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Do all identically conserved geometric tensors come from actions? A status report

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Abstract

Noether's theorem, that local gauge variations of gauge invariant actions are identically conserved (more tautologically, that gauge variations of gauge invariants vanish) was established a century ago. Its converse, in the geometric context: are all identically conserved local symmetric tensors variations of some coordinate invariant action? remains unsolved to this day. We survey its present state and discuss some of our concrete attempts at a solution.

1 Introduction

Noether's theorem is a textbook truism that the field equations of gauge theories–Maxwell, Yang-Mills, Einstein *et.al*–obey conservation,"Bianchi", identities as a consequence of their Lagrangian origins: The actions being invariant, their local gauge variations vanish. But the latter are just the divergences of the action's field variations: It suffices for models to be Lagrangian for them to obey gauge identities. But is it also necessary–are all identically conserved currents derived from actions? This converse hypothesis is almost as old as Noether's (for recent attempts, see [1]; for a history, see e.g., [2]) and remains unsolved–for the gravitational case–despite its simple form and intuitive appeal. This is not merely a formal conjecture, but has direct physical consequences: Non-Lagrangian terms have recently been proposed as alternative geometrical models. But the physics requires them to be separately conserved: Since coordinate invariant matter actions' stress-tensors are identically conserved (on matter shell), irrespective of their couplings, if any, to gravity, the proposed field equations,

$$G_{\mu\nu}(g) + E_{\mu\nu}(g) = T_{\mu\nu}(\text{matt}; g) \tag{1}$$

imply that the non-Lagrangian gravitational addition $E_{\mu\nu}$ must be identically conserved, since both the Lagrangian gravity part $G_{\mu\nu}$ and–as we saw[1] $-T_{\mu\nu}$(matt) both are. Hence counterexamples to the necessity hypothesis, if they existed, would be of physical interest and conversely their absence would remove a sea of models. We shall first review the vector gauge theories, where there are manifold counter-examples to the conjecture, before coming to the gravitational story. Concentrating on the most elementary geometrical systems, those in $D = 2$ where only the scalar curvature enters, we will discuss some differential and integral approaches to exhibit the nature of some of the obstacles involved as well as all-order versus perturbative attempts; in the latter case we have succeeded in reaching a few derivative order improvements over past results. Higher-dimensional similarities and differences will also be discussed. Given the simplicity and plausibility of the hypothesis, we cannot help but feel some obvious proof is being overlooked; perhaps this résumé will attract one!

[1] A recent suggestion [3] that a matter Lagrangian is not needed to specify matter systems, but only conservation of the stress-tensors, can be understood in this light as being entirely equivalent to the standard lore: A correct stress tensor is always the metric variation of an action, and is conserved IFF the matter field equations are invoked.

2 Vectors

A sufficiently general set of field equations, first in the abelian, $D = 4$ Maxwell, case, is

$$M^{\nu} = \partial_{\mu}\Big[X(F^2, \widetilde{F}F)F^{\mu\nu}\Big] = 0\,, \tag{2}$$

where $\widetilde{F}^{\mu\nu}$ is the $(D = 4)$ dual of $F_{\mu\nu}$ and we have used only its two simplest, algebraic, invariants in the arbitrary function X. The divergence identities $\partial_{\nu}M^{\nu} = 0$ are manifest from the antisymmetry of F contracted with the symmetric $\partial_{\mu}\partial_{\nu}$, irrespective of X. However, not all such M are A_{μ} variation of a Lagrangian: they must obey the usual Helmholz integrability conditions, which set stringent limits on the X. So here identical conservation does NOT require an action. Perhaps surprisingly this is not some purely linear, abelian property, but holds also for non-abelian fields: there, we replace ∂_{μ} by the usual covariant color derivatives D_{μ} whose commutator is now the non-abelian field strength, $[D_{\mu}\,, D_{\nu}] \sim F_{\mu\nu}$. Yet the generalization of (2) remains transverse, since $f_{abc}F^{b\mu\nu}F^{c}_{\mu\nu} = 0$ (the arguments of X are now the (color-singlet) traces of F^2 and $\widetilde{F}F$). Again, only the algebraic factor: antisymmetry, is relevant.

3 Gravity

We now come to our problem: the origin of identically conserved geometric tensors. The formalism is enormously simplified by working first in $D = 2$, where all essentials are already present, index proliferation is at a minimum and the issues are manifest. Only the scalar curvature R and its covariant derivatives, $\nabla^n R$, (since $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$), and explicit metrics contracting indices are present. Our convention is

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}{}_{\mu\alpha\nu} = g^{\mu\nu}\left(\partial_{\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \partial_{\nu}\Gamma^{\alpha}{}_{\alpha\mu} + \Gamma^{\alpha}{}_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\beta}\Gamma^{\beta}{}_{\mu\alpha}\right)\,. \tag{3}$$

Its variation is

$$\frac{\delta R(x)}{\delta g^{\mu\nu}(y)} = \left[\frac{1}{2}g_{\mu\nu}R + (g_{\mu\nu}\nabla^2 - \nabla_{\mu}\nabla_{\nu})\right]\delta^{(2)}(x-y). \tag{4}$$

Note that the $\nabla\nabla$ part of δR is the covariant version of the flat space transverse projector $O^{\mu\nu} = [\eta^{\mu\nu}\partial^2 - \partial^{\mu}\partial^{\nu}]$, but it is of course no longer transverse; there are none in curved space. Indeed this is the 2D version of the flat superpotentials $V^{\mu\nu} = \partial_{\alpha}\partial_{\beta}H^{[\mu\alpha][\nu\beta]}$, where H has the algebraic symmetries of the Riemann tensor, so V is identically conserved. In $D = 2$, H degenerates into $\varepsilon^{\mu\alpha}\varepsilon^{\nu\beta}S$ where S is a scalar, namely into the $O^{\mu\nu}$ above. First, a reminder of why invariant action-based tensors are conserved here (non-invariant actions' variations are of course not even tensors). The variation of

$$A = \int d^2x L\left(g_{\mu\nu}; R, \nabla^n R\right)\,, \tag{5}$$

is

$$\frac{\delta A}{\delta g_{\mu\nu}(x)}\Big|_{\text{total}} = \frac{\delta A}{\delta g_{\mu\nu}(x)}\Big|_{R\,\text{const}} + \int d^2y \frac{\delta R(y)}{\delta g_{\mu\nu}(x)} \frac{\delta A}{\delta R(y)}\Big|_{g\,\text{const}}\,, \tag{6}$$

and of course the Noether identity $\nabla_\nu \frac{\delta A}{\sqrt{-g}\delta g_{\mu\nu}}\Big|_{\text{total}} = 0$ holds because A is invariant under arbitrary coordinate variations, $\delta g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$. Note that both terms in (6) are "normal" tensors, as against "projector" ones, $O^{\mu\nu}S$–this point is critical to our problem, so we explain it. (Ex-)projectors are of course tensors, but strange ones whose divergences are NOT in general total derivatives: despite the notation, $\nabla_\nu(O^{\mu\nu}S)$ is of the form $S\partial R$ (or $R\partial S$, depending on choice); that is manifestly NOT always the divergence of any regular, NON-OS, tensor–for example if $S = (\partial R)^2$. The Lagrangian case is the one where OS is normal, because it also can be written as $\delta R/\delta g$, so for $S = \frac{\delta A}{\sqrt{-g}\delta R}|_g$ we recover (6).

The above illustrates sufficiency; Now for necessity: are there NON-Lagrangian identically conserved $X^{\mu\nu}(g_{\mu\nu}; R, \nabla^n R)$? In the vector cases, we saw that such (vector) terms existed because one merely algebraically contracted antisymmetric with symmetric indices, unlike the differential nature of the present problem. The lowest-level cases are easy: if $X^{\mu\nu}$ is R-independent, it must be proportional to $g^{\mu\nu}$, namely to a cosmological action $L = \sqrt{-g}$. Likewise, $X = X(g; R)$ obviously comes from an $L = \sqrt{-g}f(R)$. This is no longer so obvious when X does depend on derivatives of R. We must fall back on the projector basis of flat space conservation for inspiration. As we saw above, if the R-dependence is such that a scalar S is of the form $\frac{\delta A}{\sqrt{-g}\delta R}|_g$, then $\int d^2y\sqrt{-g}\delta R(y)/\delta g_{\mu\nu}(x) S(y)$ is the R-variation of an action and the total conserved current is its sum with $\frac{\delta A}{\sqrt{-g}\delta g_{\mu\nu}(x)}|_R$. The inspiration is of course (4), showing that the flat $O^{\mu\nu}$ must be extended to the curved one, plus the (natural) gR-term. We can now state the general problem in its tersest form, at least in the present approach. Are there NON-Lagrangian solutions of the local equation $\nabla_\nu(O^{\mu\nu}S + Z^{\mu\nu}) = 0$, where Z is a "normal" tensor, S a scalar and O the $\delta R/\delta g_{\mu\nu}$ of (4)? So far the only way a compensating "normal" Z can exist is for OS to have a normal divergence as discussed above. Although we have not succeeded in settling the question, it seems so intuitively simple that these lines may inspire a resolution. In higher D, there are a few novel wrinkles, such as the existence of 4-index $O^{\mu\nu\rho\lambda}$ from the variations of the–identically conserved–Einstein tensor, multiplied by a 2-tensor $S_{\rho\lambda}$ and of course the complications of dependence on the index-rich (covariant derivatives of) Ricci and Riemann tensors. These are all examples of the general superpotential $\partial_\alpha\partial_\beta H^{[\mu\alpha][\nu\beta]}$ mentioned earlier. Then there are Chern-Simons like operators in odd D, and finally for $D > 4$ the Lanczos-Lovelock [4, 5][2] actions' variations have no contributions from their curvature dependence, but rather entirely from their explicit metric dependence, in complete contrast with $D = 2$, where the latter is trivial. Let us now look, at the problem in a "perturbative" way.

Still in $D = 2$, our notation remains the same: we seek an identically conserved tensor $X_{\mu\nu}$

$$X_{\mu\nu} = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) S - \tfrac{1}{2} S R g_{\mu\nu} + Z_{\mu\nu} \,, \tag{7}$$

whose vanishing divergence means that

$$\nabla^\mu Z_{\mu\nu} = \tfrac{1}{2} S \partial_\nu R \,, \tag{8}$$

an equation that resembles that of a scalar-tensor model with R an independent scalar. In a weak field expansion about flat space,

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \quad \epsilon \ll 1 \,, \tag{9}$$

the leading term in (8) becomes

$$\partial^\mu Z^{(L)}_{\mu\nu} = \tfrac{1}{2} S^{(L)} \partial_\nu R^{(L)} \,, \tag{10}$$

in an obvious notation; all covariant derivatives are here replaced by partials. If $Z\,, S$ are polynomial in R and its derivatives, $Z^{(L)}_{\mu\nu}$ and $S^{(L)}$ would contain the least number of $R^{(L)}\,, \partial R^{(L)}\,, \partial^2 R^{(L)} \cdots$. In the flat limit, as no further R dependent terms can be generated from commuting two covariant derivatives, $R^{(L)}$ can thus be viewed as a metric-independent scalar, and $Z^{(L)}_{\mu\nu}$ and $S^{(L)}$ play the role of the stress tensor and scalar field equation respectively. While at flat space, any $\partial^\mu (O_{\mu\nu} S)$ is conserved, Eq.(10) does impose strong conditions on allowed S. This can be seen by applying the Euler-Lagrange operator with respect to $R^{(L)}$,

$$0 = \sum_{k=0}^{M} (-)^k \partial_{\mu_1} \ldots \partial_{\mu_k} \Big(\frac{\partial (S^{(L)} \partial_\nu R^{(L)})}{\partial \partial_{\mu_1} \ldots \partial_{\mu_k} R^{(L)}} \Big) \,, \tag{11}$$

where M is the maximal number, $\partial^M R^{(L)}$, of $R^{(L)}$ derivatives in $S^{(L)}$. This operators vanishing is a necessary, rather than sufficient, condition for conservation. One might wonder at applying it to a vector rather than to a scalar like a Lagrangian, but that is not an obstacle for us; we can imagine contracting it with some constant vector v_μ without altering the

[2] [4] merely noted the quadratic curvature topological invariants in $D = 4$, namely Gauss-Bonnet and its axial counterpart $\int d^4x \widetilde{R} R$, while [5] showed that the G-B action becomes dynamical for $D > 4$ and listed all such extensions.

conclusion. Explicitly, then we have

$$\sum_{k=0}^{M}(-)^k\sum_{p=0}^{k}\binom{k}{p}\Big(\partial_{\mu_{p+1}\cdots\mu_k}\frac{\partial S^{(L)}}{\partial\partial_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_k}R^{(L)}}\Big)\partial_{\mu_1\cdots\mu_p\nu}R^{(L)}=\sum_{k=0}^{M}\frac{\partial S^{(L)}}{\partial\partial_{\mu_1\cdots\mu_k}R^{(L)}}\partial_{\mu_1\cdots\mu_k\nu}R^{(L)}\,, \tag{12}$$

This can be recast algebraically into the compact form

$$\begin{aligned}0 &= \sum_{p=0}^{M}\partial_{\mu_1\cdots\mu_p\nu}R^{(L)}\,C^{\mu_1\cdots\mu_p}\,,\\ C^{\mu_1\cdots\mu_p} &= \sum_{k=p}^{M}(-)^k\binom{k}{p}\Big(\partial_{\mu_{p+1}\cdots\mu_k}\frac{\partial S^{(L)}}{\partial\partial_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_k}R^{(L)}}\Big)-\frac{\partial S^{(L)}}{\partial\partial_{\mu_1\cdots\mu_p}R^{(L)}}\,.\end{aligned} \tag{13}$$

Note that in order for $S^{(L)}$ to be the variation of some action, ALL coefficients $C^{\mu_1\cdots\mu_p}$ must vanish [6], while (13) only requires the seemingly weaker sum to do so. This is due to the fact that $C^{\mu_1\cdots\mu_p}$ are also composed by derivatives of $R^{(L)}$ and not completely independent of their prefactors in the sum. If one could solve (13) iteratively, first showing that $S^{(L)}$ arises from $R^{(L)}$ variation of some action $I^{(L)}$, one could then covariantize the latter by turning its ∂ into ∇, then vary it and denote the outcome by $Z^{(1)\mu\nu}$ and $S^{(1)}$. As $Z^{(1)\mu\nu}$ and $S^{(1)}$ arise from an action, they automatically satisfy

$$\nabla^\mu Z^{(1)}_{\mu\nu}=\tfrac{1}{2}S^{(1)}\partial_\nu R\,. \tag{14}$$

Thus by taking the difference of the equation above and (8), we have

$$\nabla^\mu(Z_{\mu\nu}-Z^{(1)}_{\mu\nu})=\tfrac{1}{2}(S-S^{(1)})\partial_\nu R\,. \tag{15}$$

Repeating the process starting from (9), we obtain an equation similar to (10) but now the leading terms of $Z_{\mu\nu}-Z^{(1)}_{\mu\nu}$ and $S-S^{(1)}$ in the weak field expansion would depend on higher power of $R^{(L)}$ and its partial derivatives. Assuming that the original $X^{\mu\nu}$ is a polynomial in R and its covariant derivatives of some finite degree, the procedure must terminate in a finite number of steps. We have not yet been able to find an all-orders solution, but it is straightforward to find all solutions of (13) for which S depends on at most two derivatives of R, which in turn means that all $X^{\mu\nu}$ depending on sixth derivatives of the metric are indeed Lagrangian. This is a several orders improvement on previous results [1]. [A counter-example to our hypothesis would most likely be provided by solutions of (13) for which not all C vanish.]

A different approach to the problem would be to establish that $X^{\mu\nu}$ obeys the integrability condition

$$\frac{\delta\sqrt{g}X^{\mu\nu}(x)}{\delta g_{\rho\sigma}(y)}=\frac{\delta\sqrt{g}X^{\rho\sigma}(y)}{\delta g_{\mu\nu}(x)}\,, \tag{16}$$

namely, $\frac{\delta\sqrt{g}X^{\mu\nu}(x)}{\delta g_{\rho\sigma}(y)}$ is a formally self-adjoint differential operator comprised of the Riemann tensor and its covariant derivatives. The integral form of (16) can be expressed as

$$\int_M \big(\delta_2(\sqrt{g}X^{\mu\nu})\delta_1 g_{\mu\nu} - \delta_1(\sqrt{g}X^{\mu\nu})\delta_2 g_{\mu\nu}\big) = 0\,, \tag{17}$$

for arbitrary variations $\delta_1 g$ and $\delta_2 g$. To approach our goal (17), first define the functional

$$A_X(Y) := \int_M \sqrt{g}X^{\mu\nu}Y_{\mu\nu}\,, \tag{18}$$

in which the tensor $Y_{\mu\nu}$ has finite support on M. Conservation of $X^{\mu\nu}$ implies this functional vanishes when Y is the Lie derivative of the metric with respect to a compactly supported vector field:

$$A_X(\mathcal{L}_\xi g) := 2\int_M \sqrt{g}\,X^{\mu\nu}\nabla_\mu\xi_\nu = 0\,. \tag{19}$$

Here $\mathcal{L}$ denotes the Lie derivative and we have used that $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$. Hence, the variation of $A(\mathcal{L}_\xi g)$ also vanishes so that

$$\delta_1 A_X(\mathcal{L}_\xi g) = \int_M \big(\delta_1(\sqrt{g}X^{\mu\nu})\mathcal{L}_\xi g_{\mu\nu} + \sqrt{g}X^{\mu\nu}\mathcal{L}_\xi \delta_1 g_{\mu\nu}\big) = 0\,. \tag{20}$$

The functional $A_X(Y)$ is diffeomorphism-invariant, so a variation $\delta_2 A_X(\delta_1 g)$ with $\delta_2\delta_1 g = \mathcal{L}_\xi(\delta_1 g)$ also vanishes. This gives

$$\int_M \big(\delta_2(\sqrt{g}X^{\mu\nu})\delta_1 g_{\mu\nu} + \sqrt{g}X^{\mu\nu}\mathcal{L}_\xi(\delta_1 g)\big) = 0\,. \tag{21}$$

The difference of the above two displays

$$\int_M \big(\delta_2(\sqrt{g}X^{\mu\nu})\delta_1 g_{\mu\nu} - \delta_1(\sqrt{g}X^{\mu\nu})\delta_2 g_{\mu\nu}\big)\Big|_{\delta_2 g = \mathcal{L}_\xi g} = 0\,. \tag{22}$$

Were $\delta_2 g$ not restricted to variations of the form $\mathcal{L}_\xi g$, this would complete the proof. However, in $D = 2$, this restriction is not particularly strong: one can always decompose $Y_{\mu\nu}$ as

$$Y_{\mu\nu} = \nabla_\mu Y_\nu + \nabla_\nu Y_\mu + \frac{1}{2}g_{\mu\nu}Y\,, \tag{23}$$

where the vector Y_μ and scalar Y are uniquely solved via the system

$$\big(\Delta + \tfrac{1}{2}R\big)Y_\nu = \nabla^\mu\big(Y_{\mu\nu} - \frac{1}{2}g_{\mu\nu}Y^\rho_\rho\big)\,, \quad Y = Y_{\mu\nu} - \frac{1}{2}g_{\mu\nu}Y^\nu_\nu\,. \tag{24}$$

Thus, in general $\delta_2 g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} + \frac{1}{2}\rho g_{\mu\nu}$ for some vector ξ^μ and scalar ρ. Applying the decomposition (23) to δg in (17) and comparing with (22), we conclude that in $D = 2$, the integrability of $X^{\mu\nu}$ with respect to the general metric variation is reduced to showing that the trace part of $X^{\mu\nu}$ is integrable with respect to the variation of the trace part of the metric. Evidently, if $X^{\mu\nu}$ is trace-free, the proof is completed. When $X^{\mu\nu}$ is pure trace, the divergence-free condition constrains such an $X^{\mu\nu}$ to be a constant times the metric. The generic, three-component case is thus the only relevant one; to date, we have not succeeded in this "integral" approach either.

4 Comments

We have reviewed and summarized the current standing of a century-old conjecture-validity of the converse of Noether's theorem: are all identically conserved geometrical 2-tensors the metric variations of some invariant action? This intuitively attractive proposition has proved remarkably recalcitrant to date, although some relatively minor perturbative (in powers of derivatives of curvature) results are established-notably to sixth derivative order here. A number of quite different approaches have been pursued and we have summarized them by concentrating on the simplest curved space dimension, $D = 2$, where the problem is most clearly stated without the obscuring higher D index proliferation. A proof (or indeed disproof) in $D = 2$ all but guarantees the same for all D. There are important physical consequences of this seemingly formal question to real physics: Of the many attempts to go beyond GR, addition of non-Lagrangian terms on the "left hand side" of the field equations requires them to be identically conserved, since both $G_{\mu\nu}$ and the (Lagrangian-based) matter stress tensors on their mass shell are. This would close the floodgates to a wide range of speculation. [Conversely, in the unlikely event that there are such tensors, a whole new field would open up!] In string theory, one always obtains $DX{=}0$ equation for the target space fields from the world-sheet BRST invariance. So if our conjecture is true, it also implies that all stringy gravity models are lagrangian. We have used locality as a physical demand. If that is lifted, it is trivial to provide counter-examples, albeit non-symmetric ones, such as $X^{\mu\nu} = (\nabla^\mu \Box^{-1} \nabla^\nu - g^{\mu\nu})S$ (conserved on one index). Finally, we have not investigated the recently proposed [7, 8] amusing $D = 3$ (infinity of [8]) models whose X-divergences do NOT vanish identically, but do so on shell.

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