STABILITY OF PARALLEL BUBBLY AND CAVITATING FLOWS

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ABSTRACT

This paper examines the bubble dynamic effects on the stability of parallel bubbly and cavitating flows of low void fraction. Inertial effects associated with the bubble response and energy dissipation due to the viscosity of the liquid, the heat transfer between the two phases, and the liquid compressibility are included. The equations of motion are linearized for small perturbations and a modified Rayleigh equation for the inviscid stability of the two-dimensional parallel flow is derived. Numerical solutions of the characteristic problem for the modified Rayleigh equation of a free shear layer are obtained by means of a multiple shooting method. Depending on the dispersion of the gaseous phase in the bubbly mixture, the ambient pressure and the free stream velocities, the presence of air bubbles can induce significant departures from the classical solution for a single phase fluid. Results are presented to illustrate the influence of the relevant flow parameters.

1. INTRODUCTION

The central role played by the stability of parallel flows in the analysis of a wide class of flow configurations (shear layers, boundary layers, jets, wakes, internal flows, etc.) is well documented in literature [16, 19]. Both incompressible and compressible homogeneous fluids have been investigated in a wide variety of configurations [2, 10]. Recently the stability of two-phase fluids has begun to be investigated, particularly in the context of liquid-solid suspensions [12, 17, and 20].

In the present paper we consider the problem of the inviscid stability of parallel bubbly and cavitating flows when effects associated with the dynamic response of the bubbles are taken into account. Even at very low void fractions, the presence of the bubbles drastically modifies the dynamic properties of the liquid such as the acoustic speed. Quite complex interactions can occur between the mean flow and the compliant, inertial and dissipative nature of the bubble dynamics [14, 15]. In particular, inertial effects in the bubble dynamics become important when the unstable frequencies of the flow approach the natural frequency of oscillation of individual bubbles. Then the bubbly mixture no longer behaves like a compressible barotropic fluid and significant deviations from the classical compressible flow solution are to be expected. A more detailed understanding of the phenomena associated with the bubble response is therefore necessary in order to improve the prediction, scaling, and control of the stability of parallel flows. Moreover, experiments have shown that the turbulent transition, noise spectrum, and the bubble response depend on the free stream velocity and pressure in parallel flows with traveling bubble cavitation [1, 3].

This paper is a natural extension of previous work on the dynamics of bubbly flows [6, 7, and 8] and utilizes the same system of equations. The linearized perturbation equations for a bubbly mixture are derived and some preliminary results are presented for the inviscid stability of two-dimensional unbounded shear layers. The computed eigenvalues clearly exhibit significant deviations from the single-phase, incompressible flow solutions for the flow parameters typical of low pressure bubbly and cavitating flows.

2. GOVERNING EQUATIONS

The basic equations used are described by d'Agostino *et al.* [4, 5, and 6]. If $\boldsymbol{u}$ is the velocity of the liquid, with pressure, p, unperturbed density, ρ, speed of sound, c, and bubble concentration, β, per unit liquid volume, the continuity equation for the mixture (neglecting the mass of the bubbles) can be written as:

$$\nabla\cdot\boldsymbol{u}=\frac{1}{1+\beta\tau}\frac{D(\beta\tau)}{Dt}-\frac{1}{\rho c^2}\frac{Dp}{Dt} \tag{1}$$

where $D/Dt=\partial/\partial t+\boldsymbol{u}\cdot\nabla$ is the Lagrangian time derivative, and $\tau=4\pi R^3/3$ is the volume of a bubble, assumed spherical with radius, $R(\boldsymbol{x},t)$. Note that the void fraction, $\alpha=\beta\tau/(1+\beta\tau)$, is assumed to be very small compared with unity. We shall neglect the relative motion between the bubbles and the liquid; an approximate evaluation of this effect by d'Agostino *et al.* [8] suggests that it is small. If the relative motion is negligible and the initial population, β, is uniform, then β is a simple constant in the bubbly fluid. Then, neglecting body forces and viscous effects in the large-scale flow, the momentum equation for the fluid motion becomes:

(2) $$\rho(1-\alpha)\frac{Du}{Dt} = -\nabla p$$

The bubble radius is determined by the Rayleigh-Plesset equation modified as indicated by Prosperetti [15] to account for the effects of liquid compressibility in the bubble dynamics:

(3) $$\left(1-\frac{1}{c}\frac{DR}{Dt}\right)R\frac{D^2R}{Dt^2}+\frac{3}{2}\left(\frac{DR}{Dt}\right)^2\left(1-\frac{1}{3c}\frac{DR}{Dt}\right)= =\left(1+\frac{1}{c}\frac{DR}{Dt}\right)\frac{p_R(t)+p(t+R/c)}{\rho}+\frac{R}{\rho c}\frac{dp_R(t)}{dt}$$

where the Lagrangian time derivatives follow the bubbles, and $p_R(t)$ is the liquid pressure at the bubble surface. This is related to the pressure, p_B, in the bubble (assumed uniform) by

(4) $$p_B(t) = p_R(t)+\frac{2S}{R}+4\mu\frac{1}{R}\frac{DR}{Dt}$$

where S is the surface tension at the bubble interface and μ is the viscosity of the liquid. Clearly, for the closure of the problem, the above equations must be supplemented by the mechanical and thermal equations of state and by the energy conservation equations for the two phases with the relevant boundary conditions. In the present work we shall use an "effective" viscosity, μ, which, in addition to the liquid viscosity, includes contributions to the bubble damping due to thermal and acoustic effects [4].

3. LINEAR STABILITY EQUATIONS

We now construct the equations governing the stability of a two-dimensional parallel flow. The continuity and momentum equations of the two phases are separated into mean values and small linear perturbations denoted by a hat accent:

$$u = U(y)+\hat{u}(y)e^{i(kx-\omega t)}, \quad v=\hat{v}(y)e^{i(kx-\omega t)}$$

$$p = p_o+\hat{p}(y)e^{i(kx-\omega t)} \text{ and } R = R_o+\hat{R}(y)e^{i(kx-\omega t)}$$

where ω and k are the perturbation frequency and wave number, respectively. Then, in the limit of small void fraction, equation (1) is linearized to give:

(5) $$ik\hat{u}+\hat{v}' = -i\omega_L\frac{3\alpha}{R_o}\hat{R}+i\omega_L\frac{1}{\rho c^2}\hat{p}$$

while equation (2) yields the two scalar equations:

(6) $$\rho(1-\alpha)(-i\omega_L\hat{u}-U'\hat{v}) = -ik\hat{p}$$

(7) $$\rho(1-\alpha)i\omega_L\hat{v} = \hat{p}'$$

where $\omega_L = \omega - kU$ is the Lagrangian frequency experienced by the bubbles in their motion relative to the mean flow, and primes indicate differentiation with respect to the independent variable y.

The perturbation terms in the Rayleigh-Plesset equation yield the following relation:

(8) $$\left(-\omega^2-i\omega 2\lambda+\omega_B^2\right)\hat{R} = -\left(1+i\omega\frac{R_o}{c}\right)\frac{\hat{p}}{\rho R_o}$$

in which it can be observed that each individual bubble behaves as an harmonic oscillator with natural frequency, ω_B [15]. The internal bubble pressure is

(9) $$p_B(t) = p_{B0}+\hat{p}_B(t) = p_{Bo}\left(1-\phi\hat{R}/R_o\right).$$

The reader is referred to d'Agostino and Brennen [7] for a detailed explanation of the terms appearing in the above expressions. Here we only mention that the damping coefficient, λ, is given by the sum of three terms accounting for the viscous, acoustical, and thermal dissipation. The quantity, $Re(\phi)$, can be regarded as the effective polytropic exponent of the gas in the bubble. It, respectively, tends to 1 and to the ratio of the specific heats, γ, in the isothermal and isentropic limits as $\omega \to 0$ and $\omega \to +\infty$.

Elimination of $\hat{R}$ and $\hat{p}$ from equations (5), (6), (7), and (8) yields the following system of equations governing the inviscid stability of a bubbly flow (and equivalent to a modified Rayleigh equation):

(10) $$\hat{u}' = ik\hat{v} - i\frac{U''}{\omega_L}\hat{v}-i\frac{U'}{kc_M^2}\left(i\omega_L\hat{u}-U'\hat{v}\right)$$

(11) $$\hat{v}' = -ik\hat{u}+\frac{\omega_L}{kc_M^2}\left(+i\omega_L\hat{u}-U'\hat{v}\right)$$

Here $c_M = c_M(\omega_L)$ is the complex and dispersive (frequency dependent) speed of propagation of an harmonic disturbance of angular frequency, ω_L, in the bubbly mixture. This is determined by the dispersion relation, where

(12) $$\frac{1}{c_M^2} = \frac{\omega_{Bo}^2}{c_{Mo}^2}\frac{1+i\omega R_o/c}{\omega_B^2-\omega^2-i\omega 2\lambda}+\frac{1-\alpha}{c^2}$$

where

(13) $$\omega_{Bo}^2 = \frac{3p_{Bo}}{\rho R_o^2}-\frac{2S}{\rho R_o^3} \quad \text{and} \quad c_{Mo}^2 = \frac{\omega_{Bo}^2R_o^2}{3\alpha(1-\alpha)}$$

ω_{B0} is the natural frequency of oscillation of a single bubble under isothermal conditions in an unbounded liquid and c_{M0} is the low-frequency sound speed in a free bubbly flow with incompressible liquid ($\omega_L \to 0$ and $c \to \infty$). Notice that we obtain the classical Rayleigh stability equation for a homogeneous fluid by eliminating the velocity component $\hat{u}$ from equations (10) and (11), and setting $c_M \to \infty$.

For the closure of the mathematical problem, equations (10) and (11) must be supplemented by two appropriate boundary conditions for $\hat{u}$ or $\hat{v}$ for the specific flow configuration under consideration. As will be seen below, this leads to a linear, second order eigenvalue problem for the free parameters ω or k. Just as in the homogeneous fluid formulation, the set of admissible (generally complex) values of ω or k (the eigenvalues) is uniquely determined by the condition that the corresponding nontrivial solutions (the eigenfunctions) satisfy the assigned boundary conditions. Any two of the real and imaginary parts of the complex frequency and wave number can be specified and the remaining parts are then determined. Spatially growing oscillations are studied by assigning the real frequency, ω, and solving for the complex wave number, k, which is the eigenvalue of the problem; the imaginary part of, k, namely k_i is the spatial attenuation rate of the perturbation while its real part is the wave number. A negative value of k_i therefore implies amplification of the perturbation. On the other hand temporally growing oscillations are studied by assigning a real value to k and solving for the complex ω. The two cases become identical at neutral stability.

4. INVISCID STABILITY OF SHEAR LAYERS

In order to illustrate the impact of bubble dynamic effects on the stability of parallel flows, we consider the simple classical case of a two-dimensional inviscid free shear layer between two parallel streams of velocities U_1 $(y<0)$ and U_2 $(y>0)$. It is assumed that the unperturbed velocity profile can be approximated by the hyperbolic tangent profile:

$$U(y)=\frac{U_1+U_2}{2}+\frac{U_2-U_1}{2}\tanh y$$

Inside the shear layer the equations governing the perturbations must, in general, be integrated numerically. Outside of the shear layer, where U, ω_L, and c_M are constant, the perturbation equations reduce to

$$\hat{u}'=ik\hat{v} \qquad \text{and} \qquad \hat{v}'=-ik\hat{u}+i\frac{\omega_L^2}{kc_M^2}\hat{u}$$

or

$$\hat{v}''+\left(-k^2+\frac{\omega_L^2}{c_M^2}\right)\hat{v}=0$$

and can be integrated in closed form to obtain

$$\hat{v}=A_{1,2}e^{\pm y\sqrt{k^2-\omega_L^2/c_M^2}}$$

and

$$\hat{u}=\pm A_{1,2}\frac{ik}{\sqrt{k^2-\omega_L^2/c_M^2}}e^{\pm y\sqrt{k^2-\omega_L^2/c_M^2}}$$

where $A_{1,2}$ are arbitrary complex constants, the complex square root is computed with its principal branch, and the appropriate sign is determined by requiring that the solution not diverge as $y\to\pm\infty$. From the practical standpoint, the stability problem is first transformed into a boundary value problem on $-n\delta\le y\le n\delta$ by assigning zero derivatives to the eigenvalues with respect to y within the integration range. The boundary value problem is then solved numerically using a multiple shooting method [18]. The integration is carried out with a fourth-order Runge-Kutta method (extrapolated to the fifth order), with self-adaptive step-size for meeting the required accuracy. Finally, the eigenvalues are corrected using a multidimensional modified Newton-Raphson method, in order to improve the convergence of the algorithm. The code has been validated against the results reported by Betchov and Criminale [2] and Michalke [13] for both the spatially and temporally growing oscillations in single phase flow.

The computation starts with some tentative candidate for the complex eigenvalue, k, in the case of spatial stability calculations. The arbitrary constant, A_1, is chosen to give the simple initial conditions at $y=-n\delta$ $(n>>1)$:

$$\hat{u}=\frac{ik}{\sqrt{k^2-\omega_L^2/c_M^2}}$$

and

$$\hat{v}=1$$

The equations are then integrated from $y=-n\delta$ to $y=n\delta$, where the computed values of $\hat{u}$ or $\hat{v}$ must be continuous with the upper asymptotic solution. Therefore, at $y=n\delta$, the condition:

$$\hat{u}=-\frac{ik}{\sqrt{k^2-\omega_L^2/c_M^2}}\hat{v}$$

must be satisfied. This relation is then used to iteratively correct the assumed complex eigenvalue and the process is repeated to convergence.

5. RESULTS AND DISCUSSION

In this paper, we present results for the spatial stability characteristics of a free shear layer involving a mixture of air bubbles $(\gamma=1.4)$ and water $(\rho=1000\ \text{kg/m}^3$, $\mu=0.001\ \text{Ns/m}^2$, $S=0.0728\ \text{N/m}$, $c=1485\ \text{m/s})$. The results are presented in dimensionless form by using the shear layer velocity difference, ΔU, and width, δ, for the non-dimensionalizing velocity and length. The nondimensional quantities will be denoted by an asterisk.

The three nondimensional parameters in the present analysis are the bubble natural frequency, ω_{B0}^*, the bubble radius, R^*, and the void fraction, α. When the bubble natural frequency, ω_{B0}^*, is large compared to the perturbation frequency, ω^*, the effect of uniformly dispersed bubbles on the stability characteristics of the liquid is negligible. This is true no matter how high the void fraction.

For values of ω_{B0}^* closer to ω^* the bubble dynamic effects begin to modify the stability characteristics of the flow. This is illustrated in Fig. 1, where the attenuation rate, given by the imaginary part of the wave number, k_i^*, is plotted as function the perturbation frequency, ω^*, for a void fraction, α, of 0.005 and a

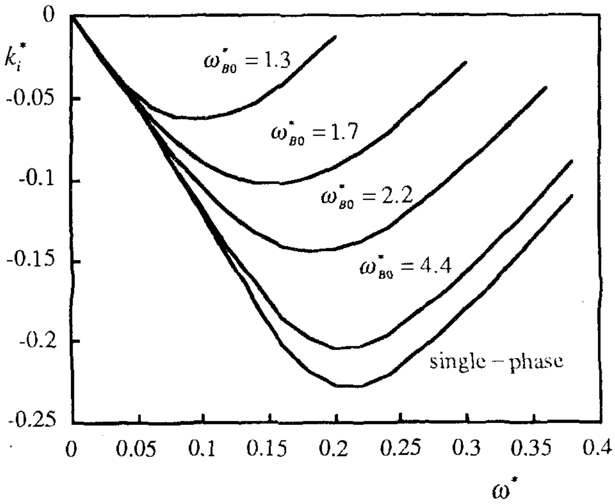

Figure 1. The attenuation rate (k_i^*) as a function of the frequency, ω^*, for a void fraction of $\alpha = 0.005$ and various bubble natural frequencies: $\omega_{B0}^* = 4.4$, $\omega_{B0}^* = 2.2$, $\omega_{B0}^* = 1.7$, and $\omega_{B0}^* = 1.35$ Also shown is the single-phase flow result.

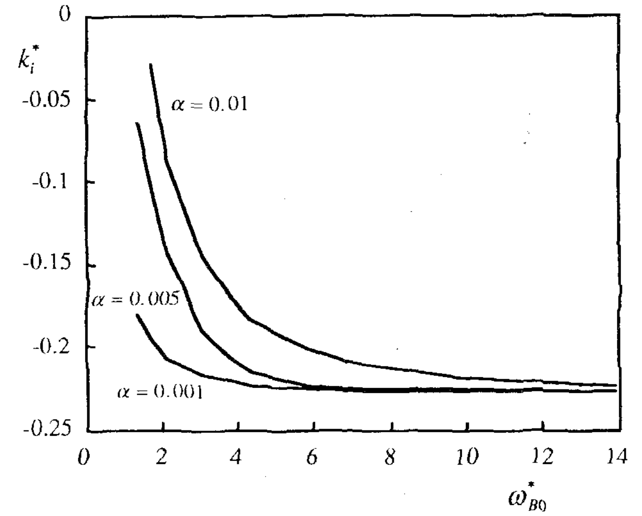

Figure 3. The most unstable attenuation rate (k_i^*) as a function of the bubble natural frequency, ω_{B0}^*, for $\alpha = 0.001$, $\alpha = 0.005$, $\alpha = 0.01$.

bubble radius R^* of 0.05. It is readily seen that the bubbles have a stabilizing effect and the closer the two frequencies the greater the stabilizing effect. Furthermore the most unstable frequency shifts towards smaller values (the most unstable frequency for the single-phase fluid is ω^*=0.206692 [13]).

The effects of the other flow parameters, namely the bubble radius, R^*, and the void fraction, α, have also been investigated. The effect of the bubble radius is shown in Figure 2, where the maximum amplification rate is plotted as a function of the natural frequency of the bubble, ω_{B0}^*. An increase in radius is shown to be destabilizing. Figure 3 illustrates the stabilizing effect of an increase in the void fraction. Figure 4 shows the decrease in the most unstable frequency, ω_m^*, with the bubble natural frequency, ω_{B0}^*, for different values of the void fraction. The effect of the bubble natural frequency, ω_{B0}^*, on the phase velocity, defined as $c_r^* = \omega^*/k_r^*$, is illustrated in Figure 5.

Figures 1, 2, 3, 4, and 5 are pertinent to those situations in which the magnitude of the bubble resonance frequency is considerably higher than the most unstable frequency for the single phase flow. Such situations are more likely to be found in practical applications. From this we deduce that bubble dynamic effects play a secondary role in this kind of flow, when compared to compressibility effects.

Other computations were carried out to examine the role played by bubble dynamics at or near resonance ($\omega^* \approx \omega_{B0}^*$). Figure 6 illustrates the bubble dynamic effects for $\alpha = 0.002$, for several bubble natural frequencies. It can be seen that the attenuation rate does not uniformly increase with the bubble natural frequency. Depending on the value of the perturbation frequency, flows characterized by a lower bubble natural frequency can be less stable than others. This

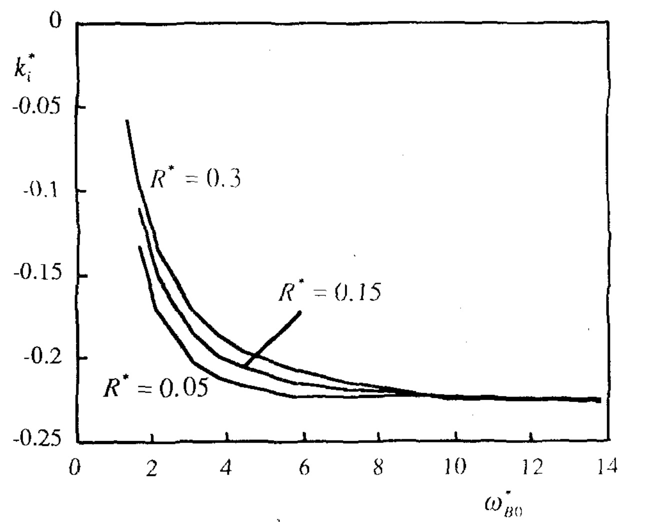

Figure 2. The most unstable attenuation rate (k_i^*) as a function of the bubble natural frequency, ω_{B0}^*, for $\alpha = 0.005$, and $R^* = 0.05$, $R^* = 0.15$, and $R^* = 0.3$.

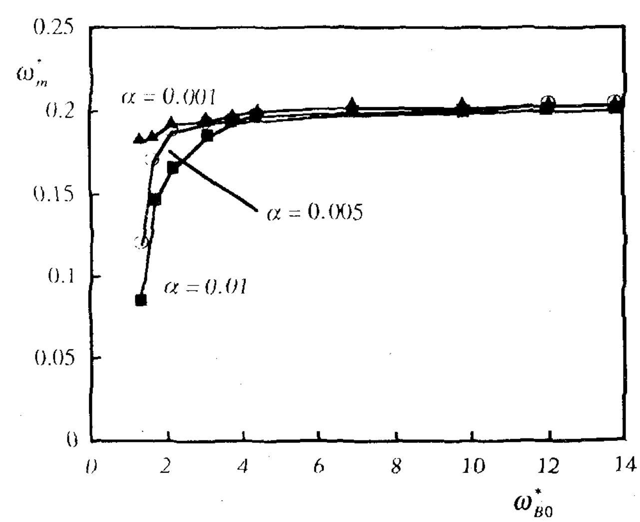

Figure 4. The frequency, ω_m^*, corresponding to most unstable attenuation rate, as a function of the bubble natural frequency, ω_{B0}^*, for $\alpha = 0.001$, $\alpha = 0.005$, and $\alpha = 0.01$.