THE STRUCTURE OF AN ELECTROMAGNETIC FIELD

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1. Starting with the fundamental equations

$$c \text{ rot } \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t} + \rho \mathbf{v}, \qquad \text{div } \mathbf{E} = \rho,$$

$$c \text{ rot } \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \qquad \text{div } \mathbf{H} = 0,$$

we adopt as an elementary solution

$$\mathbf{H} = \text{rot } \mathbf{A}, \ \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi, \ \Psi = \text{div } \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}, \ \rho = -\frac{1}{c}\frac{\partial \Psi}{\partial t}, \ \rho\mathbf{v} = c\nabla\Psi,$$

where

$$\mathbf{A} = c\int_{-\infty}^{\alpha} f(\tau)\, \nabla \log\left[\mathbf{s}.\mathbf{r} - c\,(t-\tau)\right] d\tau,$$

$$\Phi = -\int_{-\infty}^{\alpha} f(\tau)\, \frac{\partial}{\partial t} \log\left[\mathbf{s}.\mathbf{r} - c\,(t-\tau)\right] d\tau.$$

In these expressions $\mathbf{s}$ is a unit vector depending on τ, $\mathbf{r}$ denotes the radius vector from the point with co-ordinates $\xi(\tau)$, $\eta(\tau)$ $\zeta(\tau)$, to the point with co-ordinates x, y, z, while α is defined by the equation

$$[x - \xi(\alpha)]^2 + [y - \eta(\alpha)]^2 + [x - \zeta(\alpha)]^2 = c^2(t-\alpha)^2; \ \alpha \leqq t.$$

The function Ψ is given by the formula

$$\Psi = \frac{c}{\nu} f(\alpha),$$

where

$$\nu = \xi'(\alpha)\,[x - \xi(\alpha)] + \eta'(\alpha)\,[y - \eta(\alpha)] + \zeta'(\alpha)\,[z - \zeta(\alpha)] - c^2(t-\alpha).$$

This elementary field corresponds to a state of affairs in which electric charges of a concentrated form are created and travel along straight lines with the velocity of light, the directions of these lines being specified by the different values of the unit vector $\mathbf{s}$. Whenever a concentrated electric charge is created an amount of electricity which will just compensate it is fired out in all directions and provides an elementary 'aether' which is the seat of the electromagnetic field of the concentrated charge. A concentrated electric charge and its elementary aether lie at any instant on a sphere whose centre is at the point where these charges originated[1]; if now this point moves with a velocity less than the velocity of light the different spheres bearing electricity that exist at time t do not intersect and if the arbitrary function $f(\alpha)$ is never zero there will be a sphere through each point of space so that our elementary aethers will fill the whole of space; if however the function $f(\alpha)$ is sometimes zero, for axemple if it is zero when α is less than α_0, then the elementary aethers will not fill the whole of space.

If we subtract from the above field another one of the same type in which the unit vector function **s** has a different value we obtain a field in which $\rho\mathbf{v}$ and ρ are zero except in the neighbourhood of the concentrated electric charges, there is thus a cancelling of electricity when the two elementary aethers are superposed and we get an aether in the ordinary sense of the word. The field is now one in which concentrated charges of opposite signs are continually produced by a process of separation analogous to that described by Heaviside in 1901. The field thus obtained belongs to the type in which there is a rectilinear flow of energy and no accumulation of energy at any point of space: the energy in such a field may therefore be regarded as kinetic energy or energy of motion.

The most general field that possesses the property just mentioned and the additional property that the volume charge and current in the aether are zero outside the singularities of the field is obtained by writing

$$\mathbf{M} = \mathbf{H} + i\mathbf{E} = cF(\alpha,\beta)\{\nabla\alpha \times \nabla\beta\} = iF(\alpha,\beta)\left\{\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right\},$$

where α and β are defined by equations of type

$$z - ct = f(\alpha,\beta) + (x + iy)\,\theta(\alpha,\beta), \quad z + ct = g(\alpha,\beta) - \frac{x - iy}{\theta(\alpha,\beta)},$$

f, g and θ being arbitrary functions of α and β. It may be remarked that $\mathbf{M}.\mathbf{M} = 0$ and that θ is a solution of the wave equation.

In all the fields of the above type electricity or magnetism travels along straight lines with the velocity of light (the case of a plane wave of light is, however, an exception). To obtain fields in which electricity or magnetism travels with a velocity less than that of light we must superpose fields of the above type in such a way that there is a cancelling of nearly all the concentrated electric or magnetic charges. It is fairly easy to prove that the field of an isolated electric pole moving with a velocity less than that of light can be regarded as the limit of two superposed radiant fields of the type obtained by subtracting two of our elementary solutions. According to this idea the electricity at an electric pole is continually being renewed, moreover, it is the electric charge itself which is directly responsible for the effects produced at a distance, but to understand fully the production of these effects we must consider how this charge is constituted remembering how the field was built up from four of our elementary fields. It is important to notice in this connection that the volume density of electricity and electric current in an elementary field are independent of the direction in which the concentrated charges move; this simplifies matters when we want to ascertain the structure of the aether for an electric pole that is built up in a specified way.

When a number of radiant fields are superposed a cancelling of concentrated electric charges seems to be necessary in order that the principle of the conservation of energy may hold and in order that steady states of motion may be possible. Even if the cancelling is not complete everywhere it must at least be sufficient to prevent any free electricity from going to infinity. If this less stringent condition is adopted it is possible to admit fields in which the charge associated with an electric pole fluctuates slightly and electric charges are fired from one electric pole to another. It may be possible to explain gravitation in this way for it will soon be realised that the necessary fluctuation of charge is exceedingly small. The question may also be raised whether it is necessary for there to be a complete cancelling of the charges in the elementary aethers when a number of elementary fields are superposed. If the answer is in the negative the volume densities of electricity and convection current will be derivable from a function Ψ which satisfies the wave-equation at points not occupied by matter.

Summing up the essential features of the present theory, we may say that all electric charges are supposed to really travel along rectilinear paths with the velocity of light; this implies that when electricity appears to move with a smaller velocity it is made up of different entities at different times being constantly renewed so to speak. The fact that an electric charge which has been moving along a rectilinear path with the velocity of light has no surrounding field[2] is quite consistent with the present view, for all electric charge arises from electric separation[3] and its aether is created in the process, consequently an electric charge which has moved along a rectilinear path for all time with the velocity of light would have no aether to support a field.

When we admit the mathematical possibility that the electric charges in the universe have not existed in the free state for the whole of time we find that it is by no means certain that the aether fills the whole of space and this raises some interesting philosophical questions. Some of the logical difficulties[4] in the ideas of contact action and action at a distance are avoided in the present theory because an electric charge at a point P produces an effect at a distance point Q for the simple reason that either a portion of the charge itself, or a portion of the compensating charge that was created at the same time, actually goes to Q and helps to produce the particle that is acted upon, or rather the entity that represents the particle at the moment under consideration.

Passing on to a brief consideration of some more familiar types of fields we shall superpose the fields of point charges using Lienard's potentials

$$\mathbf{A} = -\frac{e}{4\pi}\frac{\mathbf{v}(\alpha)}{\nu}, \qquad \Phi = -\frac{e}{4\pi}\frac{c}{\nu}.$$

We shall be interested chiefly in the effect of an operation analogous to differentiation. Let us suppose that the co-ordinates of the point charge at time α depend on a parameter β as well as α and let $\mathbf{b}$ denote the vector with components $\partial\xi/\partial\beta$, $\partial\eta/\partial\beta$, $\partial\zeta/\partial\beta$, then if $\mathbf{f}$ is a function of α and β it is easy to prove that

$$\frac{\partial}{\partial\beta}\left(\frac{f}{\nu}\right) = \frac{1}{\nu}\frac{\partial f}{\partial\beta} - \operatorname{div}\left(\frac{f\mathbf{b}}{\nu}\right).$$

When $f = \partial\omega(\alpha,\beta)/\partial\alpha$ this formula may be written in a more convenient form by making use of the relations

$$\frac{\partial}{\partial x}\left(\frac{\xi'}{\nu}\right) + \frac{\partial}{\partial y}\left(\frac{\eta'}{\nu}\right) + \frac{\partial}{\partial z}\left(\frac{\zeta'}{\nu}\right) + \frac{\partial}{\partial t}\left(\frac{1}{\nu}\right) = 0,$$

$$\xi'\frac{\partial}{\partial x} + \eta'\frac{\partial}{\partial y} + \zeta'\frac{\partial}{\partial z} + \frac{\partial}{\partial t} = \frac{\partial}{\partial\alpha},$$

where the operators in the last equation are supposed to act on a function of α and β.

We thus obtain the equation

$$\frac{\partial}{\partial\beta}\left(\frac{1}{\nu}\frac{\partial\omega}{\partial\alpha}\right) = \frac{\partial}{\partial x}\left[\frac{1}{\nu}\frac{\partial(\omega,\xi)}{\partial(\beta,\alpha)}\right] + \frac{\partial}{\partial y}\left[\frac{1}{\nu}\frac{\partial(\omega,\eta)}{\partial(\beta,\alpha)}\right] + \frac{\partial}{\partial z}\left[\frac{1}{\nu}\frac{\partial(\omega,\zeta)}{\partial(\beta,\alpha)}\right] + \frac{\partial}{\partial t}\left[\frac{1}{\nu}\frac{\partial(\omega,\alpha)}{\partial(\beta,\alpha)}\right].$$

In particular we have

$$\frac{\partial A_x}{\partial\beta} = -\frac{e}{4\pi}\left[\frac{\partial}{\partial y}\left\{\frac{1}{\nu}\left(\eta'\frac{\partial\xi}{\partial\beta} - \xi'\frac{\partial\eta}{\partial\beta}\right)\right\} - \frac{\partial}{\partial z}\left\{\frac{1}{\nu}\left(\xi'\frac{\partial\zeta}{\partial\beta} - \xi'\frac{\partial\xi}{\partial\beta}\right)\right\} + \frac{\partial}{\partial t}\left(\frac{1}{\nu}\frac{\partial\xi}{\partial\beta}\right)\right],$$

$$\frac{\partial\Phi}{\partial\beta} = \frac{ec}{4\pi}\left[\frac{\partial}{\partial x}\left\{\frac{1}{\nu}\frac{\partial\xi}{\partial\beta}\right\} + \frac{\partial}{\partial y}\left\{\frac{1}{\nu}\frac{\partial\eta}{\partial\beta}\right\} + \frac{\partial}{\partial z}\left\{\frac{1}{\nu}\frac{\partial\zeta}{\partial\beta}\right\}\right],$$

Writing $\mathbf{p} = e\mathbf{b}/4\pi$, we find that for an electric doublet of moment $4\pi\mathbf{p}$ the electric potentials are

$$\mathbf{A} = \operatorname{rot}\left\{\frac{1}{\nu}(\mathbf{v}\times\mathbf{p})\right\} - \frac{\partial}{\partial t}\left\{\frac{1}{\nu}\mathbf{p}\right\},\ \Phi = c\operatorname{div}\left\{\frac{1}{\nu}\mathbf{p}\right\}.$$

This of course is a simple generalisation of the well known result due to Hertz and Righi.

It is well known and easy to verify that the same electromagnetic field may be derived from the magnetic potentials.

$$\mathbf{B} = -c\operatorname{rot}\left\{\frac{1}{\nu}\mathbf{p}\right\} - \frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{1}{\nu}(\mathbf{v}\times\mathbf{p})\right\},\ \Omega = \operatorname{div}\left\{\frac{1}{\nu}(\mathbf{v}\times\mathbf{p})\right\},$$

with the aid of the formulae

$$\mathbf{E} = \text{rot}\,\mathbf{B}, \qquad \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla\Omega.$$

Let us now write

$$\mathbf{M} = \mathbf{H} + i\mathbf{E} = i\,\text{rot}\,\mathbf{L} \equiv \frac{1}{c}\frac{\partial \mathbf{L}}{\partial t} + \nabla\Lambda,$$

$$\mathbf{L} = \mathbf{B} - i\mathbf{A} = \frac{\partial \mathbf{G}}{\partial t} + ic\,\text{rot}\,\mathbf{G}, \qquad \Lambda = \Omega - i\Phi = -c\,\text{div}\,\mathbf{G};$$

then in the case of an electric doublet we have

$$\mathbf{G} = \frac{i}{\nu}\left[\mathbf{p} + \frac{i}{e}(\mathbf{v} \times \mathbf{p})\right].$$

To obtain the field of a magnetic doublet we write $i\mathbf{q}$ instead of $\mathbf{p}$ for the field of a magnetic pole of strength μ is derived from the magnetic potentials

$$\mathbf{B} = \frac{\mu}{4\pi}\frac{\mathbf{v}}{\nu}, \qquad \Omega = \frac{\mu}{4\pi}\frac{c}{\nu}.$$

If $\mathbf{m} = \mathbf{q} - i\mathbf{p}$ and

$$\mathbf{G} = -\frac{1}{\nu}\left[\mathbf{m} + \frac{i}{c}(\mathbf{v} \times \mathbf{m})\right] = \frac{\mathbf{g}(\alpha)}{\nu}, \text{ say},$$

the derived field is that of an electric doublet and magnetic doublet which move together. When the vector $\mathbf{g}$ is given it is easy to determine the moment $4\pi\mathbf{p}$ of the electric doublet and the moment $4\pi\mathbf{q}$ of the magnetic doublet.

The function $\mathbf{g}(\alpha)/\nu$ may be regarded as analogous to the fundamental potential function $1/r$ of electrostatics and Hertzian functions of higher order may be derived from it by differentiation just as potential functions involving spherical harmonics are derived from $1/r$ by differentiation according to a method developed by Maxwell.

Let us regard $\xi, \eta, \zeta, \mathbf{g}$ as functions of α and a parameter β then we obtain by differentiation a new Hertzian function whose x-component is

$$G'_x = \frac{\partial}{\partial \beta}\left\{\frac{1}{\nu}g_x(\alpha)\right\} = \frac{1}{\nu}\frac{\partial g_x}{\partial \beta} - \text{div}\left\{\frac{\mathbf{b}g_x}{\nu}\right\}.$$

It should be noticed that this expression for G'_x contains differentiation with regard to x, y and z but not t so that there is apparently a lack of symmetry. This is due to the fact that α is taken to be independent of β; we can easily introduce a term involving a differentiation with regard to t by making use of the identity

$$\frac{1}{\nu}f'(\alpha) = \text{div}\left(\frac{f\mathbf{v}}{\nu}\right) + \frac{\partial}{\partial t}\left(\frac{f}{\nu}\right),$$

but it is generally simpler and more convenient to retain the former expression.

The process of differentiation may be carried out any number of times with respect to different parameters using formulae of differentiation analogous to the above. When the various derivatives are added together the result indicates that the natural generalisation of a series of spherical harmonics of form

$$\frac{S_0(\theta,\phi)}{r}+\frac{S_1(\theta,\phi)}{r^2}+\frac{S_2(\theta,\phi)}{r^3}+$$

is the following type of series of Hertzian functions of different orders

$$G=\frac{\mathbf{g}_0}{\nu}+\operatorname{div}\left(\frac{\mathbf{a}_0\mathbf{a}_1}{\nu}\right)+\operatorname{div}\operatorname{div}\left(\frac{\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2}{\nu}\right)+\operatorname{div}\operatorname{div}\operatorname{div}\left(\frac{\mathbf{c}_0\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3}{\nu}\right)+.$$

Here $\mathbf{g}_0, \mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2 \ldots.$ are arbitrary vector functions of α. It should be remarked that the vector with suffix n is treated as the vector in forming the n divergence while the other vectors are treated for the moment as scalar quantities. The product of k vectors which occurs in the $(k+1)$th term is to be regarded as a tensor of the kth order with k components each of which is a product of components of the separate vectors; there appear to be enough arbitrary functions in a sum of products of this type with $k = 0, 1, 2, \ldots., K$ for the representation of the sum of a number of Hertzian functions up to order K.

[1] As each shell of electricity moves outwards it induces a secondary separation of electricity so that electricity flows back to a new position of the primary singularity (ξ, η, ζ) and tends to maintain the electric separation. The volume density of the compensating electricity created at the primary singularity isthus not ρ but is proportional to Ψ/r, it is this electricity which is regarded as forming the elementary æther associated with the primary singularity and it is this electricity which, on account of its displacement from the concentrated charge, is directly responsible for the field.

[2] See for instance Wilson, E. B., *Washington Acad. Sci.*, **6**, 1916, (665–669).

[3] Larmor, J., *London, Proc. Mathe. Soc.*, **13**, 1913, p. 51.

[4] Whitehead, A. N., *The Anatomy of some Scientific Ideas, The Organization of Thought*, London, 1917, p. 182.

INVARIANTS WHICH ARE FUNCTIONS OF PARAMETERS OF THE TRANSFORMATION

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A systematic theory and interpretation of invariantive functions which contain the parameters of the linear transformations to which a quantic