6], however.

In this context, use of high-order integrators is necessary to guarantee accurate results but generally not sufficient: in large scattering problems direct evaluation of a simple-minded discretization scheme would usually lead to inordinately long computing times. We thus present a fast, high-order algorithm for the solution of problems of acoustic scattering from smooth surfaces in three dimensions. The present algorithm computes scattered fields in $\mathcal{O}\left(N^{3/2}\log N\right)$ operations, where N is the number of variables in the discretized problem. A variety of numerical experiments indicate that this algorithm performs exceptionally well, and, in fact, that it outperforms some of the most competitive algorithms available.

Mathematical Formulation. We consider a problem of acoustic scattering by a sound-soft obstacle. This problem, governed by the Helmholtz equation, can be treated using the acoustic single- and double- layer potentials (see e.g. [3])

$$(S\varphi)(\mathbf{r}') = \int_{\partial D} \Phi(\mathbf{r}',\mathbf{r})\varphi(\mathbf{r})ds(\mathbf{r}) \quad \text{and} \quad (K\varphi)(\mathbf{r}') = \int_{\partial D} \frac{\partial\Phi(\mathbf{r}',\mathbf{r})}{\partial\nu(\mathbf{r})}\varphi(\mathbf{r})ds(\mathbf{r}), \quad (1)$$

Here $\Phi(\mathbf{r}',\mathbf{r}) = e^{ik|\mathbf{r}'-\mathbf{r}|}/4\pi\,|\mathbf{r}'-\mathbf{r}|$ is the Green function for the Helmholtz equation, and $\nu(\mathbf{r})$ is the external normal to the surface ∂D at point $\mathbf{r}$. Explicitly, given the values of the incoming wave $\psi^i(\mathbf{r})$ on ∂D, the scattered field can be obtained easily once the integral equation for the unknown density $\varphi(\mathbf{r})$

$$\frac{1}{2}\varphi(\mathbf{r}) + (K\varphi)(\mathbf{r}) - i\eta\,(S\varphi)(\mathbf{r}) = \psi^i(\mathbf{r}), \qquad \mathbf{r}\in\partial D \quad (2)$$

has been solved. Naturally, the possibility of producing fast and accurate solutions for our problems hinges on our ability to evaluate the integrals (1) accurately and efficiently. In attempting to develop such accurate and efficient integrators one faces two main problems, namely, accurate evaluation of the singular *adjacent interactions* — without undue compromise of speed — and fast evaluation

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of the voluminous number of *nonadjacent interactions* — without compromise in accuracy. Our approach to resolving these problems and the most important features of the presented technique are described below.

Partitions of unity, Discretizations. In order to deal with topological characteristics of closed surfaces which, possibly, can be described only locally, we utilize partitions of unity. In detail, we use a covering of the surface ∂D by a number K of overlapping two-dimensional patches $\mathcal{P}^j, j = 1, \cdots, K$, (called local charts in differential geometry). The patches $\mathcal{P}^j$ are then smoothly mapped to coordinate sets $\mathcal{H}^j$ in two-dimensional space, where actual integrations are performed. Further, we utilize a partition of unity subordinated to this covering of ∂D., i.e. we introduce a set of non-negative smooth functions $\{w^j, j = 1, \ldots, K\}$, such that (i) w^j is defined, smooth and non-negative in ∂D, and it vanishes outside $\mathcal{P}^j$, and (ii) $\sum_{j=1}^{K} w^j = 1$ throughout ∂D. This allows us to reduce the problem of integration of the density $\varphi(\mathbf{r})$ over the surface to a calculation of integrals of smooth functions φ^j compactly supported in the planar sets $\mathcal{H}^j$.

Adjacent integration. Substantial difficulties in the high-order evaluation of *adjacent interactions* are caused by the singular nature of the integral kernels $\Phi(\mathbf{r}', \mathbf{r})$ and $\partial\Phi(\mathbf{r}', \mathbf{r})/\partial\nu(\mathbf{r})$ at $\mathbf{r}' = \mathbf{r}$. While, certainly, the well-known strategy of "singularity subtraction" gives rise to bounded integrands, integration of such bounded functions by means of classical high-order methods does not exhibit high-order accuracy — since the subsequent derivatives of the integrand are themselves unbounded. Thus, specialized quadrature rules must be developed and used to achieve high-order integration. The new basic high-order integrator we present is based on analytical resolution of singularities. The resolution is achieved by integration in polar coordinates centered around each singular point. The Jacobian of the corresponding change of variables has the effect of cancelling the singularity, so that high order integration in the both radial and angular directions can be performed using the trapezoidal rule. Since the corresponding radial quadrature points do not lie on the Cartesian grid, a high-order, fast interpolation technique has been developed for evaluation of the necessary function values at the radial integrations points. Efficiency is of utmost importance here, since we use one such polar coordinate transformation *at each target point.*

Our high order integrator exhibits a super-algebraic convergence for the infinitely smooth scattering surfaces (see [1]). Use of this algorithm without acceleration would lead to the customary $O(N^2)$ operation count (where N is the size of the surface grid). The constant of proportionality in this complexity estimate is rather small, however, so that, even without acceleration, our high-order integrator is an efficient solver for small to medium-sized problems.

Non-adjacent integration and acceleration. Although the evaluation of nonadjacent interactions does not present challenges from the point of view of accuracy, an efficient acceleration.strategy is required to deal with the voluminous number of these interactions. Evaluation of these interactions is equivalent to computation of certain convolution-like sums of sources lying on a non-

Patch size	Unknowns	Type	# of It.	T/It.	Error
65×65	21966	Non-Accelerated	13	34m	$1.4 \cdot 10^{-4}$
129×129	87318	Accelerated	21	108m	$1.5 \cdot 10^{-6}$

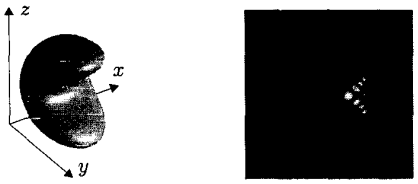
Table 1: Solution of a scattering problem from a sphere with $kR = 64.4$.

cartesian grid. We show that it is possible to find "equivalent sources" on a *three-dimensional cartesian grid* whose nonadjacent field values are effectively identical to those generated by the surface sources. The discrete convolution of these equivalent sources can then be evaluated efficiently by means of the FFT and an $\mathcal{O}\left(N^{3/2} \log N\right)$ algorithm can thus be obtained. A number of issues need to be carefully considered to obtain an accurate and feasible algorithm, including subtraction of nearly adjacent FFT contributions, evaluation of surface values from those in the rectangular grid, etc. as discussed in [1].

Our acceleration scheme is very different from the one used in the Fast Multipole Method (FMM) [5, 6]. In particular, the FMM approach depends critically on certain mappings which contain multiplication by Hankel functions of high order. These operations are associated with a substantial amount of ill conditioning, which leads to accuracy limitations known as the "subwavelength breakdown problem", see [4, p. 51]. Our approach, in contrast, is very stable, as demonstrated by error analysis provided in [1] and by the high order convergence exhibited in our numerical experiments for a wide variety of scattering problems.

Numerical Results In Table 1 we present the results of an application of our algorithm to the problem of scattering from an electrically large sphere with $kR = 64.4$; the computed far field is compared with the exact Mie series solution. (A number of iterations it took for the iterative solver GMRES to converge and time spent per iteration is also shown). A high-order nature of the algorithm is clearly seen; one can also notice that the use of acceleration allows for a substantial increase in accuracy without a significant increase in computing time. (A use of the non-accelerated scheme on the fine grid of Table 1 would require T/It. $\approx 8hs$.) In [1], other results are presented for a scattering by spheres of various sizes computed by our algorithm. These results compare very favorably to the similar results obtained using some of the leading solvers. Of course, no aspect of our algorithm is restricted to consideration of simple shapes such as spheres. To demonstrate this we consider a bean-shaped scatterer depicted in Figure 1(a), with boundary conditions corresponding to a point source inside the body. (The exact solution to this problem outside the body equals the field created by the source used). The results of this experiment are given in Table 2. (A value R refers to a half of the largest dimension of the obstacle). As in the previous example, both high-order behavior of the algorithm and efficiency of the accelerator are clearly demonstrated.

Finally, we compute the forward scattering from the bean-shaped object ($kR = 71$) under plane wave incidence. Figure 1(b) shows the near field behind the object (on a plane located on a distance $2R$ from the origin).



(a) (b)

Figure 1: Scattering by a bean-shaped obstacle; (a) the obstacle; (b) scattered field behind the obstacle from a plane wave propagating in the x direction.

kR	Patch size	Unknowns	Type	# of It.	T/It.	Error
1.57	17×17	1622	Non-accelerated	11	8s	$7.3 \cdot 10^{-4}$
17	33×33	5766	Non-accelerated	13	82s	$1.3 \cdot 10^{-3}$
17	65×65	22350	Non-accelerated	18	17m	$9.2 \cdot 10^{-6}$
17	129×129	89910	Accelerated	18	90m	$5.7 \cdot 10^{-6}$
71	129×129	89910	Accelerated	16	90m	$2.1 \cdot 10^{-4}$

Table 2: Scattering by the bean-shaped obstacle; point source

The presented combined fast high-order method performs exceptionally well: nowhere have we found results of a similar quality. The examples above suggest that, with proper parallelization, the present methods should allow for accurate solution of some of the most complex scattering problems arising in practice.

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