where

$$p_{\min} = \min_k p_k, \qquad p_{\max} = \max_k p_k$$

occur together on every line. Lemma 1 applies and hence the system (7) reduces to one equation,

$$\delta^2(Kp_{\min} - 1) - \delta(\theta K p_{\max} - p_{\min}) - \theta p_{\max} \geq 0. \quad (12)$$

The coefficient of δ^2 and the free term both are negative. Hence, (12) admits positive solutions if and only if the discriminant is positive and the coefficient of $-\delta$ is negative. We get

$$\theta^2 K^2 p_{\max}^2 - 2\theta(2p_{\max} - Kp_{\max}p_{\min}) + p_{\min}^2 \geq 0 \quad (13a)$$

$$\theta K p_{\max} - p_{\min} \leq 0, \text{ or } \theta \leq p_{\min}/(Kp_{\max}). \quad (13b)$$

The coefficient of θ^2 in (13a) is positive, and the left side of (13a) is negative for $\theta = p_{\min}/(Kp_{\max})$. Hence, the solution of (13) is the smaller of the two roots, so

$$\theta \leq \frac{2 - Kp_{\min} - 2\sqrt{1 - Kp_{\min}}}{K^2 p_{\max}},$$

and D_c^* is given by

$$D_c^* = \frac{(K-1)}{K^2 p_{\max}}\left(2 - Kp_{\min} - 2\sqrt{1 - Kp_{\min}}\right). \quad (14)$$

It is easy to check that $D_c^* > 0$ if $p_{\min} > 0$, i.e., if there are no nonzero transitions. For $K = 2$, D_c^* equals Gray's formula [2] for the binary symmetric source, namely,

$$D_c^* = \frac{1}{2}\left[1 - \frac{\sqrt{2p_{\max} - 1}}{p_{\max}}\right).$$

B. The Asymmetric Binary Source

Write

$$Q = \begin{pmatrix} d & c \\ 1 - M & M \end{pmatrix},$$

where M is the biggest element. Then note that

$$\max_{k,j} M_{k,j} = M_{2,2} = M + \delta(M - c), \qquad \text{for all } \delta.$$

This obviously is satisfied for $\delta = 0$. For other values of δ, note that the slopes of $M_{k,j}(\delta)$ are differences of coefficients in the same column, that the slopes of $M_{1,2}$ and $M_{2,1}$ are negative, and that $M_{1,1}$ has the same slope as $M_{2,2}$; therefore, none of them can exceed $M_{2,2}$. It is then obvious that

$$M_{2,1} = 1 - M + \delta(c - M) = \min_{k,j} M_{k,j}$$

for all δ, since $M_{2,1} + M_{2,2} = 1$. Lemma 1 applies again and (11) becomes

$$\delta^2(c - M) + \delta[1 - M - \theta(M + d)] - \theta M \geq 0, \quad (15)$$

which admits positive solutions if and only if

$$\theta^2(M+d)^2 - 2\theta[(M+1)(M+d) - 2M] + (1-M)^2 \geq 0 \quad (16a)$$

$$1 - M - \theta(M + d) \geq 0. \quad (16b)$$

As in (13) we obtain

$$D_c^* = \frac{(M+1)(M+d) - 2M - 2\sqrt{Md(M-c)}}{(M+d)^2}.$$

Again D_c^* coincides with Gray's D_c [3, eq. (44)] apart from a typographical error therein.

A simple upper bound to D_c can be obtained as follows. Let $R_n(D)$ denote the error-frequency rate-distortion function of a source that produces n-tuples of successive letters independently according to the nth-order marginal of the Markov source; and let D_n denote the value below which the Shannon lower bound to $R_n(D)$ is tight. It is not difficult to show that $(K-1)\min_{k,j} Q_{k,j}$ is an upper bound to D_2 and that D_n is monotonic nonincreasing with n. It follows that $(K-1)\min_{k,j} Q_{k,j}$ is an upper bound to D_n for all n and hence to D_c.

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Coding Protection for Magnetic Tapes: A Generalization of the Patel–Hong Code

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***Abstract*—Patel and Hong have constructed a code that can correct any track error or two track erasures in a 9-track magnetic tape. Here the construction is extended to a code that can correct a track error and a track erasure or three track erasures. A generalization is given.**

I. Introduction

Patel and Hong [1], [2] devised an error-correcting scheme that was successfully used in the IBM 3420 series tape units with a recording density of 6250 b/in. This error-correcting scheme is capable of correcting any error pattern on a single track or any error patterns on two tracks provided that the erroneous tracks i and j are identified by some external pointers (that is, two track erasures). Here we shall present a subcode of the Patel–Hong code capable of correcting a track error together with a track erasure, or three track erasures.

An IBM 3420 series tape unit writes characters in parallel across nine tracks on a half-inch tape, as shown in Fig. 1. Each character consists of eight information bits and one overall parity-check bit.

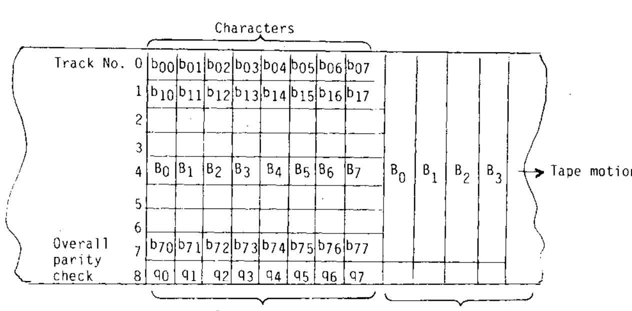

Fig. 1.

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The rows and columns of this array will be considered as elements of the Galois field of order 2^8, GF(2^8). As in the case of the Patel–Hong code the irreducible polynomial used to define GF(2^8) is $g(x) = 1 + x^3 + x^4 + x^5 + x^8$. Denote the first eight bits of each column by B_i, $0 \le i \le 7$, and each row by Z_j, $0 \le j \le 8$. Z_8 is also denoted Q and is a parity-check row. In our code B_0, B_1, and Z_8 will contain parity- check bits; hence, the rate of the code is 2/3. The code is defined as

$$\sum_{i=0}^{8} Z_i = 0, \tag{1}$$

$$\sum_{i=0}^{7} x^i B_i = 0, \tag{2}$$

$$\sum_{i=0}^{7} x^{2i} B_i = 0, \tag{3}$$

where

$$Z_i = \sum_{k=0}^{7} b_{ik} x^k \in \mathrm{GF}(2^8),$$

$$B_j = \sum_{k=0}^{7} b_{kj} x^k \in \mathrm{GF}(2^8).$$

Of course, the "polynomial" operations in (1), (2), and (3) are taken modulo $g(x)$, i.e., they are operations in GF(2^8). Equations (1) and (2) define the Patel–Hong code, so our code is a subcode of the Patel–Hong code.

II. Encoding

Z_8 is easily obtained using the procedure described in [1]. B_2, B_3, B_4, B_5, B_6, and B_7 are given, since they contain the information symbols. From (2) and (3)

$$B_0 + xB_1 = \sum_{i=2}^{7} x^i B_i$$

$$B_0 + x^2 B_1 = \sum_{i=2}^{7} x^{2i} B_i. \tag{4}$$

Solving system (4) we obtain

$$B_0 = \sum_{i=2}^{7} x^{i+1}(x^{i-2} + \cdots + 1) B_i \tag{5}$$

$$B_1 = \sum_{i=2}^{7} x^{i-1}(x^{i-1} + \cdots + 1) B_i. \tag{6}$$

Circuits performing operations (5) and (6) are easily constructed.

III. Decoding

Assume rows $\hat{Z}_0, \hat{Z}_1, \cdots, \hat{Z}_8$ are received (columns $\hat{B}_0, \hat{B}_1, \cdots, \hat{B}_7$, respectively). The decoder's first step is to calculate the three syndromes

$$S_0 = \sum_{i=0}^{8} \hat{Z}_i \tag{7}$$

$$S_1 = \sum_{i=0}^{7} x^i \hat{B}_i \tag{8}$$

$$S_2 = \sum_{i=0}^{7} x^{2i} \hat{B}_i. \tag{9}$$

If no errors occur, by (1), (2), and (3) we have $S_0 = S_1 = S_2 = 0$. The key property needed for the decoding is given by the following lemma.

Lemma 1:

$$S_1 = \sum_{i=0}^{7} x^i \hat{Z}_i \tag{10}$$

$$S_2 = \sum_{i=0}^{7} x^i \hat{Z}_i^2. \tag{11}$$

Proof: Equation (10) was proved by Patel and Hong [2]. Let us prove (11). From (9)

$$S_2 = \sum_{i=0}^{7} x^{2i} \hat{B}_i = \sum_{i=0}^{7} x^{2i} \sum_{j=0}^{7} \hat{b}_{ji} x^j$$

$$= \sum_{j=0}^{7} x^j \sum_{i=0}^{7} \hat{b}_{ji} x^{2i}$$

$$= \sum_{j=0}^{7} x^j \left(\sum_{i=0}^{7} \hat{b}_{ji} x^i \right)^2 \quad \text{(since the field has characteristic 2)}$$

$$= \sum_{j=0}^{7} x^j \hat{Z}_j^2.$$

As we stated at the beginning the code can correct either a track error and a track erasure, or three track erasures. Hence we need two decoding modes.

A. Mode I: Correction of a Track Error and a Track Erasure

Assume that an error pattern e_i occurs in track i and e_j occurs in track j, j is known, and all the other tracks are correctly transmitted. If $j \le 7$, from (7), (10), and (11) we obtain

$$S_0 = e_i + e_j$$

$$S_1 = x^i e_i + x^j e_j$$

$$S_2 = x^i e_i^2 + x^j e_j^2. \tag{12}$$

Solving system (12) we obtain

$$x^i\left(x^{-j} S_2 + S_0^2\right) = x^{-j} S_1^2 + S_2. \tag{13}$$

First we need to construct circuits that will find $x^{-j}S_2 + S_0^2$ and $x^{-j}S_1^2 + S_2$. Then we multiply $x^{-j}S_2 + S_0^2$ by x until we obtain $x^{-j}S_1^2 + S_2$. We now count how many times we had to multiply by x; in this way i is obtained. Once we know i we are in the Patel–Hong case of two erasures, i.e., we have to solve the system

$$S_0 = e_i + e_j$$

$$S_1 = x^i e_i + x^j e_j. \tag{14}$$

Solving this system we obtain

$$e_i = \frac{x^j S_0 + S_1}{x^i + x^j} \tag{15}$$

$$e_j = \frac{x^i S_0 + S_1}{x^i + x^j}. \tag{16}$$

Circuits to obtain e_i and e_j can be implemented as described in [1].

Assume $j = 8$. Then we have to solve (we are not interested in e_8)

$$S_1 = x^i e_i$$

$$S_2 = x^i e_i^2. \tag{17}$$

Since $x^i S_2 = S_1^2$ we easily obtain i, and then $e_i = x^{-i} S_1$ gives us e_i. The decoder's final step is to add e_i and e_j to the corresponding tracks.

B. Mode II: Correction of a Triple Track Erasure

Assume erasure patterns e_i, e_j, e_k occur in tracks i, j, and k, where $0 \le i < j < k \le 8$. If $k < 8$ we have

$$S_0 = e_i + e_j + e_k$$
$$S_1 = x^i e_i + x^j e_j + x^k e_k$$
$$S_2 = x^i e_i^2 + x^j e_j^2 + x^k e_k^2. \tag{18}$$

The solution of this system is given by

$$e_i^2 = \frac{x^{j+k}S_0^2 + S_1^2 + (x^j + x^k) S_2}{(x^i + x^j)(x^i + x^k)}$$
$$e_j^2 = \frac{x^{i+k}S_0^2 + S_1^2 + (x^i + x^k) S_2}{(x^i + x^j)(x^j + x^k)}$$
$$e_k^2 = \frac{x^{i+j}S_0^2 + S_1^2 + (x^i + x^j) S_2}{(x^i + x^k)(x^j + x^k)} \tag{19}$$

Circuits to solve (19) are more complicated than in the case of two erasures, but still are feasible. To find e_i, e_j, and e_k, we need to take the square root. But this is easily done, since square root is a linear $1-1$ operation.

Finally, if $k = 8$ we have to solve the system

$$S_1 = x^i e_i + x^j e_j$$
$$S_2 = x_i e_i^2 + x^j e_j^2, \tag{20}$$

and the solution is given by

$$e_i^2 = \frac{S_1^2 + x^j S_2}{x^i(x^i + x^j)}$$
$$e_j^2 = \frac{S_1^2 + x^i X_2}{x^j(x^i + x^j)}. \tag{21}$$

IV. Generalization

The construction can be generalized to an $(n+1) \times n$ array, i.e., an $(n+1)$-track tape. As before, denote by Z_i the rows $0 \le i \le n$, and by B_j the first n bits in each column, $0 \le j \le n-1$. Z_i and B_j are considered elements in GF(2^n), so we have to choose an irreducible polynomial $g(X)$ of degree n in $\mathbf{Z}_2[x]$. Hence, GF(2^n) is defined by $g(x)$, $Z_i = \sum_{k=0}^{n-1} b_{ik} x^k$, $B_j = \sum_{k=0}^{n-1} b_{kj} x^k$ (notice that our array is now (b_{ij}) $0 \le i \le n$, $0 \le j \le n-1$, $b_{ij} \in$ GF(2)). Take $0 \le m \le n-1$. Columns $B_0, B_1, \cdots, B_m$ will contain parity-check bits together with row Z_n. Hence, the code has rate $n - m - 1/(n+1)$ and is defined by the $m+2$ equations in GF(2^n);

$$\sum_{i=0}^{n} Z_i = 0$$
$$\sum_{i=0}^{n-1} x^{2^j i} B_i = 0, \qquad 0 \le j \le m. \tag{22}$$

Call this code $B(n,m)$-code. Using this notation, the Patel–Hong code is a $B(8,0)$-code, while the code described in the previous section is a $B(8,1)$-code. Whenever $2s + t \le m + 2$ the $B(n,m)$-code can correct s track errors and t track erasures. Assuming $\hat{Z}_i$, $0 \le i \le n$, is received, we have the syndromes

$$S_0 = \sum_{i=0}^{n} \hat{Z}_i$$
$$S_{j+1} = \sum_{i=0}^{n-1} x^{2^j i} \hat{B}_i, \qquad 0 \le j \le m. \tag{23}$$

The key property necessary for decoding is

$$S_{j+1} = \sum_{i=0}^{n-1} x^i \hat{Z}_i^{2^j}, \qquad 0 \le j \le m. \tag{24}$$

Equation (24) is proved in the same way as Lemma 1. From (23) and (24), if $2s + t \le m + 2$ and s track errors and t track erasures occur, we have to solve a system of $m+2$ equations with $m+2$ unknowns. We will have $\lfloor (m+2)/2 \rfloor + 1$ decoding modes, depending on the number $s = 0, 1, \cdots, \lfloor (m+2)/2 \rfloor$ of track errors that $B(n,m)$ can correct.

It remains to be shown that the solution of this system of $m+2$ equations exists and is unique. To see this observe from (22), (23), and (24) that a $\beta(n,m)$ code can be defined as the set of vectors $(Z_0, Z_1, \cdots, Z_n)$ satisfying

$$\sum_{i=0}^{n} Z_i = 0$$
$$\sum_{i=0}^{n-1} x^i Z_i^{2^j} = 0, \qquad 0 \le j \le m. \tag{25}$$

This is a linear code over GF(2^n) of length $n+1$. The result is proved if we show that this code has minimum distance $m+3$. From (25) note that the parity-check matrix of the code is (λ_{ji}) $-1 \le j \le m$, $0 \le i \le n$, where

$$\lambda_{-1,i} = 1,$$
$$(\lambda_{ji})^{2^j} = x^i, \qquad \text{for } 0 \le j \le m,\ 0 \le i \le n-1,$$
$$\lambda_{jn} = 0, \qquad \text{for } 0 \le j \le m.$$

The code has minimum distance $m+3$ and dimension $n - m - 1$ if and only if any $m+2$ columns in the parity-check matrix are linearly independent. Choose any $m+2$ columns $0 \le i_0 \le i_1 < \cdots < i_{m+1} \le n-1$. We must show that $\det[(\lambda_{j i_k})] \ne 0$.

Taking each $\lambda_{j i_k}$ to the power 2^m, this is equivalent to showing that

$$\det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x^{i_0} & x^{i_1} & x^{i_2} & \cdots & x^{i_{m+1}} \\ (x^{i_0})^2 & (x^{i_1})^2 & (x^{i_2})^2 & \cdots & (x^{i_{m+1}})^2 \\ & & & & \\ (x^{i_0})^{2^m} & (x^{i_1})^{2^m} & (x^{i_2})^{2^m} & \cdots & (x^{8_{m+1}})^{2m} \end{pmatrix} \ne 0. \tag{26}$$

We prove this result by induction. Replace x^{i_0} in the first column by the variable y. Then

$$f(y) = \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ y & x^{i_1} & x^{i_2} & \cdots & x^{i_{m+1}} \\ y^2 & (x^{i_1})^2 & (x^{i_2})^2 & \cdots & (x^{i_{m+1}})^2 \\ & & & & \\ y^{2m} & (x^{i_1})^{2^m} & (x^{i_2})^{2^m} & \cdots & (x^{i_{m+1}})^{2^m} \end{pmatrix}$$

is a polynomial of degree 2^m in y. Since we are in a field of characteristic 2, $f(y)$ is divisible by the following linear factors:

$$m+1 \quad \text{factors } (y + x^{i_k}), \qquad 1 \le k \le m+1$$
$$\binom{m+1}{3} \quad \text{factors } (y + \alpha_1 + \alpha_2 + \alpha_3), \qquad \alpha_1, \alpha_2, \alpha_3 \in \{x^{i_1}, x^{i_2}, \cdots, x^{i_{m+1}}\}$$

and so on. This gives a total of $\binom{m+1}{1} + \binom{m+1}{3} + \binom{m+1}{5} + \cdots = 2^m$ linear terms in y. Since the degrees match, $f(y)$ can be factored as a constant c times the linear factors described above. The constant c is given by

$$c = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x^{i_1} & x^{i_2} & \cdots & x^{i_{m+1}} \\ (x^{i_1})^2 & (x^{i_2})^2 & \cdots & (x^{i_{m+1}})^2 \\ & & & \\ (x^{i_1})^{2^{m-1}} & (x^{i_2})^{2^{m-1}} & \cdots & (x^{i_{m+1}})^{2^{m-1}} \end{pmatrix}$$

which is nonzero by induction. Replacing y by x^{i_0}, we have an explicit factorization of (26). Since the remaining factors are polynomials in x of degree smaller than n, they are all nonzero. This proves our claim.

Example: Consider $B(8,2)$. Our field is $GF(2^8)$ defined by $g(x) = 1 + x^3 + x^4 + x^5 + x^8$, and the parity-check bits are B_0, B_1, B_2, and Z_8 (see Fig. 1). The rate of this code if 5/9, and it is defined by the equations

$$\sum_{i=0}^{8} Z_i = 0$$

$$\sum_{i=0}^{7} x^i B_i = 0$$

$$\sum_{i=0}^{7} x^{2i} B_i = 0$$

$$\sum_{i=0}^{7} x^{4i} B_i = 0. \tag{27}$$

$B(8,2)$ has three decoding modes: correction of two track errors; correction of a track error and two track erasures; correction of four track erasures.

V. Conclusion

A family $B(n,m)$ of $(n+1) \times n$ array codes has been presented. $B(n,m)$ has rate $(n-m-1)/(n+1)$ and, in an $(n+1)$-track tape, can correct s track errors and t track erasures, whenever $2s + t \le m + 2$. The Patel–Hong code used in the IBM 3420 is a particular case of this family, $B(8,0)$.

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Comparison of the SPRT and the Sequential Linear Detector in Autoregressive Noise

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***Abstract*—The sequential probability ratio test (SPRT) with constant thresholds and the sequential linear detector (SLD) are compared under an autoregressive noise assumption in terms of the average sample number (ASN) when both detectors have the same error probabilities. For small ASN their performances are evaluated numerically. Results show that the SLD is better when ρ (the parameter in the autoregressive noise) is positive and the SPRT is better when ρ is negative. However, their asymptotic efficiencies (as the signal approaches zero) are found to be the same.**

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I. Introduction

The problem of detecting a constant signal corrupted by noise can be formulated as testing a pair of hypotheses,

$$H_0: \quad X_i = Y_i, \qquad i = 1, 2, \cdots$$

versus

$$H_1: \quad X_i = Y_i + \theta, \qquad i = 1, 2, \cdots, \tag{1}$$

where $X_1, X_2, \cdots$ are (discrete) observation random variables; $Y_1, Y_2, \cdots$ are zero-mean noise random variables; and θ is a constant signal. We denote a realization of a random variable by its lowercase letter. For example, y_i is a realization of Y_i.

The sequential probability ratio test (SPRT) with constant thresholds for (1) is described as follows. Continue testing as long as

$$B < \frac{f_Y(x_1 - \theta, x_2 - \theta, \cdots, x_n - \theta)}{f_Y(x_1, x_2, \cdots, x_n)} < A, \tag{2}$$

where A and B are the test thresholds and where $f_Y(t_1, \cdots, t_n)$ is the n-variate joint probability density function (pdf) of $Y_1, \cdots, Y_n$. If the upper (lower) threshold is violated, the test terminates with an acceptance of $H_1(H_0)$. The sample size, the number of samples at which the test terminates, is a random variable. Let it be denoted by N. Suppose that the actual signal strength is not necessarily the design value θ but rather $\delta\theta$, where $\delta > 0$. Note the difference between the actual signal strength (which is $\delta\theta$) and the hypothesized signal strength (which is θ). The introduction of δ serves two purposes. One is to conveniently express the detector performances under both H_0 and H_1 by the same expression. The other is to see the effect of signal mismatch between the actual signal strength and the hypothesized signal strength. If $\delta = 0$, the hypothesis H_0 is true, and if $\delta = 1$ hypothesis H_1 is true, i.e., the actual signal coincides with the hypothesized signal strength; $\delta \neq 0$ and $\delta \neq 1$ means a signal mismatch. The expected value of N as a function of δ is called the average sample number (ASN) function, and is denoted by $E(N|\delta)$. The probability of accepting H_0 as a function of δ is called the operating characteristic (OC) function, and it is denoted by $L(\delta)$. We shall use the symbols α and 1-β for the error probabilities under H_0 and H_1, respectively. Therefore, $\alpha = 1 - L(0)$ and $1\text{-}\beta = L(1)$.

If $Y_1, Y_2, \cdots$ are independent and identically distributed (iid), then (2) can be written as

$$\ln(B) < \sum_{i=1}^{n} \ln[f_Y(x_i - \theta)/f_Y(x_i)] < \ln(A), \tag{3}$$

where $f_Y(\cdot)$ is the pdf of Y_i. Wald [1] derived approximations for the ASN and OC functions of (2), and it has been proved [2] that (3) is optimum when $Y_1, Y_2, \cdots$ are iid. The test (3) is optimum in the sense that it minimizes the ASN under H_0 and H_1 among all tests that have error probabilities no larger than α and 1-β. For dependent noise an optimum detector is in the form of a generalized sequential probability ratio test (GSPRT) [3], [4], namely,

$$B_n < \frac{f_Y(x_1 - \theta, x_2 - \theta, \cdots, x_n - \theta)}{f_Y(x_1, x_2, \cdots, x_n)} < A_n, \tag{4}$$

where B_n and A_n are thresholds that are functions of n. However, the determination of A_n and B_n is still an open problem. Equation (4) does not rule out the case that A_n and B_n are constants. Since there is no procedure for evaluating A_n and B_n,