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S. J. van Enk, G. Nienhuis, "Rotating photons," Proc. SPIE 6905, Complex Light and Optical Forces II, 69050E (25 January 2008); doi: 10.1117/12.765476

Event: Integrated Optoelectronic Devices 2008, 2008, San Jose, California, United States

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## ABSTRACT

We propose a quantum theory of rotating light beams and study some of its properties. Such beams are polychromatic and have either a slowly rotating polarization or a slowly rotating transverse mode pattern. We show there are, for both cases, three different natural types of modes that qualify as rotating, one of which is a new type not previously considered. We discuss differences between these three types of rotating modes on the one hand and non-rotating modes as viewed from a rotating frame of reference on the other, thus resolving some paradoxes mentioned recently.

**Keywords:** Angular momentum, photons, rotation

## 1. INTRODUCTION

Several recent papers describe "rotating beams of light".[1–3] Such beams may have, for example, a rotating linear or slightly elliptical polarization, and should not be confused with circularly polarized light. At a fixed instant of time the direction of the (linear or elliptical) polarization vector rotates as a function of the propagation coordinate, and in a fixed plane perpendicular to the propagation direction the polarization rotates as a function of time. In a different type of rotating light beams it is the transverse intensity pattern rather than the polarization direction that is rotating. Such rotating beams of light may be produced by passing stationary beams through rotating optical elements, such as astigmatic lenses and half-wave plates.[4–7] The rotational frequency $\Omega$ of the optical elements (and hence of the light beams) will typically be very slow compared to the optical frequency $\omega$ of the light beam.

The theory discussed in the cited papers[1–3] is classical, although it is noted that some effects are more conveniently understood in terms of photons. Some interesting paradoxes and even some contradictions are mentioned in,[2] but the contradictions are not resolved there. The contradictions arise when one expects properties of rotating light beams to equal those of nonrotating light beams as seen from a rotating frame of reference. Here we give a quantum description of rotating light beams. We show that there are in fact several different natural ways of defining "rotating photons". Moreover, we show that a *different* type of rotating photons arises by applying a rotation operator to standard (non-rotating) quantized modes. The latter photons describe photons seen from a rotating frame. Carefully distinguishing these different types of rotating photons thus removes the contradictions mentioned above from Ref.[2]

We will be particularly interested in the angular momentum of rotating light beams. As we will show below, it will be easier to calculate the average angular momentum in a quantum description than in the classical descriptions of.[1–3] For example, it is pointed out in Refs.[1,3] that one should be careful when applying expressions for the angular momentum derived for *monochromatic* light beams.[8,9] Indeed, rotating light beams are necessarily *polychromatic*. In the formalism we use here no such problems arise and the quantum formalism takes care of polychromatic modes automatically.

Naturally, most salient features of rotating photons can be described in terms of angular momentum of light, simply because angular momentum operators generate rotations in space.[10] This angular momentum can have a spin or an orbital nature.[11] We thus start out by defining a complete set of electromagnetic field modes as follows: we use monochromatic modes with definite values of both spin and orbital angular momentum in the $z$ direction, $S_z$ and $L_z$. The corresponding quantum numbers are denoted by $\omega$ (for energy), $m$ (for orbital angular momentum) and $s=\pm1$ (for spin angular momentum, or, more precisely, for helicity). The modes may be exact

Complex Light and Optical Forces II, edited by David L. Andrews, Enrique J. Galvez, Gerard Nienhuis,
Proc. of SPIE Vol. 6905, 69050E, (2008) · 0277-786X/08/$18 · doi: 10.1117/12.765476

solutions of the Maxwell equations (Bessel modes[12]), or they may be exact solutions of the paraxial equation, for modes propagating in the $z$ direction (Laguerre-Gaussian modes). We must assume for the exact Bessel modes, however, that they, too, are propagating mostly in the $z$ direction. That is, we assume $k_T \ll \omega/c$, with $k_T$ the magnitude of the *transverse* components of the wave vector. This condition is needed in order for $L_z$ and $S_z$ to be well-defined angular momenta with integer eigenvalues.[13] So we use paraxial modes in either case.

A rotating mode or photon is then defined as an (almost) equal superposition of two opposite angular momenta $l$ and $-l$, with different frequencies $\omega \pm l\Omega$. Photons with a rotating polarization are superpositions of two opposite spin angular momenta; photons with a rotating transverse mode pattern are superpositions of opposite orbital angular momenta.

Besides the three quantum numbers mentioned so far, there is a fourth quantum number necessary to fully specify an arbitrary mode. This fourth quantum number describes the remaining transverse spatial degree of freedom. It could be the number of zeros $n_T$ in the transverse mode pattern of a Laguerre-Gaussian mode or the transverse momentum $\hbar k_T$ of a Bessel mode.[13] For our purposes we do not have to specify the transverse degrees of freedom any further. We thus assume that the fourth quantum number is fixed, so that we can use a simplified notation and denote the modes by the indices $(\omega, m, s)$.

In this paper we have space only to give a technical discussion of rotating photons. Illustrative xamples are considered in the extended version of this paper.[14]

## 2. PRELIMINARIES

For a given mode, the negative-frequency component of the (dimensionless) classical electric field can be written in cylindrical coordinates as

$$\vec{F}_{\omega,m,s}(z,\rho,\phi,t) = \exp(im\phi)\exp(-i\omega t)F(\rho,z)\vec{e}_s, \tag{1}$$

which is valid for the free field. The polarization vectors are $\vec{e}_\pm = (\vec{e}_x \pm i\vec{e}_y)/\sqrt{2}$. There are other nonzero components of the electric field but they are small in the paraxial approximation. We focus our attention on the main component (1).

In Eq. (1) we left the dependence of the field on $\rho$ and $z$ unspecified. The precise form of $F$ depends on whether we use exact or paraxial modes. For example, in the case of the exact Bessel modes we have[13]

$$F(\rho,z) \propto J_{|m|}(k_T\rho)\exp(ik_z z), \tag{2}$$

where $k_z$ is the longitudinal component of the wave vector, $k_z^2 = k^2 - k_T^2$, and $J_m$ is the $m$th-order Bessel function. $F$ depends on the quantum numbers $k_T$ and $\omega$ in this case, but not on $s$ and only on the absolute value $|m|$. For paraxial modes we have the more involved expressions for the Laguerre-Gaussian modes.[15] Also in that case, $F$ depends on the quantum numbers $\omega$, the absolute value $|m|$, and the quantum number $n_T$, but not on the polarization index. This observation plays an important role later on, when we define modes as superpositions of different modes that always have the same value of $|m|$.

The Bessel modes do not diffract, and the transverse intensity pattern is independent of $z$. There is a $z$-dependent phase factor, and by choosing it equal to $\exp(ik_z(z - z_0))$ for some fixed $z_0$ for *all* modes, we ensure that the mode functions in the plane $z = z_0$ are independent of the frequency $\omega$, if we fix the remaining quantum numbers $k_T, m, s$. We will refer to the plane $z = z_0$ as the reference plane. For Bessel modes each plane has the same intensity configuration. For paraxial modes, on the other hand, we do have to define a particular location of the reference plane, and we choose the same value $z_0$ for *all* paraxial modes. Such modes form a complete set of (paraxial) modes. Usually, the reference plane will be chosen as the focal plane, where the wavefronts are flat.

## 3. TIME-DEPENDENT MODES

### 3.1 Field operators and mode functions

Here we consider the theory of quantized modes, with as starting point the mode functions (1). We use the Heisenberg picture, so that operators rather than states depend on time. The (time-dependent) creation and annihilation operators for modes with quantum numbers $\omega, m, s$ are denoted by $\hat{a}^\dagger_{\omega,m,s}(t)$ and $\hat{a}_{\omega,m,s}(t)$. (Recall that we leave out the transverse quantum numbers). The frequency is a continuous variable, and the mode operators are assumed to obey the standard bosonic commutation rules $[\hat{a}_{\omega',m,s}(t), \hat{a}^\dagger_{\omega,m,s}(t)] = \delta(\omega - \omega')$. Likewise, the single-photon states $|\omega, m, s\rangle = \hat{a}^\dagger_{\omega,m,s}|\text{vac}\rangle$ are delta-function normalized. For a free field, the time dependence of the mode operators is simply

$$\hat{a}_{\omega,m,s}(t) = \exp(-i\omega t)\hat{a}_{\omega,m,s}(0). \tag{3}$$

We need the electric field operator from the relevant modes. In the present paper, we can restrict ourselves to the paraxial modes propagating in the positive $z$ direction. The contribution of a single paraxial mode to the positive-frequency part of the electric field operator is

$$\hat{\vec{E}}^{(+)}_{\omega,m,s}(t) = \sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c}}\vec{F}_{\omega,m,s}\hat{a}_{\omega,m,s}(t) =: \vec{E}_{\omega,m,s}\hat{a}_{\omega,m,s}(t) \ , \tag{4}$$

with $\vec{F}$ given by (1). Here we indicate operators by adorning them with hats. The normalization factor proportional to the square root of the frequency $\omega$ ensures the proper form of the Hamiltonian, in the form

$$\hat{H} = \sum_{m,s} \int d\omega \ \hbar\omega \ \hat{a}^\dagger_{\omega,m,s}\hat{a}_{\omega,m,s} \ . \tag{5}$$

This normalization is based on the assumption that the mode functions $\vec{F}_{\omega,m,s}$ are normalized in each transverse plane, as is common for paraxial modes. This means that $\int \rho d\rho d\phi |\vec{F}_{\omega,m,s}|^2 = 1$. The presence of the $\sqrt{\omega}$ term is responsible for the existence of various different types of rotating photons, as we will see below.

For later use we display the expression for contribution of a mode to the positive-frequency part of the vector potential in the Coulomb gauge as

$$\hat{\vec{A}}^{(+)}_{\omega,m,s}(t) = -i\sqrt{\frac{\hbar}{4\pi\epsilon_0 \omega c}}\vec{F}_{\omega,m,s}\hat{a}_{\omega,m,s}(t) =: \vec{A}_{\omega,m,s}\hat{a}_{\omega,m,s}(t) \ . \tag{6}$$

Finally, we also give here the expression for the spin and orbital angular momentum operators that will play a crucial role in the rest of the paper:

$$\begin{aligned} \hat{S}_z &= \sum_{m,s} \int d\omega \ \hbar s \ \hat{a}^\dagger_{\omega,m,s}\hat{a}_{\omega,m,s} \ , \\ \hat{L}_z &= \sum_{m,s} \int d\omega \ \hbar m \ \hat{a}^\dagger_{\omega,m,s}\hat{a}_{\omega,m,s} \ . \end{aligned} \tag{7}$$

The operator for the total angular momentum is denoted as $\hat{J}_z = \hat{S}_z + \hat{L}_z$.

### 3.2 Unitary transformations of modes

From now on we shall use for simplicity a generic subscript $i$ to indicate the full set of quantum numbers $\omega$, $m$, $s$ and the remaining transverse mode number, so that the summation over $i$ represents a summation over $m$ and $s$ and an integration over $\omega$. In this notation, the operators for the positive-frequency part of the electric field and the vector potential are denoted as

$$\hat{\vec{E}}(t) = \sum_i \vec{E}_i\hat{a}_i(t) \ , \ \hat{\vec{A}}(t) = \sum_i \vec{A}_i\hat{a}_i(t) \ . \tag{8}$$

Given a complete set of field modes $\vec{E}_i$ and corresponding mode operators $\hat{a}_i$, we can define a different complete set of (orthonormal) modes and mode operators in a general way. For this purpose we transform the field modes $\vec{E}_i$ as

$$\vec{E}'_i = \sum_j U_{ij}\vec{E}_j, \tag{9}$$

where $U_{ij}$ is a unitary matrix. As will become obvious in the next section, the specific transformations we will consider couple only a limited number of modes, so that the summation in (9) extends over a few discrete indices $j$. A key feature of the transformation is that it couples modes with different frequencies, so that the primed modes are not monochromatic. In order that the electric field operator can be expanded as

$$\hat{\vec{E}}(t) = \sum_i \vec{E}'_i \hat{a}'_i(t) \; , \tag{10}$$

we must transform the set of annihilation operators by

$$\hat{a}'_i(t) = \sum_j U^*_{ij} \hat{a}_j(t) \; . \tag{11}$$

The unitarity of $U$ ensures that the new modes are still orthogonal and that the new creation and annihilation operators still satisfy the correct equal-time commutation relations. Due to the unitarity of $U$, the inverse expansion of (11) is given by the transpose matrix, and we find for time zero

$$\hat{a}_j(0) = \sum_i U_{ij} \hat{a}'_i(0) \; . \tag{12}$$

For the *free* field the time dependence of the electric-field operator can be explicitly taken into account by incorporating the time dependence in the mode functions. Substituting the inverse expansion (12) into Eq. (8) gives the resulting expression

$$\begin{aligned} \hat{\vec{E}}(t) &= \sum_{i,j} \vec{E}_j \exp(-i\omega_j t) U_{ij} \hat{a}'_i(0) \\ &=: \sum_i \vec{E}'_i(t) \hat{a}'_i(0) \; , \end{aligned} \tag{13}$$

where the last line defines time-dependent and possibly non-monochromatic mode functions $\vec{E}'_i(t)$. By this transformation we can easily make a connection between the quantum theory of rotating photons and the classical theory of rotating light beams. On the basis of the new modes, the electric-field operator $\hat{\vec{E}}(t)$ is now expressed either as an expansion (10) with time-dependent mode operators, or as an expansion (13) in time-dependent non-monochromatic mode functions. It is noteworthy that although the summations are the same, the summands are not. Only when the transformation does not couple modes with different frequencies are the two expansions (10) and (13) the same term by term.

## 4. ROTATION WITHOUT ANGULAR MOMENTUM

To describe modes rotating at a frequency $\Omega$ around the $z$ axis, we start with monochromatic modes at some frequency $\omega$. We take equal superpositions of *two* fields with opposite angular momenta (either spin or orbital) and shift their frequencies by opposite amounts, proportional to the angular momentum. In fact, we can choose to have a rotating transverse intensity pattern by shifting the frequency proportional to the orbital angular momentum, or a rotating polarization by shifting in proportion to the spin angular momentum. It is convenient to consider these cases separately.

### 4.1 Rotating polarization

As an example of Eq. (11), we define new mode operators

$$\hat{b}_\pm := (\hat{a}_{\omega+\Omega s,m,s} \pm \hat{a}_{\omega-\Omega s,m,-s})/\sqrt{2}. \tag{14}$$

This transformation is, by Eq. (9), accompanied by the mode definitions

$$\vec{E}^b_\pm := (\vec{E}_{\omega+\Omega s,m,s} \pm \vec{E}_{\omega-\Omega s,m,-s})/\sqrt{2}. \tag{15}$$

For ease of notation we indicate the various transformed modes by different letters, rather than by primes. The new modes $b$ are described by new quantum numbers. For instance, $\omega$ is a nominal frequency now, and no longer the eigenfrequency of the mode. Indeed, the new mode has no eigenfrequency anymore. The index $\pm$ replaces the bi-valued polarization index $s$. The transverse quantum number, not displayed explicitly, stays the same, and so does $m$, the orbital angular momentum. The rotation frequency $\Omega$ is *not* an additional quantum number, but rather a parameter labeling the complete set of modes defined by (15). Indeed, whereas fixing a particular quantum number always restricts the set of modes to some smaller subset, fixing $\Omega$ still leaves one with a complete set of modes. On the other hand, we could try to consider $\Omega$ a quantum number if we *fix* a particular value of $\omega = \omega_0$. This would not be a natural choice, though, especially when $\Omega \ll \omega$. Moreover, modes with frequencies larger than $2\omega_0$ would not be included *.

The reason for calling these new modes "rotating" is as follows: the extra time-dependent terms in the electric field operator rotating at a frequency $\pm\Omega$ due to the change in frequency can be absorbed into the polarization part. For instance, take $s = 1$ and consider the reference plane $z = z_0$. In that plane the mode functions $F(\rho, z_0)$ for the 2 modes appearing in the definition for $b$ are identical, by construction. The transformation (13) for the electric-field operators and the $b_\pm$ modes takes the form

$$\vec{E}^b_+(t)\hat{b}_+(0) + \vec{E}^b_-(t)\hat{b}_-(0) = \vec{E}^b_+\hat{b}_+(t) + \vec{E}^b_-\hat{b}_-(t)\ . \tag{16}$$

In the reference plane these modes can be written as

$$\begin{aligned}\vec{E}^b_\pm(t)\hat{b}_\pm &= \sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c}}\exp(-i\omega t)\exp(im\phi)F(\rho, z_0)\hat{b}_\pm \\ &\times \ [\cos\theta\vec{e}_+\exp(-i\Omega t) \pm \sin\theta\vec{e}_-\exp(i\Omega t)]\ ,\end{aligned} \tag{17}$$

where we define

$$\begin{aligned}\cos\theta &= \sqrt{\frac{\omega+\Delta}{2\omega}} \\ \sin\theta &= \sqrt{\frac{\omega-\Delta}{2\omega}},\end{aligned} \tag{18}$$

with $\Delta = \Omega$ the frequency shift. The last line in Eq. (17) is the time-dependent polarization vector

$$\begin{aligned}\vec{e}(t) &= A_\pm(\vec{e}_x\cos\Omega t + \vec{e}_y\sin\Omega t) \\ &+ iA_\mp(-\vec{e}_x\sin\Omega t + \vec{e}_y\cos\Omega t)\ ,\end{aligned} \tag{19}$$

where

$$A_\pm = \frac{\cos\theta \pm \sin\theta}{\sqrt{2}}\ . \tag{20}$$

Since typically we will have $\Omega \ll \omega$ (so that $A_+$ is very close to 1, and $A_-$ is close to 0), the rotating polarization is almost linear for both the $b_+$ mode and the $b_-$ mode. Both modes $b_\pm$ describe an elliptical polarization whose axes rotate in the same direction at a frequency $\Omega$ around the $z$ axis. In fact, it is easy to verify that a time shift by $\tau_1 = \pi/(2\Omega)$ transforms the $b_+$ mode into the $b_-$ mode and *vice versa*. More precisely,

$$\vec{E}^b_\pm(t + \pi/(2\Omega)) = -i\exp(-i\pi\omega/(2\Omega))\vec{E}^b_\mp(t). \tag{21}$$

Nevertheless these two modes are distinct, and the mode operators $\hat{b}_+$ and $\hat{b}^\dagger_-$ commute.

So far we considered the field in the reference plane only. If we go outside the reference plane $z = z_0$ then for Bessel modes we still find the polarization is rotating in the same way in each plane $z$ =constant. For solutions

*On the flipside, our mode definition requires that $\Omega < \omega$, so there is always a range of frequencies $\omega$ for which rotating modes at angular velocity $\Omega$ cannot be defined.

of the paraxial equations (Gaussian beams), however, the two components of the field at different frequencies diffract in slightly different ways. Nevertheless, as long as we do not stray too far from the reference plane, the polarization still rotates in more or less the same way. Similar conclusions will hold for all modes discussed below. We will always display the field in the focal plane $z = z_0$, and it should be kept in mind that our descriptions are in general meant to apply near the reference plane.

### 4.2 Rotating transverse mode pattern

In a similar way we may define new modes by

$$\hat{c}_\pm := (\hat{a}_{\omega+\Omega m,m,s} \pm \hat{a}_{\omega-\Omega m,-m,s})/\sqrt{2}. \tag{22}$$

This transformation is, by Eqs. (9) and (11), accompanied by the mode definitions

$$\vec{E}^c_\pm := (\vec{E}_{\omega+\Omega m,m,s} \pm \vec{E}_{\omega-\Omega m,-m,s})/\sqrt{2}. \tag{23}$$

These modes obey a relation similar as Eq. (16), with $b$ replaced by $c$. Now the extra time-dependent terms in the electric field operator rotating at a frequency $\pm\Omega m$ due to the change in frequency can be absorbed into the azimuthal part. Just as before, let us take $s = 1$ and consider the reference plane $z = z_0$. The electric-field operators for the $c_\pm$ modes in that plane can (using (13)) be written as

$$\begin{aligned} \vec{E}^c_\pm(t)\hat{c}_\pm &= \sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c}}\exp(-i\omega t)F(\rho,z_0)\vec{e}_+\hat{c}_\pm \\ &\times \; [\cos\theta\exp(im(\phi-\Omega t)) \\ &\quad \pm\sin\theta\exp(-im(\phi-\Omega t))] \end{aligned} \tag{24}$$

with the same definition (18) of the angle $\theta$ as before, except that now $\Delta = m\Omega$. Clearly, the time-dependent field has a transverse mode pattern that rotates with a frequency $\Omega$ around the $z$ axis. Again, for both modes $c_\pm$ the direction of the rotation is the same. And just as for the polarization case, a time shift interchanges the modes $c_\pm$. Now the time shift that accomplishes this is a shift by $\tau_m = \pi/(2m\Omega)$, so that

$$\vec{E}^c_\pm(t+\pi/(2m\Omega)) = -i\exp(-i\pi\omega/(2m\Omega))\vec{E}^c_\mp(t). \tag{25}$$

Finally, we note that the quantum numbers of the modes $c_\pm$ are different than those of the original modes $a$: in particular, instead of the quantum number $m$ we have now both $m$ and $-m$, while keeping $s$ and $\omega$ (although the latter is no longer the eigenfrequency of the modes).

### 4.3 Rotating polarization and mode pattern

There is nothing to prevent us from defining modes where both the transverse mode profile *and* the polarization are rotating at a frequency $\Omega$. We just define

$$\hat{d}_\pm := (\hat{a}_{\omega+\Omega(m+s),m,s} \pm \hat{a}_{\omega-\Omega(m+s),-m,-s})/\sqrt{2}. \tag{26}$$

We can even define modes where the polarization is rotating at a different frequency than the transverse mode pattern,

$$\hat{e}_\pm := (\hat{a}_{\omega+\Omega m+\Omega' s,m,s} \pm \hat{a}_{\omega-\Omega m-\Omega' s,-m,-s})/\sqrt{2}. \tag{27}$$

Since all these redefinitions are unitary, the corresponding electric field amplitudes will, by construction, still be valid normalized solutions of the appropriate wave equations.

In order to see what it means to have the polarization and the transverse mode profile rotating at different frequencies, let us consider one explicit example. Suppose we define

$$\hat{f}_\pm = (\hat{a}_{\omega+\Omega,+1,-1} \pm \hat{a}_{\omega-\Omega,-1,+1})/\sqrt{2}. \tag{28}$$

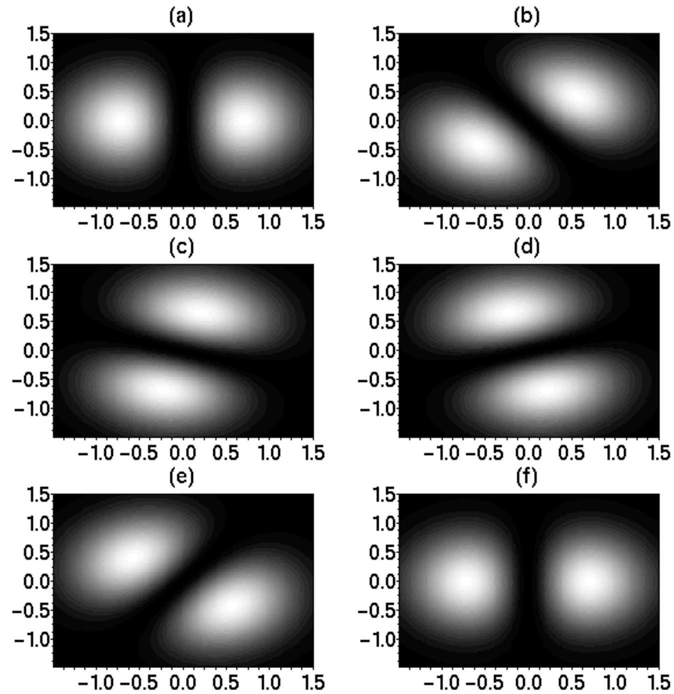


Figure 1. Transverse intensity profile of the $x$ component of the field corresponding to the mode $f_+$, as a function of $x$ and $y$ (in arbitrary units) in the focal plane $z = 0$. Snapshots are shown at 6 different times: for figures (a)–(f) we have $\Omega t = n\pi/5, n = 0 \ldots 5$, respectively. Here $\Omega > 0$ and the sense of rotation of the intensity profile is positive.

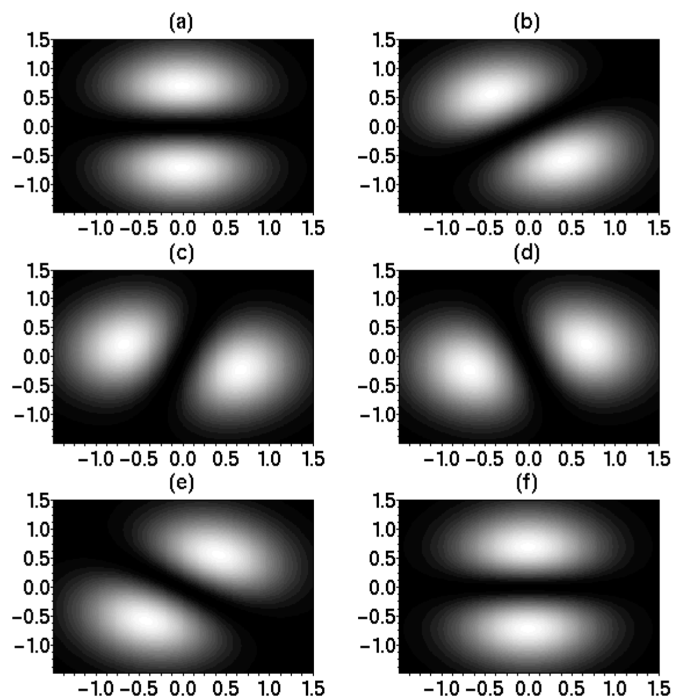


Figure 2. Same as for Fig. 1, but for the $y$ component.

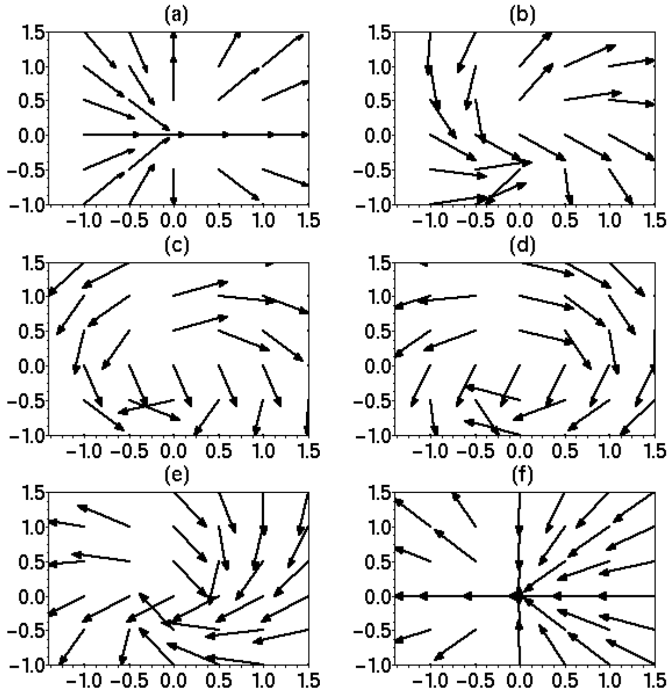


Figure 3. The (linear) polarization vector of the field corresponding to the mode $f_+$. Snapshots are shown at the same instants as in Figs. 1 and 2. Note the direction of rotation of the polarization vector is opposite (negative) to that of the transverse intensity patterns of Figs. 1 and 2, although we still have $\Omega > 0$.

Then this mode can be viewed as having a transverse mode pattern that rotates in the positive direction at frequency $\Omega$. Indeed, for any fixed linear polarization component, its field distribution rotates in the positive direction. On the other hand, the same mode can also be seen as having a polarization vector that rotates in the *negative* direction. That is, if we fix any point in the reference plane, then the local polarization vector rotates in the negative direction. The reason is simple: the extra time-dependent phase factors $\exp(\pm i\Omega t)$ can be absorbed either in the polarization vector or in the transverse mode pattern, and the choice is arbitrary, of course. We illustrate this behavior in Figs. 1–3. We plot snapshots at different times of the intensity profiles for the $x$ and $y$ components of the field in Figs. 1 and 2, respectively. The polarization direction is plotted in Fig. 3 at the same instants of time. Clearly, the polarization direction rotates clock-wise, the intensity patterns counter-clock-wise.

### 4.4 Rotating single photons produced by rotating mode inverters

Now consider a single photon in any one of the modes we have defined so far. For instance, consider a state of the form

$$|1\rangle_b = \hat{b}_+^\dagger|\text{vac}\rangle, \tag{29}$$

where $|\text{vac}\rangle$ denotes the vacuum state, with all modes unoccupied by photons. The coherence properties of a single-photon state are characterized by the complex matrix element of the electric-field operator

$$\langle\text{vac}|\hat{\vec{E}}(\vec{r},t)|1\rangle_b = \vec{E}_+^b(\vec{r},t)\ . \tag{30}$$

This quantity, which is the quantum analog of the classical electric field, is the detection amplitude function of the photon. It determines the second-order coherence of a one-photon field.[16]

This photon has an average spin angular momentum of zero, although its polarization is rotating at a frequency $\Omega$, according to (19). The simple reason is that the photon is in an *equal* superposition of spin angular momentum eigenstates with eigenvalues $+\hbar$ and $-\hbar$. There is no contradiction in having a rotating polarization and yet zero spin, as there is no simple *linear* relation between polarization and spin angular momentum. The expectation value of the electric-field amplitude is in fact always zero for any single-photon state, but the spin angular momentum is determined, in both the classical case and the quantum case, by a bilinear function of the field amplitudes; the spin is nonzero for any single photon with a definite polarization that is not linear. The average energy $\langle E \rangle$ of the $b$ photon is $\hbar\omega$, again because it is in an *equal* superposition of energy eigenstates with energies $\hbar\omega \pm \hbar\Omega$.

Similar conclusions hold for the other modes, $c$, $d$, $e$, and $f$ defined in the previous subsections. That is, for each such mode the average energy of a single photon is $\hbar\omega$. For the mode $c$, the orbital angular momentum vanishes, while for the other modes the total angular momentum is zero, even though these modes obviously display rotation.

In[3] it was shown that rotating photons are generated by rotating mode inverters. Suppose one has an optical element that "inverts" the polarization vector of a light beam according to

$$\vec{e}_s \mapsto \vec{e}_{-s} \tag{31}$$

for $s = \pm 1$. This is the effect of a half-wave plate. Then a plate that rotates at an angular frequency $\Omega/2$ will generate a mode with a polarization vector rotating at a frequency $\Omega$. The doubling of the rotation frequency can be understood by noting that in a rotating frame the mapping (31) becomes

$$\vec{e}_s \exp(is\Omega t/2) \mapsto \vec{e}_{-s} \exp(-is\Omega t/2)\ . \tag{32}$$

The quantum equivalent of this mapping is

$$\hat{a}_{\omega,m,s} \mapsto \hat{a}_{\omega - s\Omega, m, -s}\ . \tag{33}$$

The linear superposition $(\hat{a}_{\omega,m,1} + \hat{a}_{\omega,m,-1})/\sqrt{2}$ of mode operators corresponds to a mode with linear polarization. This linear superposition is mapped onto

$$(\hat{a}_{\omega,m,1} + \hat{a}_{\omega,m,-1})/\sqrt{2} \mapsto \hat{b}_+\ , \tag{34}$$

so that a linearly polarized single-photon state is mapped onto the state $|1\rangle_b$. This shows that a rotating half-wave plate with linear polarized photons as input generates $b_+$ photons as output.

We can also take as input a linear superposition of modes with orbital mode indices $m$ and $-m$, onto a mode converter rotating at an angular frequency $\Omega/2$. The corresponding classical mapping is

$$\vec{E}_{\omega,m,s} \exp(im\Omega t/2) \mapsto \vec{E}_{\omega - m\Omega, -m, s} \exp(-im\Omega t/2)\ . \tag{35}$$

There is, however, an ambiguity here: a mode converter will have an input plane $z = z_i$ and a different output plane $z = z_o$. Since between those planes modes with different frequencies will diffract differently, a rotating mode converter works properly only when $z_i$ and $z_o$ are sufficiently close for those diffraction effects to be negligible. Assuming this is the case, the quantum equivalent of the mapping by a rotating mode converter is

$$\hat{a}_{\omega,m,s} \mapsto \hat{a}_{\omega - m\Omega, -m, s}\ . \tag{36}$$

The mode operator corresponding to the superposition of two modes with opposite orbital angular momentum $\pm m$ is mapped as

$$(\hat{a}_{\omega,m,s} + \hat{a}_{\omega,-m,s})/\sqrt{2} \mapsto \hat{c}_+\ . \tag{37}$$

A single photon in this superposition mode is therefore converted into the single-photon state $|1\rangle_c$, with the single-photon wave function $\vec{E}^c_+(\vec{r}, t)$. Again, since this photon is in an equal superposition of two states with orbital angular momentum $\pm\hbar m$, its average orbital angular momentum is zero, even though the mode pattern is rotating. This agrees with the conclusions for a classical rotating field created by a rotating mode inverter.[3]

## 5. ROTATION WITH ANGULAR MOMENTUM

There is an alternative way of defining mode transformations. If we consider the expression for the rotating polarization of the mode of Eq. (17), then we see extra prefactors $\sin\theta$ and $\cos\theta$ appearing because of the (quantum) normalization factor proportional to $\sqrt{\hbar\omega}$. Similar factors appear in Eq. (24) for the rotating transverse mode pattern, for the same reason. In order to compensate for those prefactors we replace the mode operators (14) by the definition

$$\begin{aligned} \hat{g}_+ &= \sin\theta\hat{a}_{\omega+\Omega s,m,s} + \cos\theta\hat{a}_{\omega-\Omega s,m,-s}, \\ \hat{g}_- &= \cos\theta\hat{a}_{\omega+\Omega s,m,s} - \sin\theta\hat{a}_{\omega-\Omega s,m,-s}\ . \end{aligned} \tag{38}$$

The compensation of the factor $\sqrt{\omega}$ works only for the '$+$' mode of the pair of modes (38), but the companion '$-$' modes are necessary to make the redefinition unitary.

For the '$+$' modes we get instead of (17) the expression for the electric-field operator

$$\begin{aligned} \vec{E}^g_+(t)\hat{g}_+ &= \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\exp(-i\omega t)\sin\theta\cos\theta\exp(im\phi)F(\rho,z_0) \\ &\times\ \hat{g}_+\left[\vec{e}_+\exp(-i\Omega t)+\vec{e}_-\exp(i\Omega t)\right]\ . \end{aligned} \tag{39}$$

The last line now describes a rotating *linear* polarization

$$\vec{e}(t) = \vec{e}_x\cos\Omega t + \vec{e}_y\sin\Omega t\ . \tag{40}$$

A single photon from this mode now does possess a nonzero average spin angular momentum, equal to

$$\langle\hat{S}_z\rangle = (\sin^2\theta - \cos^2\theta)\hbar = -\hbar\Omega/\omega\ . \tag{41}$$

This result is exact, not perturbative, even though typically we do have $\Omega/\omega \ll 1$. The photon is in an unbalanced superposition of states with angular momentum $+\hbar$ with relative weight $\sin^2\theta$ and with angular momentum $-\hbar$ with weight $\cos^2\theta$. Similarly, the energy of a $g_+$ photon is

$$\langle E\rangle = \hbar\omega - \hbar\Omega^2/\omega\ . \tag{42}$$

Of course we can define analogous modes with rotating transverse mode patterns, while also compensating for the extra factors $\sin\theta$ and $\cos\theta$ in Eq. (24), by defining

$$\begin{aligned} \hat{h}_+ &= \sin\theta\hat{a}_{\omega+\Omega m,m,s} + \cos\theta\hat{a}_{\omega-\Omega m,-m,s}\ , \\ \hat{h}_- &= \cos\theta\hat{a}_{\omega+\Omega m,m,s} - \sin\theta\hat{a}_{\omega-\Omega m,-m,s}\ . \end{aligned} \tag{43}$$

It is again only the $h_+$ modes for which the $\theta$-dependent prefactors in the electric-field amplitude cancel. That is, the electric field of such a mode rotates around the $z$ axis with the same shape as in the non-rotating case $\Omega = 0$. A single photon in the $h_+$ mode has an orbital angular momentum equal to

$$\langle\hat{L}_z\rangle = -\hbar m^2\Omega/\omega\ , \tag{44}$$

and the energy is $\langle E\rangle = \hbar\omega - \hbar m^2\Omega^2/\omega$. Thus for both modes $g_+$ and $h_+$the angular momentum is, perhaps counter-intuitively, negative for a mode rotating in the positive direction around the $z$ axis.

Interestingly, the '$-$' modes that we were forced to define by requiring unitarity also have a nice property: for these modes it is the $\omega$-dependent prefactors in the expression for the vector potential that cancel, rather than in the expression for the electric field. Thus the $g_-$ mode describes a vector potential whose direction rotates uniformly and without changing length around the $z$ axis. Similarly, for the $h_-$ modes the transverse mode pattern of the vector potential rotates around the $z$ axis without changing shape. For single photons in the $'-'$ modes we find now that the angular momentum has the opposite value as for the $'+'$ modes. Thus we have

$$\langle \hat{S}_z \rangle = \hbar\Omega/\omega \,, \tag{45}$$

for a $g_-$ photon and

$$\langle \hat{L}_z \rangle = \hbar m^2 \Omega/\omega, \tag{46}$$

for a $h_-$ photon. The energy of the photons is $\langle E \rangle = \hbar\omega + \hbar\Omega^2/\omega$ and $\langle E \rangle = \hbar\omega + \hbar m^2\Omega^2/\omega$, respectively. So here the energy per photon is higher than $\hbar\omega$, while for the $'+'$ modes it was lower by the same amount. As far as the authors are aware, the $'-'$ modes have not been discussed before.

Of course, these other types of rotating photons can be generated with rotating mode inverters by taking different superpositions as input.

## 6. PHOTONS AS SEEN FROM A ROTATING FRAME

The modes we have constructed so far are "rotating modes". The field modes satisfy the Maxwell equations or the paraxial equations, and the mode operators satisfy the Heisenberg equations of motion. It is useful to compare those modes to the modes we get by applying a rotation operator of the form

$$\hat{R}(t) = \exp(i\hat{J}_z\Omega t) \tag{47}$$

to non-rotating modes. The transformed mode operators

$$\hat{a}'(t) = \hat{R}^\dagger \hat{a}_{\omega,m,s}(t)\hat{R} = \exp(i\Omega(m+s)t)\hat{a}_{\omega,m,s}(t) \tag{48}$$

no longer satisfy the correct Heisenberg equations of motion for a free field, because the unitary rotation operator depends on time. Instead the mode operators and the corresponding field operators describe modes as seen from a rotating frame, rotating at an angular frequency $\Omega$ around the $z$ axis. One may easily confuse operators like $\exp(i\Omega(m+s)t)\hat{a}_{\omega,m,s}(t)$ with the similar operators $\hat{a}_{\omega-\Omega(m+s),m,s}(t)$. Their time dependence is the same, but since the corresponding mode functions have different frequencies, they satisfy different equations of motion, and display different diffraction behavior.

On the other hand, the fact that a rotating beam of light can be described by taking superpositions of modes with different values of angular momentum and shifting the frequency in proportion to the angular momentum can be explained by this very analogy. The frequency shift can be seen as a rotational version of the Doppler shift.[6]

Alternatively, the frequency shift proportional to angular momentum can be seen as a time-dependent manifestation of a geometric phase,[5] with the time derivative of the phase equaling the frequency shift. For a rotating polarization this shift arises from the Pancharatnam phase,[19] for a rotating transverse mode pattern it arises from the similar "orbital" geometric phase associated with mode transformations.[20] The latter geometric phase was measured recently in its time-independent form[21] by using the mode converter from.[22]

It may be that there is a deeper connection between angular momentum of light and the various geometric phases of light: according to Refs.[?] the geometric phase arises only when angular momentum is exchanged, and this was confirmed in special cases in.[?;20] A recent experiment[?] indicates that this connection between angular momentum exchange and the occurrence of a geometric phase may be more general.

## 7. INTERFERENCE BETWEEN ROTATING PHOTONS

A typical quantum effect is the appearance of a dip at zero delay in the number of coincidences of detector clicks behind a beam splitter as a function of delay between two input photons that enter the two input ports: the Hong-Ou-Mandel (HOM) dip.[23] If we consider the interference of two, say, $b_+$ photons as a function of a time delay between them, then we find the standard HOM curve is modulated by an extra time-dependent factor. Namely, we have

$$b_+(t+\tau) = \cos\Omega\tau b_+(t) - \sin\Omega\tau b_-(t), \tag{49}$$

so that the modulation factor is simply

$$\cos^2 \Omega\tau = \frac{1}{2} + \frac{1}{2}\cos 2\Omega\tau. \tag{50}$$

This implies the HOM curve will display an oscillation at a frequency $2\Omega$ as a result of the rotating character of the polarization. But, of course, it can also be viewed as a beat frequency between the $\omega \pm \Omega$ components of the modes. Finally, we note the modulation factor is slightly different from the overlap of the two time-dependent polarizations,

$$|\vec{e}(t) \cdot \vec{e}^*(t+\tau)|^2 = \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta\cos(2\Omega\tau). \tag{51}$$

## 8. SUMMARY

We developed a quantum theory of rotating photons for which either the (linear or slightly elliptical) polarization vector or the transverse mode pattern rotates slowly around the propagation ($z$) direction. The rotational frequency $\Omega$ is independent of the optical frequency $\omega$, and will typically be much smaller than $\omega$. We found that there are, in each case, *three* natural types of rotating photons: they can have spin angular momentum $-\hbar\Omega/\omega$, 0, or $+\hbar\Omega/\omega$, if the photon has a rotating polarization, and an orbital angular momentum $-m\hbar\Omega/\omega$, 0, or $+m\hbar\Omega/\omega$ if the photon has a rotating transverse mode pattern composed of modes with orbital angular momenta $\pm m\hbar$. These three types of rotating photons correspond to modes with a rotating unchanging electric field vector, an *equal* superposition of opposite angular momenta, and a rotating unchanging vector potential, respectively. We distinguished these rotating photons from nonrotating photons as viewed from a frame rotating at $-\Omega$ around the $z$ axis.

In an extended version of this paper[14] we provide many examples and applications involving rotating photons, mostly having to do with quantum information processing.

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