

Similarity Solution for Cylindrical Magnetohydrodynamic Blast Waves

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A similarity solution is obtained for the flow behind a very strong (in the hydrodynamic sense) cylindrical magnetohydrodynamic shock wave produced by the sudden release of energy along a line of infinite extent in a plasma. The plasma is assumed to be an ideal gas with infinite electrical conductivity, and to be permeated by the azimuthal magnetic field of a line current. It is shown that it is of critical importance to take into account the ambient magnetic pressure, no matter how small. It is found that, to preserve similarity, the external circuit is required to maintain a constant axial current; this result also appears in the related problem, treated by Greenspan, where the ambient plasma is nonconducting. This boundary condition is shown to have some interesting consequences, especially with regard to the energy content of the system. The dependence of the shock speed on the explosive energy is determined as a function of the ambient magnetic field both for the present case and for Greenspan's case, and interesting differences are noted. Other differences between the two cases are also discussed.

I. INTRODUCTION

IT is well known that a similarity solution exists for the flow behind a very strong cylindrical shock wave produced by the sudden release of a finite amount of energy per unit length, along a straight line of infinite extent in a uniform gas.¹ A corresponding magnetohydrodynamic problem also has the same similarity.²⁻⁴ Here the shocked gas is an electrically conducting plasma which interacts with an applied magnetic field. The gas is assumed to be initially in a purely azimuthal magnetic field produced by a constant line current, the return for which is a concentric cylindrical conductor of very large radius. The flow is again produced by the sudden release of energy along the axis (by exploding the wire carrying the current, for example).

The flows considered here are those for which the conductivity of the shocked plasma is infinite. For this class of flows, as Greenspan⁴ has pointed out, two limiting cases arise. In the first case, the quiescent gas has zero conductivity but the gas behind the shock has infinite conductivity. The magnetic field across the shock is continuous and

the boundary conditions at the shock are the ordinary hydrodynamic jump conditions for a strong shock. This situation arises if the magnetic shock transition zone is much wider than the viscous transition zone, which is replaced by a shock jump. The entire effect of the magnetic field, in this case, is contained in its interaction with the flow behind the shock. This limiting case, hereinafter referred to as "modified hydrodynamic," has been discussed by Greenspan. In the second limiting case, the magnetic field is discontinuous across the shock, and the boundary conditions at the shock are those appropriate to a magnetohydrodynamic shock. This situation results if both the magnetic and viscous transition zones are very thin and if the quiescent gas is also a perfect conductor (as a result of pre-ionization, for example). In this case, hereinafter referred to as "pure magnetohydrodynamic," the magnetic field affects conditions at the shock itself as well as those behind it.

In this paper, the pure magnetohydrodynamic flow is considered in detail. This case was first discussed by Pai,³ but the results obtained below are found to disagree significantly with his. The discrepancy is shown to arise from an approximation made by Pai to the pressure jump condition at the shock. The particular approximation was to neglect the magnetic pressure in front of the shock. While it is true that, for small values of this magnetic pressure, the errors introduced at the shock are small, nevertheless the inclusion of such a pressure,

¹ S. C. Lin, *J. Appl. Phys.* **25**, 54 (1954).

² V. P. Korobeinikov, *Doklady Akad. Nauk S.S.S.R.* **121**, No. 4 (1958).

³ S. I. Pai, "Cylindrical Shock Waves Produced by Instantaneous Energy Release in Magneto-Gas Dynamics," Air Force Office Scientific Research, TN 58-215 (1958).

⁴ H. P. Greenspan, *Phys. Fluids* **5**, 225 (1962).

no matter how small, produces a profound effect at the axis and, concomitantly, in the energy content of the gas.⁵

The question of the energy content of the gas is also considered below, for the modified hydrodynamic as well as for the pure magnetohydrodynamic case. It is shown that in both cases, in contrast to the ordinary blast-wave solution, the increment in the total (gas dynamic plus magnetic) energy of the gas is not equal to the energy supplied by the explosion. The two cases differ markedly, however, in that in the pure magnetohydrodynamic case the increment is infinite, whereas in the modified hydrodynamic case it is finite but less than the energy supplied by the explosion. The difference between the increment and the explosion energy represents a concentration of magnetic energy, in one case supplied by, and in the other case removed by the external circuit in maintaining the constant current required by the similarity solution. It is possible in both cases to calculate the contribution from the external circuit, and thereby to relate the shock speed and the mechanical energy in the gas to the energy supplied by the explosion. The analysis reveals an interesting difference between the two cases. In both cases only part of the energy supplied by the explosion appears as mechanical energy, the remainder being converted to magnetic energy. In the pure magnetohydrodynamic case, this converted part of the explosion energy appears in the compressed magnetic field behind the shock, while in the modified hydrodynamic case, it is delivered instead to the external circuit.

It seems reasonable to believe that, despite the divergence mentioned above, the results obtained have physical significance. The divergence appears because of the unphysical assumption of a dimensionless axial conductor. One would expect that in any actual experiment, where the axial conductor has a finite radius, the true flow would approach that given by the similarity solution at distances from the axis much greater than the radius of the conductor. One would also expect that any relations obtained between finite quantities, such as the dependence of the shock speed on the explosion energy or the ratio of mechanical energy to explosion energy, would also be approximately valid. These are questions which, of course, must ultimately be decided experimentally.

⁵ In a related problem, brought to the authors' attention by the reviewer, J. D. Murray [Mathematika 8, 99 (1961)] has also included the effect of an ambient magnetic pressure.

II. FORMULATION OF THE PROBLEM

A solution of the magnetohydrodynamic equations is desired for the flow behind a cylindrical shock wave produced by the sudden release of a finite amount of energy E_0 per unit length, along an axis of infinite extent in a plasma. The plasma is assumed to be an ideal gas with infinite electrical conductivity; viscosity and heat conduction are neglected. An azimuthal magnetic field is assumed to exist initially in the plasma. As a consequence of the assumption of infinite conductivity, the magnetic field in the plasma remains azimuthal. Under the assumptions made, all quantities are functions only of the radius r and time t , and the motion of the gas is governed by the following equations:

$$r \partial \rho / \partial t + (\partial / \partial r)(r \rho u) = 0 \quad (\text{continuity}), \quad (1a)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{1}{\rho} \frac{\partial}{\partial r} \left(\frac{B_\theta^2}{2\mu} \right) + \frac{B_\theta^2}{\mu \rho r} = 0$$

(momentum), (1b)

$$\partial B_\theta / \partial t + (\partial / \partial r)(u B_\theta) = 0 \quad (\text{induction}), \quad (1c)$$

$$(\partial / \partial t + u \partial / \partial r)(p / \rho^\gamma) = 0 \quad (\text{entropy}). \quad (1d)$$

In these equations, p is the gas pressure, ρ is the density, u the radial velocity, and B_θ the azimuthal magnetic field. The specific-heat ratio γ is assumed constant and the magnetic permeability μ is taken to be that of free space (mks units are used throughout). The gas is assumed to be initially in a uniform state p_0, ρ_0 . The initial azimuthal magnetic field is assumed to be that produced by a constant current I_0 , flowing along the axis.

Entering into the initial and boundary conditions of the problem are certain dimensional constants; these are the initial pressure p_0 , the initial density ρ_0 , the initial axial current I_0 , and the energy E_0 liberated per unit length. Since the dimensions of the quantity μI_0^2 are the same as those of E_0 , only three of the four constants are dimensionally independent. If, in addition, the liberated energy E_0 is large enough to produce a very strong shock, the initial pressure p_0 may be neglected, and only two constants with independent dimensions enter into the problem— ρ_0 and E_0 . As in the ordinary blast-wave case, the only combination of length and time that can be formed from these constants is of the form

$$(\text{length})^2 / \text{time} \sim (E_0 / \rho_0)^{1/2}.$$

It now follows from general considerations that the system of equations, (1), possesses a similarity

solution; this solution is of the form

$$\rho = \rho_0 \mathcal{R}(\eta), \tag{2a}$$

$$u = (r/t) \mathcal{U}(\eta), \tag{2b}$$

$$p = \rho_0 (r/t)^2 \mathcal{P}(\eta), \tag{2c}$$

$$B_\theta = (\mu \rho_0)^{1/2} (r/t) \mathcal{B}(\eta), \tag{2d}$$

where \mathcal{R} , \mathcal{U} , \mathcal{P} , and \mathcal{B} are non-dimensional functions of a nondimensional variable. The only nondimensional variable which can be formed from the variables r and t and the dimensional constants of the problem is the variable

$$\eta = r^2/c^*t, \tag{3}$$

where c^* is some characteristic parameter with the dimensions of (length)²/time.⁶

It is convenient to introduce as independent variables, instead of r and t , the stream function ψ defined by

$$\partial\psi/\partial r = (\rho/\rho_0)r, \quad \partial\psi/\partial t = -(\rho/\rho_0)ur, \tag{4}$$

and the variable ϕ defined as

$$\phi = 2\psi/r^2. \tag{5}$$

Physically, the value of ϕ corresponding to any (r, t) is the ratio of the mass between the axis and the radius r at time t to the mass initially in the same volume. The value of ϕ at the shock, therefore, is $\phi_s = 1$. A line of constant ϕ is also a line of constant η in the (r, t) plane.

In terms of the variables (ψ, ϕ) , Eqs. (2) take the form

$$\rho = \rho_0 \sigma(\phi), \tag{6a}$$

$$u = c^*U(\phi)/\psi^{1/2}, \tag{6b}$$

$$p = \rho_0 c^{*2} P(\phi)/\psi, \tag{6c}$$

$$B_\theta = (\mu \rho_0)^{1/2} (c^*/\psi^{1/2}) \beta(\phi), \tag{6d}$$

where the quantities σ , U , P , and β are nondimensional functions of the single (nondimensional) variable ϕ . The parameter c^* in Eq. (6) is related to the speed of the shock; in general,

$$c = Ac^*/\psi^{1/2}, \tag{7}$$

where c is the shock speed and A is an arbitrary constant. In the present case, it is convenient to choose $A = 1$.

It is necessary for the initial conditions to be compatible with the assumed form, (6), of the solution. The plasma is assumed to be initially

⁶ The quantity η defined by Eq. (3) is proportional to the square of that defined by Greenspan.

at rest, to have a uniform density ρ_0 , a uniform pressure p_0 , and to be permeated by the azimuthal magnetic field of a constant line current I_0 . The first two conditions are obviously expressible in the desired form; viz.,

$$U_0 = 0 \tag{8a}$$

and

$$\sigma_0 = 1, \tag{8b}$$

respectively, while the last condition can be written in the form

$$\beta_0 = (\mu I_0^2/8\pi^2 \rho_0 c^{*2})^{1/2}. \tag{8c}$$

For the condition of uniform pressure to be compatible with (6), however, it is necessary to assume that the initial pressure is negligible; i.e.,

$$p_0 = 0. \tag{8d}$$

This is the usual blast wave assumption of a very strong shock.

The Jacobian of the transformation defined by Eqs. (4) and (5) is

$$J = \partial(\psi, \phi)/\partial(r, t) = -2\sigma u \phi. \tag{9}$$

Since the Jacobian is not zero (except at certain points), the transformation defines a one-to-one mapping between (r, t) and (ψ, ϕ) . One may, in fact, obtain an integral expression for the stream function ψ . The differential equation

$$\frac{\partial t}{\partial \psi} = -\frac{1}{J} \frac{\partial \phi}{\partial r} = \frac{1}{2^{\frac{1}{2}} c^*} \frac{\sigma - \phi}{\sigma \phi^{\frac{1}{2}} U} \tag{10}$$

is easily integrated to yield

$$\psi = 2^{\frac{1}{2}} c^* t \sigma \phi^{\frac{1}{2}} U / (\sigma - \phi). \tag{10a}$$

Finally, it is possible to express the variable η as a function (implicit) of ϕ ; by combining Eqs. (3), (5) and (10), one obtains

$$\eta = 2^{\frac{3}{2}} \sigma U / \phi^{\frac{1}{2}} (\sigma - \phi). \tag{11}$$

If one now substitutes Eq. (6) into Eqs. (1) and transforms the partial derivatives to derivatives with respect to ϕ , the equations of motion become

$$2\phi^{\frac{1}{2}} U \frac{d}{d\phi} \left(\frac{\sigma}{\phi^{\frac{1}{2}}} \right) + \frac{\sigma^2 U}{\phi} + 2(\phi - \sigma)\sigma \frac{dU}{d\phi} = 0 \tag{continuity}, \tag{12a}$$

$$\phi^2 U \frac{dU}{d\phi} + \frac{\phi}{\sigma} (\phi - \sigma) \frac{d}{d\phi} (P + \frac{1}{2}\beta^2) + (P + \frac{1}{2}\beta^2) - \frac{\phi \beta^2}{2\sigma} = 0 \text{ (momentum)}, \tag{12b}$$

$$2\phi U \frac{d\beta}{d\phi} + \frac{\beta\sigma U}{\phi} + 2(\phi - \sigma)\beta \frac{dU}{d\phi} = 0$$

(induction), (12c)

$$(d/d\phi)(P/\sigma^\gamma) = 0 \quad \text{(entropy).} \quad (12d)$$

A solution of Eqs. (12) is desired which satisfies the boundary conditions appropriate to a very strong magnetohydrodynamic shock. In terms of physical variables, these conditions may be written:

$$\rho_0/\rho_s = 1 - u_s/c \quad \text{(continuity),} \quad (13a)$$

$$p_s + (1/2\mu)(B_s^2 - B_0^2) = \rho_0 u_s c \quad \text{(momentum),} \quad (13b)$$

$$B_0/B_s = 1 - u_s/c \quad \text{(induction),} \quad (13c)$$

$$[c^2 - \frac{1}{2}(\gamma + 1)cu_s](c - u_s)^2 = (B_0^2/\mu\rho_0)$$

$$\cdot [c^2 - \frac{1}{2}(\gamma + 1)cu_s + \frac{1}{2}\gamma u_s^2] \quad \text{(wave speed),} \quad (13d)$$

where the subscript *s* denotes quantities immediately behind the shock and *B*₀ is the value of *B*_{*s*} immediately ahead of the shock. The quantity *c* is again the shock speed, given by (7) with *A* = 1. The wave-speed equation (13d) is derived by combining the usual energy equation with the continuity, momentum, and induction equations.

It should be noted that the magnetic pressure ahead of the shock has been retained in the momentum equation (13b). Although it is necessary to neglect the ambient gas pressure in this equation to preserve similarity, the same is not true of the ambient magnetic pressure which does have the proper form. This term, which was neglected by Pai, can therefore be retained and, as will be shown below, should be retained.

In terms of the nondimensional variables defined by Eq. (6), the shock relations become:

$$1/\sigma_s = 1 - U_s, \quad (14a)$$

$$P_s + \frac{1}{2}\beta_0^2(\sigma_s^2 - 1) = U_s, \quad (14b)$$

$$\beta_0/\beta_s = 1 - U_s, \quad (14c)$$

$$[1 - \frac{1}{2}(\gamma + 1)U_s](1 - U_s) = \beta_0^2[1 - (\frac{1}{2}\gamma)U_s]. \quad (14d)$$

The quantity β_0 is given by Eq. (8c), and is,

physically, the ratio of the Alfvén speed in the gas immediately ahead of the shock to the shock speed (Alfvén number).

If β_0 is considered a parameter, Eqs. (14) define a one-parameter family of shock curves. For a given value of β_0 , *U*_{*s*} is first determined as a root of the quadratic equation (14d). One then obtains σ_s and β_s directly from Eqs. (14a) and (14c), respectively, and then *P*_{*s*} from Eq. (14b).

Three algebraic integrals of the four equations of motion may be obtained. The entropy equation (12d) has the immediate integral

$$P/\sigma^\gamma = a = \text{const} \quad (15)$$

from which it follows that the entropy is constant along a streamline. A second integral may be obtained by combining Eqs. (12a) and (12c); namely

$$\beta^2\phi/\sigma^2 = 2b = \text{const} \quad (16)$$

which expresses the fact that the field is “frozen” into the fluid—a well-known consequence of the assumption of infinite conductivity. A final algebraic integral, obtained by combining Eqs. (12a) and (12b), is

$$\frac{1}{\sigma} \left(\frac{P}{\gamma - 1} + \frac{1}{2}\beta^2 + \frac{1}{2}\sigma U^2 \right) + \frac{\phi - \sigma}{\sigma\phi} (P + \frac{1}{2}\beta^2) = k = \text{const.} \quad (17)$$

This so-called “energy integral” results from the fact that, with the assumed similarity, the total energy is constant not only for the entire flow field, but also between any two similarity lines.

To establish the connection between the variables used here and those of Greenspan, one additional relation may be derived; from Eqs. (11) and (14a), the value of η at the shock is $\eta_s = 2^{\frac{1}{2}}$. It then follows from Eq. (3) that the location of the shock is at $r_s(t) = 2^{\frac{1}{2}}(c^*t)^{\frac{1}{2}}$.

The three integrals (15), (16), and (17) may now be used to eliminate the functions *P*, β , and *U* from Eqs. (12). The system (12) is thereby reduced to the single first-order equation

$$\frac{d\xi}{d\sigma} = \frac{\xi}{\sigma(1 - \xi)} \frac{a\sigma^{\gamma-1}[\xi - \gamma/(\gamma - 1)][(\gamma - 2)\xi - \gamma] - (1 + \xi)[k + b\xi(\xi - 2)]}{a\sigma^{\gamma-1}[2\xi - \gamma/(\gamma - 1)] + b\xi(3\xi - 4) + k}, \quad (18)$$

where

$$\xi \equiv \sigma/\phi. \quad (19)$$

If neither *a* nor *b* is zero, no further analytic inte-

gration appears possible. In the general case, then, the problem is reduced to the numerical integration of a single ordinary equation.

III. GENERAL REMARKS

A few general remarks can be made in connection with Eq. (18). Both cases mentioned in the Introduction, i.e., pure magnetohydrodynamic and modified hydrodynamic, as well as the ordinary blast wave, appear as special cases of Eq. (18). The solution, in each case, is that integral curve of Eq. (18) which starts at the point appropriate to the particular shock and terminates at the point which corresponds to the axis. [Each point on the integral curve maps into a line of constant η in the (r, t) plane.] In every case, the point corresponding to the shock is a regular point of the differential equation. Since only a single integral curve may pass through a regular point, the solutions obtained in this manner are unique.

A point of central importance in the analysis is the dependence of the solution on conditions at the shock. The boundary conditions at the shock enter into Eq. (18) through the constants $a, b,$ and $k,$ which are determined by evaluating the integrals (15), (16), and (17) at the shock. Because of the nonlinearity of Eq. (18), these constants affect the entire flow behind the shock in a very complicated way. The most striking effect, however, is on conditions at the axis. An axis of symmetry always corresponds to a singular point of such a differential equation, i.e., to a point at which both the numerator and denominator vanish. The behavior of physical quantities at the axis is very sensitive to the location of this singular point, which, in turn, is determined by the relationships among the constants $a, b,$ and k prescribed by conditions at the shock. As will be seen below, this sensitivity to the location of the singular point manifests itself in a profound difference at the axis, especially in the pressure distribution, between the pure magnetohydrodynamic and modified hydrodynamic cases.

IV. ORDINARY BLAST-WAVE SOLUTION

For purposes of illustration and comparison, the ordinary blast-wave solution in the present variables is briefly sketched below. The blast-wave solution is recovered in the limit of zero magnetic field. The appropriate shock conditions are Eqs. (14) with $\beta_0 = 0.$ These, applied to Eqs. (15) through (17), yield $a = [2/(\gamma + 1)][(\gamma - 1)/(\gamma + 1)]^\gamma, b = k = 0;$ the integrals (15) and (17) therefore become

$$P = [2/(\gamma + 1)][(\gamma - 1)/(\gamma + 1)]^\gamma \sigma^\gamma \tag{20}$$

and

$$\frac{1}{2}\sigma U^2 + [\gamma/(\gamma - 1) - \xi]P = 0, \tag{21}$$

respectively, with ξ defined by Eq. (19).

The singular point of Eq. (18) corresponding to the axis in this case is the saddle point $\sigma = 0, \xi = \gamma/(\gamma - 1).$ The behavior of physical quantities in the neighborhood of the axis can be deduced from an analysis of Eq. (18) in the neighborhood of the singular point. However, in this case Eq. (18) can be integrated analytically; the solution satisfying the boundary conditions at the shock is

$$\sigma = \left(\frac{\gamma - 1}{2}\right)^{2/(2-\gamma)} \left(\frac{\gamma + 1}{\gamma - 1}\right)^{(\gamma+1)/\gamma} \cdot \left[\frac{(\gamma - 1)\xi - \gamma}{\xi}\right]^{1/\gamma} [(2 - \gamma)\xi + \gamma]^{2/(2-\gamma)}. \tag{22}$$

Equations (20)–(22) give the complete solution, in parametric form, of the equations of motion. The parameter ξ decreases monotonically from the value $(\gamma + 1)/(\gamma - 1)$ at the shock to the value $\gamma/(\gamma - 1)$ at the axis.

It can be seen from these equations that P and U both become zero at the axis, with $P \sim \sigma^\gamma$ and $U \sim \sigma^{\frac{1}{2}(2\gamma-1)}.$ Since $\phi = \sigma \xi \sim \sigma$ at the axis, it follows from Eq. (10a) that $\psi \sim \sigma^\gamma t,$ and consequently from Eqs. (6) for the physical variables, that $\rho \rightarrow 0, u \rightarrow 0,$ and $pt \rightarrow \text{const}$ at the axis. These are well-known properties of the blast-wave solution.

V. PURE MAGNETOHYDRODYNAMIC CASE

In the pure magnetohydrodynamic case, the boundary conditions at the shock are given by Eqs. (14) with $0 < \beta_0 < 1.$ (The density ratio across the shock decreases with increasing $\beta_0,$ becoming unity when $\beta_0 = 1.)$ These shock conditions, applied to Eqs. (16) and (17), yield

$$b = k = \frac{1}{2}\beta_0^2. \tag{23}$$

When these values of b and k are substituted into Eq. (18), it is found that a higher-order singular point $\sigma = 0, \xi = 1$ corresponds to the axis. The parameter ξ decreases monotonically from its value at the shock to its value at the axis.

An interesting conclusion can already be drawn from the location of the singular point. It is easy to show, from the various defining relationships and the integral (16) with $b = \frac{1}{2}\beta_0^2,$ that the magnetic field behind the shock is given by

$$B_0(r, t) = (\mu I_0/2\pi r)\xi(r, t). \tag{24}$$

The factor multiplying the parameter ξ in Eq. (24) is clearly the initial magnetic field. Thus, as might

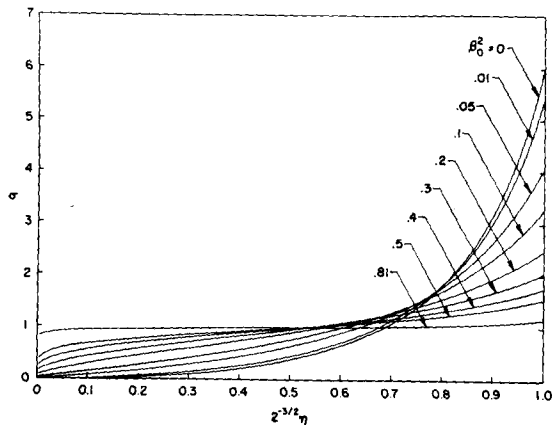


FIG. 1. Dependence of density, in units of ρ_0 , on nondimensional variable η , for various values of $\beta_0^2 = \mu I_0^2 / 8\pi^2 \rho_0 c^{*2}$; $\gamma = 1.4$.

be expected, the magnetic field is everywhere compressed between the shock and the axis. At the axis, however, the magnetic field maintains its initial value. This implies that, to preserve similarity, the external circuit must maintain a constant current I_0 . (The same result was obtained by Greenspan in the modified hydrodynamic case.) One might expect *a priori* that a flow with the same similarity would be produced if the current in the wire were instantaneously increased to a new constant value I_1 , since such a boundary condition introduces no new dimensional constants and is expressible in a form compatible with the similarity. This, however, is apparently not the case; only the maintenance of a constant current preserves similarity.

Because of the high-order singularity of Eq. (18) at $\sigma = 0$, $\xi = 1$, the dependence of the other variables on ξ in the neighborhood of the axis is most easily determined from the continuity equation (12a). An analysis of this equation shows that, in the neighborhood of $\sigma = 0$, $\xi = 1$,

$$U \cong U_0 \exp \{ -[2/(\gamma - 1)][1/(\xi - 1)] \}, \quad (25)$$

where U_0 is a constant. It then follows from equations (17), (10a), and (11) that

$$\sigma \cong \sigma_1 (\xi - 1)^{2/(\gamma - 1)}, \quad (26a)$$

$$\psi \cong \psi_0 t (\xi - 1)^{(2 - \gamma)/(\gamma - 1)} \cdot \exp \{ -[2/(\gamma - 1)][1/(\xi - 1)] \}, \quad (26b)$$

$$\eta \cong \eta_0 (\xi - 1)^{-\gamma/(\gamma - 1)} \cdot \exp \{ -[2/(\gamma - 1)][1/(\xi - 1)] \}, \quad (26c)$$

where $\sigma_1 = [(\gamma - 1)b/a]^{1/(\gamma - 1)}$, and ψ_0 and η_0 are constants. The last of these equations verifies that

this singular point indeed corresponds to the axis ($\eta = 0$).

The behavior of the physical variables in the neighborhood of the axis can now be deduced from the defining equations (6) and the integrals (15) and (16). The results are:

$$\rho \cong \rho_0 \sigma_1 (-\ln \eta^{\frac{1}{2}(\gamma - 1)})^{-2/(\gamma - 1)}, \quad (27a)$$

$$u \cong (c^*/t)^{\frac{1}{2}} [1/(\gamma - 1)] \eta^{\frac{1}{2}} (-\ln \eta)^{-1}, \quad (27b)$$

$$p \cong \frac{\rho_0 c^{*2}}{t} \frac{8b}{(\gamma - 1)} \eta^{-1} (-\ln \eta)^{-2}, \quad (27c)$$

$$B^2/2\mu \cong (\rho_0 c^*/t) 2b \eta^{-1}. \quad (27d)$$

As in the ordinary blast-wave solution, the density and velocity approach zero at the axis. The dependence on η , however, is quite different in the two cases, the logarithmic dependence being peculiar to the pure magnetohydrodynamic case. The density, in particular, approaches zero much more slowly in the pure magnetohydrodynamic case, especially for the larger values of β_0 (Fig. 1). The pressure is markedly different, becoming infinite at the axis rather than asymptotically constant as in the blast-wave solution (Fig. 3). This infinity will be discussed further in connection with the energy content of the gas.

The results of the numerical integration of Eq. (18) for various values of β_0^2 , with $\gamma = 1.4$, are shown in Figs. 1-4. It is clear that the presence of a magnetic field ahead of the shock has an important effect on conditions at the shock (and everywhere behind it) as well as those at the axis. The compression

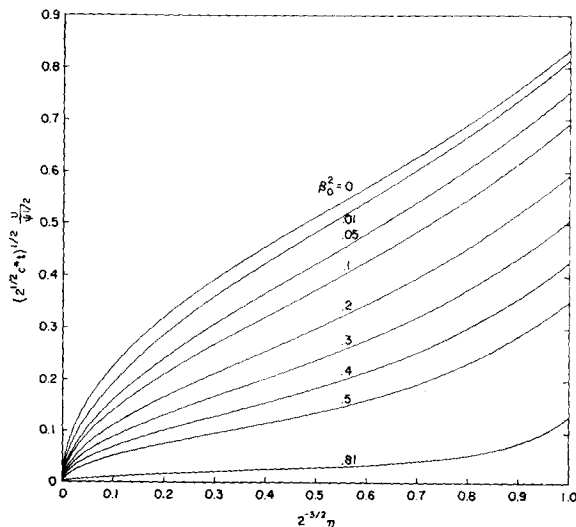


FIG. 2. Dependence of velocity, in units of $2^{-1/2} c^* t^{-1/2}$ on nondimensional variable η , for various values of $\beta_0^2 = \mu I_0^2 / 8\pi^2 \rho_0 c^{*2}$; $\gamma = 1.4$.

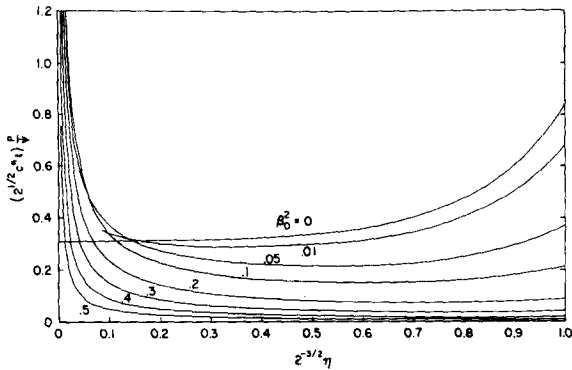


FIG. 3. Dependence of gas pressure, in units of $2^{-1}\rho_0 c^* t^{-1}$, on nondimensional variable η , for various values of $\beta_0^2 = \mu I_0^2 / 8\pi^2 \rho_0 c^{*2}$; $\gamma = 1.4$.

across the shock decreases from its strong-shock value of $[(\gamma + 1)/(\gamma - 1)]$ for zero magnetic field ($\beta_0 = 0$) to a value of unity for $\beta_0 = 1$. The velocity and gas pressure at the shock vary accordingly, decreasing from their strong-shock values to zero over the range of β_0 . (The similarity solution is valid, of course, only so long as the shock remains a strong shock in the hydrodynamic sense, that is, so long as the gas pressure ahead of the shock is negligible compared to that behind it.) The magnetic pressure at the shock, on the other hand, exhibits a maximum at a value of β_0^2 between 0.3 and 0.4. The appearance of such a maximum was noted in the inverse pinch effect.⁷ The existence of this maximum depends only on the jump conditions at the shock, which are the same as in the inverse pinch.

The importance of including the ambient magnetic pressure in the shock relations is illustrated by comparison of the above results with those of Pai, where this term was neglected. In the first place, if the ambient magnetic pressure is neglected in Eq. (14b), the Alfvén number, β_0 , is allowed to become arbitrarily large. This is in contradiction to the limits $0 < \beta_0 < 1$ obtained above for a pure magnetohydrodynamic shock. In the second place, all hydrodynamic quantities at the shock become independent of β_0 , always attaining their strong-shock values. It can be seen from Figs. 1-3 that this is a reasonable approximation only for very small values of β_0 . The same error is introduced into the magnetic field ratio across the shock which, with Pai's conditions, is always $(\gamma + 1)/(\gamma - 1)$ but which, with Eqs. (14), decreases from this value to unity at $\beta_0 = 1$. Finally, and what is responsible for an important discrepancy even for the smallest values of β_0 , Pai's conditions lead to

⁷ C. Greifinger, and J. D. Cole, Phys. Fluids 4, 527 (1961).

different values for the constants a , b , and k of Eqs. (15)-(17), and thereby alter the location of the singular point of Eq. (18) which corresponds to the axis. With Pai's conditions, the singular point is located at $\sigma = 0$, $\xi = (\gamma + 1)/(\gamma - 1)$ rather than at $\sigma = 0$, $\xi = 1$ (which is no longer even a singular point of the differential equation). Along the integral curve passing through the shock, the variable ξ decreases from its value of $(\gamma + 1)/(\gamma - 1)$ at the shock to some minimum value, and then increases back to $(\gamma + 1)/(\gamma - 1)$ at the axis. Since the magnetic field behind the shock is given by Eq. (24), the location of the singular point implies that the current in the wire has jumped to $(\gamma + 1)/(\gamma - 1)$ times its initial value, a solution in contradiction to the requirement of a constant current when the ambient magnetic pressure is retained. The difference in location of the singular point also leads to discrepancies in the behavior of the other physical variables near the axis. In Pai's solution, the density and velocity go to zero at the axis, but with a dependence on η which is quite different from that given by Eqs. (27a) and (27b). The most dramatic difference, however, is in the pressure which, rather than becoming infinite, as given by Eq. (27c), goes to zero at the axis! The solution in the neighborhood of the axis thus shows a sensitivity to the shock conditions which is more than a little surprising.

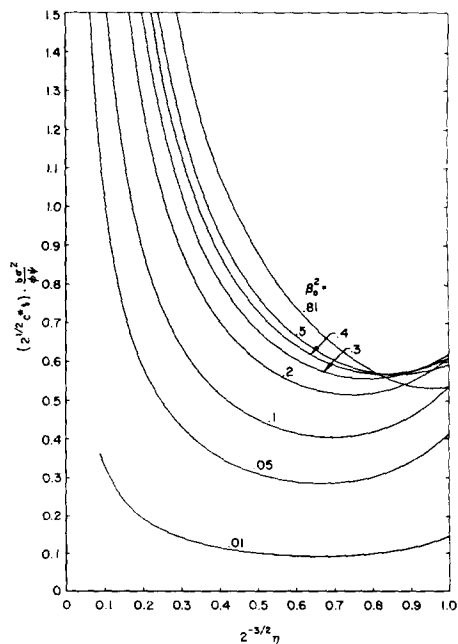


FIG. 4. Dependence of magnetic pressure, in units of $2^{-1}\rho_0 c^* t^{-1}$, on nondimensional variable η , for various values of $\beta_0^2 = \mu I_0^2 / 8\pi^2 \rho_0 c^{*2}$; $\gamma = 1.4$.

VI. MODIFIED HYDRODYNAMIC CASE

The modified hydrodynamic case has been considered in detail by Greenspan, who obtained the solution by the numerical integration of three coupled differential equations combined with one integral of the motion.⁸ The solution can also be obtained within the framework of the present formalism. Some properties of the solution of interest for comparison with the pure magnetohydrodynamic case and for the energy considerations to follow, are briefly noted at this point. In this case, the magnetic field is assumed to be continuous across the shock; the boundary conditions at the shock are therefore simply the ordinary hydrodynamic strong shock conditions, and the effect of the magnetic field appears only in the flow behind the shock. The range of the Alfvén number β_0 is now indeed unbounded, in contrast with the pure magnetohydrodynamic case.

When the strong shock conditions, together with the continuity of the magnetic field, are applied to Eqs. (16) and (17), one obtains

$$b = \frac{1}{2}\beta_0^2[(\gamma - 1)/(\gamma + 1)]^2, \tag{28a}$$

$$k = [(\gamma - 3)(\gamma + 1)/(\gamma - 1)^2]b \tag{28b}$$

[cf. Eq. (23) for the pure magnetohydrodynamic case]. The singular point of Eq. (18) which corresponds to the axis is the point $\sigma = 0$, $\xi = (\gamma + 1)/(\gamma - 1)$. Along the integral curve through the shock, the parameter ξ decreases from its value at the shock, $\xi_s = (\gamma + 1)/(\gamma - 1)$, to some minimum value, and then increases back to $(\gamma + 1)/(\gamma - 1)$ at the axis. In the modified hydrodynamic case, the magnetic field behind the shock is given by

$$B_\theta(r, t) = (\mu I_0/2\pi r)[\xi(r, t)/\xi_s] \tag{29}$$

[cf. Eq. (24)]. Thus, the field is everywhere smaller between the shock and the axis than it was initially; the field is expanded by the shock rather than compressed by it as in the pure magnetohydrodynamic case. At the axis, the field again maintains its initial value, so that, as in the pure magnetohydrodynamic case, a requirement for similarity is that the external circuit maintain a constant current.⁹

⁸ This was only a special case of the more general problem of finite conductivity, treated by Greenspan, where the entropy and frozen-field integrals do not exist.

⁹ Although the singular point of Eq. (18) for Pai's solution is the same as that for Greenspan's solution, a current jump is implied in the former case, but a constant current in the latter. The reason for this is that Pai's jump condition for the magnetic field leads to a value of b given by Eq. (23) rather than Eq. (28a), and hence to an expression for B_θ given by Eq. (24) rather than Eq. (29).

This difference in the variation of the magnetic field implies a marked difference between the two cases in the current distributions in the shocked gas. In both cases, the field ahead of the shock is assumed to remain unaltered; the integrated current through the gas must therefore be zero in both cases. In the pure magnetohydrodynamic case, where the field jumps across the shock, there is a cylindrical current sheet along the shock front, flowing in the same direction as the axial current, and balanced by opposite currents in the remainder of the gas. In the modified hydrodynamic case, where the field is continuous across the shock, there is no current sheet. Between the shock and the point where the variable ξ reaches its minimum value, there are currents in the gas flowing in the same direction as the axial current; these are balanced by the opposite currents in the remainder of the gas.

Another important difference between the two cases is in the behavior of the other physical variables in the neighborhood of the axis. The density and velocity again become zero at the axis, but not with the logarithmic dependence on η of Eqs. (27a) and (27b). The pressure, however, also goes to zero at the axis, in contrast to the pure magnetohydrodynamic case where it becomes infinite. The results for these two cases, as well as for the ordinary blast wave, are summarized in Table I.

TABLE I. Dependence of physical variables on nondimensional variable η in the neighborhood of the axis ($\eta \rightarrow 0$).

Variable	Case		
	Pure Magneto-hydrodynamic	Modified Hydrodynamic	Ordinary Blast-Wave
ρ	$(-\ln \eta^{1/(\gamma-1)})^{-2/(\gamma-1)}$	$\eta^{2/(\gamma-1)}$	$\eta^{1/(\gamma-1)}$
$ut^{1/2}$	$\eta^{1/2} (-\ln \eta)^{-1}$	$\eta^{1/2}$	$\eta^{1/2}$
pt	$\eta^{-1} (-\ln \eta)^{-2}$	η	const

VII. ENERGY CONTENT

The one remaining parameter of the problem is the explosive energy per unit length, E_0 , released along the axis at $t = 0$. This parameter enters through the fact that, with the assumed similarity, the total energy between any two similarity lines, in particular between the shock and the axis, is constant in time. In the ordinary blast-wave solution, the initial energy of the gas is assumed negligible and the gas is assumed isolated from all external energy sources. The energy content (entirely mechanical, in this case) of the gas in motion is therefore equal to the energy released at $t = 0$.

This equality allows one to obtain a relation between the shock speed and the quantity E_0 .

When a magnetic field is present, the situation is not quite so simple. One minor point is that, although the mechanical energy of the gas is initially negligible, the magnetic energy is not (being infinite, in fact) and must therefore be taken into account. Secondly, while the total mechanical energy of the gas in motion is still constant in time (as is also the magnetic energy), this energy can no longer be identified with the energy released at $t = 0$. Because of the coupling between gas and magnetic field, some fraction (not known *a priori*) of the energy E_0 appears as mechanical energy and some fraction as magnetic energy. Finally, the system is no longer isolated from external sources; it is coupled to an external circuit, which, moreover, is required to maintain a constant current. It is conceivable that such a boundary condition on the circuit implies an exchange of energy at $t = 0$ between external source and system. That this is in fact the case will be shown below.

As noted above, the mechanical energy and the magnetic energy are both constant between any two similarity lines. Another quantity, closely related to the latter, is also constant between any two lines of constant η , viz., the magnetic flux. Although the total magnetic flux between shock and axis is constant in time subsequent to the explosion, it is not, however, equal to the total magnetic flux in the same volume prior to the explosion; there has been a change in flux (per unit length) $\Delta\Phi$ in this volume given by

$$\Delta\Phi = \int_0^{r_s(t)} \left(B_\theta - \frac{\mu I_0}{2\pi r} \right) dr$$

$$= 2\pi\rho_0 c^{*2} \frac{\beta_0}{I_0} \int_0^1 \left(\frac{\beta}{\phi^{\frac{1}{2}}} - \beta_0 \right) \frac{d\phi}{(\sigma - \phi)}. \quad (30)$$

Since the flux between shock and axis is constant in time, it follows that an amount of flux $\Delta\Phi$ is supplied (or withdrawn) at the axis at the time $t = 0$, when shock and axis coincide. Associated with this flux is a quantity of energy (per unit length) $\Delta E_c = I_0 \Delta\Phi$ which is supplied to (or withdrawn from) the system at $t = 0$ by the source maintaining the constant current I_0 . The increment in the total energy of the system thus contains a part ΔE_c in addition to the part E_0 released along the axis at $t = 0$.

The increment in the total energy of the system appears in two forms: an increment ΔE_G in the mechanical energy, given by

$$\Delta E_G = 2\pi\rho_0 c^{*2} \int_0^1 \left[\frac{P}{(\gamma - 1)} + \frac{1}{2}\sigma U^2 \right] \frac{d\phi}{\phi(\sigma - \phi)} \quad (31a)$$

and an increment ΔE_B in the magnetic energy, given by

$$\Delta E_B = 2\pi\rho_0 c^{*2} \int_0^1 \frac{1}{2} \left[\frac{\beta^2}{\phi} - \beta_0^2 \right] \frac{d\phi}{(\sigma - \phi)}. \quad (31b)$$

The energy balance equation for the system then becomes

$$E_0 = \Delta E_G + \Delta E_B - \Delta E_c. \quad (32)$$

In Eq. (32), the quantity $\Delta E_B - \Delta E_c$ represents that part of E_0 which appears as magnetic energy.

The energy equation can be written in the form

$$E_0 = 2\pi\rho_0 c^{*2} \text{In}(\beta_0; \gamma). \quad (33)$$

where $\text{In}(\beta_0; \gamma)$ is a constant which depends on the parameter β_0 and the specific-heat ratio γ , and which can be evaluated only after the complete integration of the equations of motion. It is this equation which enables one to obtain the dependence of the shock speed (which is related to c^*) on the explosive energy E_0 and the Alfvén number β_0 .¹⁰

VIII. RESULTS AND CONCLUSIONS

The results of the numerical calculation of the energy parameters are shown in Figs. 5 and 6. The dependence of the quantity $\text{In}(\beta_0; \gamma)$ on β_0 is shown, for two different values of γ , in Fig. 5. In the modified hydrodynamic case $\text{In}(\beta_0; \gamma)$ increases monotonically from the ordinary blast-wave value at $\beta_0 = 0$ to the asymptotic value found by Greenspan at $\beta_0 \rightarrow \infty$. This implies, from Eqs. (33) and (7), that, for a given explosive energy, the shock speed decreases with increasing magnetic field. The strength of the shock, however, remains constant in this case. In the pure magnetohydrodynamic case, $\text{In}(\beta_0; \gamma)$ increases from the blast-wave value at $\beta_0 = 0$ to a maximum at a β_0 of about 0.2, and then decreases to zero at $\beta_0 = 1$. For a given energy release in this case, then, the shock becomes weaker with increasing magnetic field, but the shock speed first decreases to a minimum value and then increases as the magnetic field is increased.

Figure 6 shows the dependence on β_0 of the fraction of E_0 which appears as mechanical energy. In the pure magnetohydrodynamic case, this fraction decreases monotonically from unity at $\beta_0 = 0$ to $\frac{1}{2}$ at $\beta_0 = 1$. In the weak-shock limit, then,

¹⁰ In his treatment of the modified hydrodynamic case, Greenspan calculated $\text{In}(\beta_0; \gamma)$ only in the limit $\beta_0 \rightarrow \infty$.

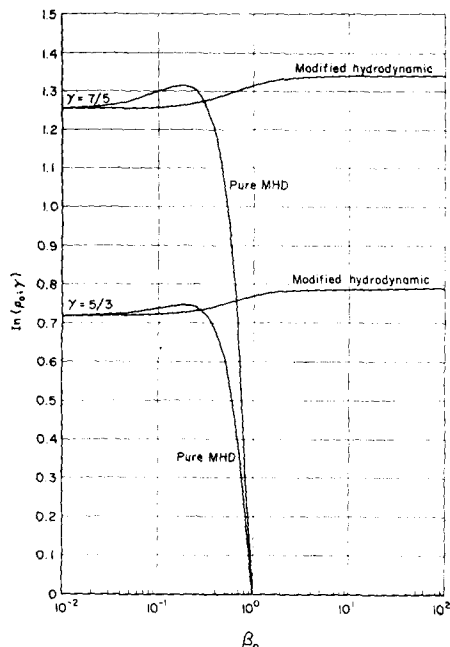


FIG. 5. Dependence of the shock-speed parameter $\ln(\beta_0; \gamma)$ on β_0 in the modified hydrodynamic and pure magnetohydrodynamic (pure MHD) cases.

there is equipartition of the explosive energy between mechanical and magnetic energy. This is in contrast to the modified hydrodynamic case where this fraction is essentially unity, independent of β_0 . Actually, this quantity decreases by about one-half of one percent at a value of β_0 of about 0.5, but the decrease is too small to be shown in the figure. The explosive energy, therefore, appears almost entirely as mechanical energy in the modified hydrodynamic case.

There are some other interesting differences in energy content in the pure magnetohydrodynamic and modified hydrodynamic cases which may be inferred from Eqs. (30) and (31), taken together

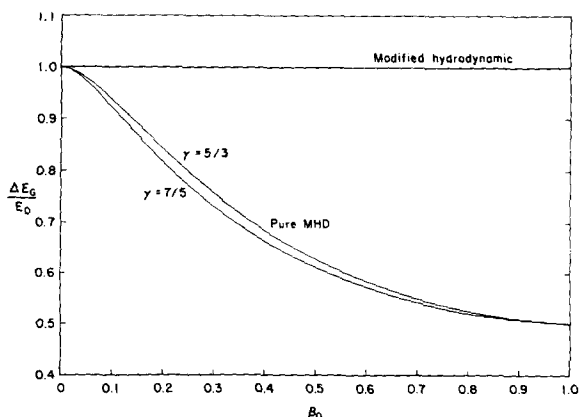


FIG. 6. Dependence on β_0 of the fraction of the explosive energy which appears as mechanical energy.

with Eqs. (24) and (29) and the discussion pertaining thereto. In the pure magnetohydrodynamic case, where the field is everywhere compressed behind the shock, the quantities ΔE_B and $\Delta E_c = I_0 \Delta \Phi$ are both positive. Both quantities are, in fact, positively infinite; the appearance of this divergence and its effect on the validity of the results has been discussed in the Introduction. The difference $\Delta E_B - \Delta E_c$ is, however, finite and positive; it represents the portion of E_0 which is converted to magnetic energy at $t=0$. Since $\Delta E_B > \Delta E_c$, this part of E_0 appears in the compressed magnetic field between shock and axis.

In the modified hydrodynamic case, where the field is everywhere expanded behind the shock, both ΔE_B and ΔE_c are negative (although finite, in this case). The difference $\Delta E_B - \Delta E_c$ is, however, again positive, and again represents the portion of E_0 which is converted to magnetic energy at $t = 0$. However, since $\Delta E_B < E_c$, this part of E_0 does not appear now in the magnetic field behind the shock but is delivered instead to the external circuit.

The difference just discussed implies a corresponding difference in the voltage induced in the external circuit by the gas in motion. In the pure magnetohydrodynamic case, ΔE_c is positive; i.e., the external source supplies a pulse of energy at $t = 0$. Thus, a positive voltage pulse (i.e., in the same sense as I_0) should be induced in the circuit at $t = 0$. Subsequently, as the shock progresses outward through the gas, ambient magnetic energy enters the region behind the shock. Since the total magnetic energy (and magnetic flux) between the shock and the axis remains constant in time, magnetic energy must leave at the axis at the same rate it is entering at the shock. Thus, the short positive voltage pulse should be followed by a long negative voltage pulse.

In the modified hydrodynamic case, the situation is just reversed. The quantity ΔE_c is now negative, which means that a pulse of energy is delivered to the circuit at $t = 0$. This implies an initial negative voltage pulse. As the shock progresses outwards, it "pushes out" some of the magnetic field from the gas through which it passes (the field is expanded as a result of the passage of the shock). There is thus an outflow of magnetic energy through the shock which must be balanced by an influx of energy at an equal rate at the axis. The short negative voltage pulse should therefore be followed by a long positive voltage pulse.

The long voltage pulses discussed above should be observable experimentally. However, whether or not the short voltage pulses would also be seen is an open

question. As mentioned in the Introduction, one would expect that, in any actual experiment, where the axial conductor has a finite radius, the true flow would approach that given by the similarity solution at distances from the axis much greater than the radius of the conductor. The short voltage pulse at $t = 0$, however, is a property of the solution which is associated with the axis itself, and one may reasonably wonder whether this feature of the solution would be retained in an actual experiment.

Finally, the energy and flux considerations outlined above help to account, at least heuristically, for the difference in the pressure distribution at the axis in the two cases. One can think of an imaginary interface at the axis across which there must be a

pressure balance if the flow is not to separate from the axis. In the pure magnetohydrodynamic case, an infinite magnetic pressure is produced on one side of this interface by the addition of magnetic flux at $t = 0$; this is balanced by an infinite gas pressure at the axis. In the modified hydrodynamic case, a "partial magnetic vacuum" is created at the axis by the removal of flux at $t = 0$; pressure balance in this case requires that the gas pressure be zero at the axis.

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