

Fermionic matrix product states and one-dimensional short-range entangled phases with antiunitary symmetries

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We extend the formalism of matrix product states (MPS) to describe one-dimensional gapped systems of fermions with both unitary and antiunitary symmetries. Additionally, systems with orientation-reversing spatial symmetries are considered. The short-ranged entangled phases of such systems are classified by three invariants, which characterize the projective action of the symmetry on edge states. We give interpretations of these invariants as properties of states on the closed chain. The relationship between fermionic MPS systems at a renormalization group fixed point and equivariant algebras is exploited to derive a group law for the stacking of fermionic phases. The result generalizes known classifications to symmetry groups that are nontrivial extensions of fermion parity and time-reversal.

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I. INTRODUCTION AND OVERVIEW

Matrix product states (MPS) have proven useful at describing the ground states of gapped local Hamiltonians in one spatial dimension [1,2]. This representation leads to a classification of interacting short-range-entangled (SRE) bosonic phases with a symmetry G in terms of the group cohomology of G [3,4]. One-dimensional systems of fermions are related to these bosonic systems by the Jordan-Wigner transformation, and this fact has been exploited to classify fermionic SRE phases [4,5]. The language of MPS was recently extended to an intrinsically fermionic formalism in what has been dubbed fermionic MPS (fMPS) [6,7]. This approach has the benefits of straightforwardly describing systems on a closed chain with twisted boundary conditions and allowing one to derive a group law for the stacking of fermionic SRE phases. It also makes manifest the relation to spin-TQFT, which is conjectured to describe the long-distance physics of fermionic systems [7]. Bosonic and fermionic systems with a time-reversal symmetry have also been classified by use of MPS, by allowing the projective action of the symmetry on edge degrees of freedom to be antiunitary [4,5]. It is natural to ask whether the formalism of fMPS can be applied to these systems. Bosonic systems with antiunitary symmetries have previously been studied with an approach similar to ours [8]. Fermionic systems were described in Ref. [6], though their results differ from ours.

The structure of the paper and its main results are as follows. In Sec. II, we review the formalism of \mathcal{G} -equivariant fMPS for a unitary on-site symmetry \mathcal{G} . We recall how fermionic SRE phases are classified by Morita classes of equivariant algebras and how the invariants α , β , and γ that characterize these algebras appear in the action of \mathcal{G} on edge degrees of freedom. We then derive interpretations of the invariants on the closed chain that extend the results of Ref. [9], which were discovered in the context of spin-TQFT. Next, time-reversing symmetries and their relation to spatial parity are discussed. The generalizations of the three invariants to

phases with such symmetries are derived and interpreted. In Sec. III, a general stacking law (44) is derived for fermionic SRE phases with a symmetry \mathcal{G} that is a central extension by fermion parity of a bosonic symmetry group that may contain antiunitary symmetries. We contrast this result with the bosonic group structure and emphasize the origin of the difference. In Sec. IV, we demonstrate our result with several examples, recovering the $\mathbb{Z}/8$ classification of fermionic SRE phases in the symmetry class BDI ($T^2 = 1$) and the $\mathbb{Z}/2$ classification in the class DIII ($T^2 = P$).

II. FERMIONIC MPS

A. Unitary symmetries

We begin by briefly recalling the fMPS formalism, leaving many of the details to Ref. [7].

The symmetries of a fermionic system form a finite supergroup (\mathcal{G}, p) ; that is, a finite group \mathcal{G} with a distinguished central involution $p \in \mathcal{G}$ called *fermion parity*. Every supergroup arises as a central extension of a group $G_b \simeq \mathcal{G}/\mathbb{Z}_2$ of bosonic symmetries by $\mathbb{Z}/2 = \{1, p\}$:

$$\mathbb{Z}_2 \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{t} \end{array} \mathcal{G} \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{s} \end{array} G_b. \quad (1)$$

Such extensions are classified by cohomology classes $[\rho] \in H^2[G_b, \mathbb{Z}/2]$. A trivialization $t : \mathcal{G} \rightarrow \mathbb{Z}/2$ defines a representative $\rho = \delta t$ of this class. In the context of fermionic phases [10], when \mathcal{G} splits, the choice of splitting t is part of the physical data, as it determines the action of G_b on fermions.

MPS is an ansatz for constructing one-dimensional gapped systems with such a symmetry. A translation-invariant \mathcal{G} -symmetric MPS system consists of the following data: a physical one-site Hilbert space A with a unitary action R of \mathcal{G} , a virtual space V with a projective action Q of \mathcal{G} , and an injective MPS tensor $T : A \rightarrow \text{End} V$ that satisfies an

equivariance condition: for all $a \in A$, $g \in \mathcal{G}$,

$$T[R(g)a] = Q(g)T(a)Q(g)^{-1}. \quad (2)$$

From this data, one may construct a gapped, symmetric, and frustration-free lattice Hamiltonian [11].

For our purposes, we are only interested in MPS at a renormalization group (RG) fixed point, where a system exhibits the physics universal to its gapped phase. Physical sites are blocked together under real-space RG. At a fixed point, this procedure endows the space A with a product $m : A \otimes A \rightarrow A$, making A into a finite-dimensional associative algebra and the space V into a faithful module over A with structure tensor T [12]; that is,

$$T(a)T(b) = T[m(a \otimes b)] \quad (3)$$

for all $a, b \in A$. It follows that $T\{m[R(g)a \otimes R(g)b]\} = T[R(g)m(a \otimes b)]$. Since T is injective, m satisfies

$$R(g)m(a \otimes b) = m[R(g)a \otimes R(g)b]. \quad (4)$$

We say A is a \mathcal{G} -equivariant algebra. Because T can be put into a canonical form, A is semisimple [12]. It can also be viewed as the algebra of linear operators on the space of low energy boundary states [4].

Our approach is to work with ground states of MPS systems, rather than their Hamiltonians. Every ground state of an MPS Hamiltonian has the form of a generalized MPS, which we now describe. Given a (\mathcal{G}, p) -equivariant MPS system described by an algebra A and tensor $T : A \rightarrow \text{End}(V)$, one obtains a generalized MPS by choosing an observable $X \in \text{End}(V)$ that *supercommutes* with $T(a)$ [13]; that is,

$$XT(a) = (-1)^{|a||X|}T(a)X \quad (5)$$

for all $a \in A$ such that $R(p)a = (-1)^{|a|}a$, where the parity of X is defined by $PXP^{-1} = (-1)^{|X|}X$, for $P := Q(p)$. A linear map X satisfying this condition is called an even or odd $\mathbb{Z}/2$ -graded module endomorphism, depending on its parity.

For X satisfying the supercommutation rule (5), the generalized MPS has conjugate wave function

$$\langle \psi_{T,g}^X | = \sum_{i_1, \dots, i_N} \text{Tr} U[Q(g)XT(e_{i_1}) \cdots T(e_{i_N})] |i_1 \cdots i_N\rangle \quad (6)$$

in the g -twisted sector. When \mathcal{G} splits as $G_b \times \{1, p\}$, the g -twisted sector for $t(g) = 0$ consists of states on the circle with NS spin structure and a G_b gauge field of holonomy $b(g)$, while the g -twisted sector for $t(g) = 1$ consists of states on the circle with R spin structure and a G_b gauge field of holonomy $b(g)$. We emphasize that one must choose a splitting t to make sense of the the g -twisted sector as a $b(g)$ -twisted NS or R sector. When \mathcal{G} is nonsplit, one cannot speak of independent spin structures and gauge fields [14].

The MPS description of a gapped system is not unique. For this reason, fixed point systems—and therefore gapped phases—are classified not by the algebras themselves but by their Morita classes [12]. After all, it is the module category (the T 's and X 's) of A that determines the system's ground states.

B. Equivariant algebras for fermionic SRE phases

For the remainder of the paper, we will restrict our attention to SRE phases. The algebras that correspond to fixed points in

such phases are either of the form $\text{End}(U)$ for U a projective representation Q of \mathcal{G} or of the form $\text{End}(U_b) \otimes \mathbb{C}\ell(1)$, for U_b a projective representation of G_b , with the following actions of \mathcal{G} [7]. We refer to algebras of the first (second) type as “even” (“odd”). Odd algebras are only present when the extension for \mathcal{G} splits.

The action of \mathcal{G} on an even algebra $A = \text{End}(U)$ is simply

$$R(g) \cdot M = Q(g)MQ(g)^{-1}. \quad (7)$$

Two even algebras are Morita equivalent if their projective representations have the same $[\omega] \in H^2[\mathcal{G}, U(1)]$ [15]. It is shown in the Appendix that $[\omega]$ is equivalent to a pair $(\alpha, \beta) \in C^2[G_b, U(1)] \times C^1[G_b, \mathbb{Z}/2]$ that satisfies $\delta\alpha = 1/2\beta \cup \rho$ and $\delta\beta = 0$, up to coboundaries [16]. In particular, when \mathcal{G} splits, ρ is trivial and the equivalence classes defining the phase are $[\alpha], [\beta] \in H^2[G_b, U(1)] \times H^1[G_b, \mathbb{Z}/2]$. The invariant $[\alpha]$ is simply $[\omega]$ pulled back by s to G_b , while β measures the parity of $Q(g)$:

$$PQ(g)P^{-1} = e^{i\pi\beta[b(g)]}Q(g). \quad (8)$$

Let Γ denote the generator of $\mathbb{C}\ell(1)$ with $\Gamma^2 = +1$. The action of \mathcal{G} on an odd algebra is

$$R(g) \cdot M \otimes \Gamma^m = (-1)^{[\beta(g)+t(g)]m} Q(g)MQ(g)^{-1} \otimes \Gamma^m. \quad (9)$$

Morita classes of odd algebras are classified by the class $[\alpha] \in H^2[G_b, U(1)]$ that describes the projective G_b -action Q and the class $[\beta] \in H^1[G_b, \mathbb{Z}/2]$ that describes the action of G_b on Γ .

It is apparent from (9) that changing the trivialization t shifts β by an arbitrary $\mu = (t' - t) \in Z^1[G_b; \mathbb{Z}/2]$. Therefore, unless t is fixed, β is not a well-defined invariant of odd algebras. On the other hand, for even algebras, redefining the action of g_b to $Q(g_b)P^{\mu(g_b)}$ leaves β unchanged but shifts α by $1/2\beta \cup \mu$.

In summary, when \mathcal{G} splits as $G_b \times \mathbb{Z}_2^F$, the \mathcal{G} -symmetric fermionic SRE phases are classified by $[\alpha], [\beta], [\gamma] \in H^2[G_b, U(1)] \times H^1[G_b, \mathbb{Z}/2] \times \mathbb{Z}/2$, where $\gamma \in \mathbb{Z}/2$ tells us whether the algebra is even ($\gamma = 0$) or odd ($\gamma = 1$). When \mathcal{G} does not split, only the invariants $[\alpha]$ (not a cocycle) and $[\beta]$ are present.

A system in an SRE phase has exactly one state per twisted sector. To see this from the algebra, count independent solutions X . The unique simple module over an even algebra $\text{End}(U)$ is U , and by Schur's lemma its only endomorphism is $X = \mathbb{1}$. The unique faithful simple module over an odd algebra $\text{End}(U_b) \otimes \mathbb{C}\ell(1)$ is $U_b \otimes \mathbb{C}^{11}$. It has two endomorphisms—an even one $X = \mathbb{1} \otimes \mathbb{1}$ and an odd one $X = \mathbb{1} \otimes \sigma_y$. The former appears in the wave function of the NS sector MPS state and the latter for the R sector MPS state.

C. Invariants of fermionic SRE phases

The invariants α , β , and γ can also be extracted from an SRE fermionic MPS system without reference to the algebra A . Below we give a physical interpretation of these invariants as observable quantities.

We begin by studying how the MPS in the g -twisted sector transforms under the action of a unitary symmetry $h \in \mathcal{G}_0$. Let

ω be the cocycle that characterizes the projective action Q on the module. Then

$$\begin{aligned} R(h) \cdot \text{Tr} Q(g) X T^i |i\rangle &= \text{Tr} Q(g) X Q(h)^{-1} T^i Q(h) |i\rangle \\ &= e^{2\pi i [\omega(h,g) + \omega(hg, h^{-1}) - \omega(h, h^{-1})]} \text{Tr} Q(hgh^{-1}) [Q(h) X Q(h)^{-1}] T^i |i\rangle. \end{aligned} \quad (10)$$

We used the fact that

$$\omega(h, h^{-1}) = \omega(h^{-1}, h), \quad (11)$$

which follows from the cocycle condition [17]. We see that under the action of a unitary symmetry h the following occurs:

- (1) The g -twisted sector maps to the hgh^{-1} -twisted sector.
- (2) The operator X is conjugated by $Q(h)$.
- (3) States also pick up a phase of

$$e^{2\pi i [\omega(h,g) + \omega(hg, h^{-1}) - \omega(h, h^{-1})]}. \quad (12)$$

We are now ready to interpret the three invariants.

1. Gamma

Suppose $h = p$ and $g \in \{1, p\}$. Then the phase (12) vanishes, but there is still a sign coming from the conjugation of X by P . It is always $+1$ if the algebra is of the form $\text{End}(U)$ (i.e., if $\gamma = 0$). If the algebra is of the form $\text{End}(U_b) \otimes \mathbb{C}\ell(1)$ (i.e., if $\gamma = 1$), this sign is $+1$ in the NS sector and -1 in the R sector. Therefore we can conclude that the invariant $(-1)^\gamma$ is detected as the fermion parity (p -charge) of the R sector state.

2. Beta

Continuing to take $h = p$, in the g -twisted sector the phase (12) becomes

$$1/2\beta(g) := \omega(p, g) - \omega(g, p). \quad (13)$$

This term satisfies $\beta(pg) = \beta(g)$ and takes values in $\{0, 1/2\}$; in fact, it defines a $\mathbb{Z}/2$ -valued cocycle of G_b . See the Appendix for a proof. When $\gamma = 0$, the sign $(-1)^{\beta(g_b)}$ is the fermion parity of the g -twisted sector for g with $b(g) = g_b$. If \mathcal{G} splits, one can equivalently say that $(-1)^{\beta(g)}$ is the parity of the $b(g)$ -twisted NS and R sectors. If \mathcal{G} splits, it is possible that $\gamma = 1$. In this case, one must choose a splitting to make sense of β . Then $(-1)^{\beta(g)}$ is still the parity of the $b(g)$ -twisted NS sector, but the parity of the $b(g)$ -twisted R sector receives a contribution of -1 from conjugation of X by P , in addition to the $\beta(g)$ term.

Note that $\beta(g)$ also describes the g -charge of the p -twisted (Ramond) sector for systems with $\gamma = 0$. This is no coincidence: the phase (12) agrees with Eq. (4.11) of Ref. [18], where it was derived from bosonic (i.e., $X = \mathbb{1}$) TQFT. If g and h commute, one can sew together the ends of the cylinder to create a torus with holonomies g and h around its cycles. This torus evaluates to the phase

$$\omega(h, g) + \omega(hg, h^{-1}) - \omega(h, h^{-1}) = \omega(h, g) - \omega(g, h). \quad (14)$$

This surface can also be evaluated as a torus with holonomies h and g^{-1} , respectively, yielding

$$\begin{aligned} &\omega(g^{-1}, h) + \omega(g^{-1}h, g) - \omega(g^{-1}, g) \\ &= \omega(h, g) + \omega(g^{-1}, hg) - \omega(g^{-1}, g) \\ &= \omega(h, g) - \omega(g, h). \end{aligned} \quad (15)$$

These are equal, as is required by consistency of the TQFT. In terms of states, the h -charge of the g -twisted sector is the same as the g^{-1} -charge of the h -twisted sector, as long as g and h commute. There is no analogous statement for systems with $\gamma = 1$. Recall that $\beta(g)$ measures whether or not g acts as σ_z on the second factor of $\text{End}(U) \otimes \mathbb{C}\ell(1)$. Then $Q(g)$ anticommutes with $X = \mathbb{1} \otimes \sigma_y$, and so the state picks up an extra charge of $\beta(g)$ which cancels with the sign (12) for a total g -charge of $+1$ in the R sector.

3. Alpha

Consider the MPS state on a circle with two adjacent domain walls, parametrized by bosonic symmetries $g_b, h_b \in G_b$, as in Fig. 1. Upon fusing them, the state picks up a phase

$$\begin{aligned} &\text{Tr} Q[s(g_b)] Q[s(h_b)] T^i |i\rangle \\ &= e^{2\pi i \omega(s(g_b), s(h_b))} \text{Tr} Q[s(g_b) s(h_b)] T^i |i\rangle. \end{aligned} \quad (16)$$

These phases define a G_b -cochain

$$\alpha(g_b, h_b) = \omega[s(g_b), s(h_b)]. \quad (17)$$

If \mathcal{G} splits, then the fact that ω is a cocycle implies that α is as well. If the extension \mathcal{G} is instead defined by a nontrivial ρ , then α has coboundary $1/2\beta \cup \rho$. See the Appendix for details. Redefining each $X = \mathbb{1}$ by a sector-dependent phase shifts α by a \mathcal{G} -coboundary with arguments in G_b , as expected.

Note that when β and γ are trivial, there are no fermionic states and the system is insensitive to spin structure. In this sense, α captures purely bosonic features of the system.

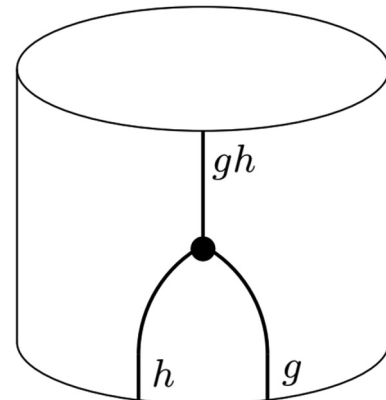


FIG. 1. Fusion of domain walls.

4. Summary

1. $(-1)^\gamma$ is the fermion parity of the untwisted R sector.
2. If $\gamma = 0$, $(-1)^{\beta(g_b)}$ is the fermion parity of the g -twisted sector for either of the two g 's with $b(g) = g_b$. Alternatively, $(-1)^{\beta(g_b)}$ is the g -charge of the untwisted R sector. If $\gamma = 1$, $(-1)^{\beta(g_b)}$ is the fermion parity of the g_b -twisted NS sector, as determined by the choice of splitting.
3. $e^{2\pi i \alpha(g_b, h_b)}$ is the phase due to fusing g_b and h_b domain walls.

D. Antiunitary and orientation-reversing symmetries

More generally, a fermionic system may be invariant under antiunitary symmetries as well as unitary ones. In this case, the full symmetry group \mathcal{G} is a central extension by \mathbb{Z}_2^F of a bosonic symmetry group G_b , which is itself an extension of \mathbb{Z}_2^T by a finite group G_0 , as in Fig. 2. The symmetry class (\mathcal{G}, p, x) is determined by a central $p \in \mathcal{G}$ and a map $x : G_b \rightarrow \mathbb{Z}/2$ that encodes whether a bosonic symmetry is unitary or antiunitary. Note that the composition $x \circ b$, which we also call “ x ,” satisfies $x(p) = 0$. Let \mathcal{G}_0 denote its kernel.

A fixed point MPS system of symmetry class (\mathcal{G}, p, x) consists of a finite-dimensional semisimple associative algebra A and a faithful module $T : A \rightarrow \text{End}(V)$, satisfying the equivariance conditions (4), (2) as before, only now the group action may be antiunitary. In particular, the projective action on V is given by a unitary operator $Q(g)$ for each $g \in \mathcal{G}_0$ and an antiunitary operator $Q(g)$ for each $g \notin \mathcal{G}_0$ that satisfy

$$Q(g)Q(h) = e^{2\pi i \omega(g, h)} Q(gh) \quad (18)$$

for phases $\omega(g, h)$. By comparing $[Q(g)Q(h)]Q(k)$ and $Q(g)[Q(h)Q(k)]$, we find the x -twisted cocycle condition

$$\omega(g, h) + \omega(gh, k) = (-1)^{x(g)} \omega(h, k) + \omega(g, hk). \quad (19)$$

Redefining each $Q(g)$ by a g -dependent phase corresponds to shifting ω by an x -twisted coboundary. Therefore the action of \mathcal{G} on the module V is characterized by a twisted cohomology class $[\omega] \in H^2[\mathcal{G}, U(1)_T]$. The group action R on A is defined via (2). It will be convenient to define linear maps $M(g)$ by

$$M(g) = \begin{cases} Q(g), & g \in \mathcal{G}_0, \\ Q(g)K, & g \notin \mathcal{G}_0, \end{cases} \quad (20)$$

where K denotes complex conjugation.

Unitary symmetries that reverse the orientation of one-dimensional space can also be described in this language. Let x measure whether a symmetry reverses orientation. The

$$\begin{array}{ccc} & \mathbb{Z}_2^F & \\ & \downarrow & \\ & \mathcal{G} & \\ & \downarrow b & \\ G_0 & \longrightarrow & G_b \xrightarrow{x} \mathbb{Z}_2^T \end{array}$$

FIG. 2. Symmetry data.

natural generalization of (2) is

$$\begin{aligned} T[R(g)a] &= M(g)T(a)M(g)^{-1} & \text{for } g \in \mathcal{G}_0, \\ T[R(g)a] &= M(g)T(a)^T M(g)^{-1} & \text{for } g \notin \mathcal{G}_0. \end{aligned} \quad (21)$$

Let us introduce the following shorthand. For a matrix $\mathcal{O} \in \text{End}(V)$, define

$$\begin{aligned} \mathcal{O}^{T0} &= \mathcal{O}, & \mathcal{O}^{T1} &= \mathcal{O}^T, \\ \{\mathcal{O}\}^0 &= \mathcal{O}, & \{\mathcal{O}\}^1 &= [\mathcal{O}^{-1}]^T. \end{aligned} \quad (22)$$

Since R is a group homomorphism,

$$\begin{aligned} M(g)\{M(h)\}^{x(g)}T(a)^{Tx(gh)}M(h)^{Tx(g)}M(g)^{-1} \\ &= T[R(g)R(h)a] \\ &= T[R(gh)a] \\ &= M(gh)T(a)^{Tx(gh)}M(gh)^{-1}. \end{aligned} \quad (23)$$

This implies there exists a number $\omega(g, h) \in \mathbb{R}/\mathbb{Z}$ such that

$$M(g)\{M(h)\}^{x(g)} = e^{2\pi i \omega(g, h)} M(gh). \quad (24)$$

By comparing the two equal expressions $M(g)\{M(h)\}^{x(g)}\{M(k)\}^{x(gh)}$ and $M(g)\{M(h)M(k)^{x(h)}\}^{x(g)}$, one recovers the x -twisted cocycle condition (19) for ω .

From the perspective of two-dimensional spacetime, it is not surprising that time-reversal [19] and space-reversal should be treated similarly. To make the connection more explicit, note that the physical Hilbert space carries the action of an antilinear involution $*$, which we regard as CPT (see Ref. [12]). Using the equivariance of the multiplication and (anti)unitarity of $R(g)$ with respect to the inner product on the Hilbert space, it may be shown that $*$ commutes with $R(g)$ for all $g \in \mathcal{G}$. With respect to the product on A , this map is an antiautomorphism. If $R(g)$ denotes the action of a time-reversing symmetry, $R(g)*$ is a unitary symmetry that reverses the orientation of space. Then

$$T[R(g)*a] = M(g)\overline{T(*a)}M(g)^{-1} = M(g)T(a)^T M(g)^{-1}. \quad (25)$$

Moreover, since $*$ commutes with $R(g)$, the equivariance condition (2) implies that $M(g)$ is unitary (up to a phase), so (18) and (24) are equivalent (up to a coboundary). For the remainder of the paper, we suppress $*$ and simply write R to denote a time-reversing or space-reversing symmetry.

E. Invariants of fermionic SRE phases with antiunitary symmetries

As in the case of unitary symmetries, fermionic SRE systems at fixed points correspond to even algebras of the form $\text{End}(U)$ and odd algebras of the form $\text{End}(U_b) \otimes \mathbb{C}\ell(1)$. However, when the symmetries may act antiunitarily, the cohomology class characterizing the Morita class (and hence the SRE phase) is twisted.

We now discuss the meaning of the invariants α , β , and γ in the antiunitary context, following the previous analysis. The form of the MPS conjugate wave function is (6) as before. Consider the action of an antiunitary symmetry $h \notin \mathcal{G}_0$ on an

MPS in the g -twisted ($g \in \mathcal{G}_0$) sector:

$$\begin{aligned}
 R(h) \cdot \text{Tr}[Q(g)XT^i|i] &= \text{Tr}[M(g)XM(h^{-1})(T^i)^T M(h^{-1})^{-1}]|i\rangle \\
 &= \text{Tr}[M(h^{-1})^T X^T M(g)^T M(h^{-1})^{-1T} T^i]|i\rangle \\
 &= e^{2\pi i \omega(h, h^{-1})} \text{Tr}[M(h^{-1})^T M(hg)^{-1} M(hg) \\
 &\quad \times X^T [M(h)M(g)^{-1T}]^{-1} T^i]|i\rangle \\
 &= e^{2\pi i [\omega(h, h^{-1}) - \omega(h, g)]} \text{Tr}\{[M(hg)M(h^{-1})^{-1T}]^{-1} \\
 &\quad \times [M(hg)X^T M(hg)^{-1}] T^i\}|i\rangle \\
 &= e^{2\pi i [\omega(h, g^{-1}) + \omega(hg^{-1}, h^{-1}) + \omega(g, g^{-1}) - \omega(h, h^{-1})]} \\
 &\quad \times \text{Tr}\{Q(hg^{-1}h^{-1})[M(hg)X^T M(hg)^{-1}] T^i\}|i\rangle, \quad (26)
 \end{aligned}$$

where in the last line we use the fact that

$$\begin{aligned}
 \omega(hg^{-1}h^{-1}, hgh^{-1}) &= -\omega(h, g) - \omega(hg, h^{-1}) \\
 &\quad - \omega(hg^{-1}, h^{-1}) - \omega(h, g^{-1}) \\
 &\quad - \omega(g^{-1}, g) - 2\omega(h^{-1}, h), \quad (27)
 \end{aligned}$$

which can be verified by repeated application of the twisted cocycle condition. We see that under the action of an antiunitary symmetry h the following occurs:

1. The g -twisted sector maps to the $hg^{-1}h^{-1}$ -twisted sector.
2. The operator X is transposed, then conjugated by $M(hg)$ [20].
3. States also pick up a phase of

$$e^{2\pi i [\omega(h, g^{-1}) + \omega(hg^{-1}, h^{-1}) + \omega(g, g^{-1}) - \omega(h, h^{-1})]}. \quad (28)$$

The phase matches Eq. (4.12) of Ref. [18]. In particular, when g acts on the R sector, it is

$$1/2\beta(g) := \omega(g, p) - \omega(p, g) + \omega(p, p), \quad g \notin \mathcal{G}_0. \quad (29)$$

This phase satisfies $\beta(pg) = \beta(g)$, takes values in $\mathbb{Z}/2$, and, together with (13), is a G_b -cocycle. Refer to the Appendix for a proof. When $\gamma = 0$, this is the g -charge of the R sector. However, when $\gamma = 1$, the charge receives an additional contribution from the transformation of X . Similar to in the unitary case detailed above, the total charge is the β -independent quantity $(-1)^{x(g)}$, so this interpretation of β fails.

The invariant β also has an interpretation in terms of edge states, like (8) [21]. A time-reversing symmetry $g \notin G_0$ maps V to its dual space V^* , on which p acts as P^{-1} , so the parity of $Q(g)$ is read off of

$$P^{-1}Q(g)P^{-1} = e^{i\pi\beta(\bar{g})}Q(g), \quad g \notin \mathcal{G}_0. \quad (30)$$

A similar interpretation holds if g reverses the orientation of space. Let $V^* \otimes V$ represent the tensor product of left and right edge state spaces. On this space, g acts as

$$\begin{aligned}
 \psi_L \otimes \psi_R &\mapsto Q(g)^{-1}(\psi_L \otimes \psi_R)^T Q(g) \\
 &= Q(g)^{-1}\psi_R \otimes Q(g)\psi_L. \quad (31)
 \end{aligned}$$

Then β appears as the result of acting by $P \otimes P^{-1}$, g , then $P^{-1} \otimes P$:

$$\psi \otimes 1 \mapsto 1 \otimes PQ(g)P\psi = e^{i\pi\beta(\bar{g})}(1 \otimes \psi). \quad (32)$$

The meaning of α (17) is more difficult to describe in Hamiltonian language [22]. The lack of twisted sectors for antiunitary symmetries means that $\alpha(g_b, h_b)$ has an interpretation as the phase due to fusing domain walls only when g_b and h_b are unitary. The rest of α appears in other places. It is convenient to first describe the invariant ω . For two unitary symmetries $g, h \in \mathcal{G}_0$, the phase $\omega(g, h)$ is due to fusing domain walls. It was shown in Ref. [18] that two extra families of phases—which we now describe—together with ω restricted to \mathcal{G}_0 , determine the full ω on \mathcal{G} . The first family is the phases (28) due to acting on the g -twisted sector by an antiunitary symmetry h . The second family consists of the relative phases due to comparing, for each antiunitary symmetry $g \notin \mathcal{G}_0$, the crosscap state (see Refs. [8,18])

$[Q(g)Q(g)^T]|i\rangle$ to the MPS state in the g^2 -twisted sector. These phases have the simple form $\omega(g, g)$. Note that these data are not gauge invariant, and the equivalence classes of them under shifting ω by a twisted coboundary do not take a simple form. Now that we described the full ω , the full α can be recovered by restricting to G_b . As we demonstrate in the Appendix, the result is a G_b cochain whose x -twisted coboundary is $\beta \cup \rho$.

Finally, γ is the fermion parity of the untwisted Ramond sector, as in the unitary case.

III. FERMIONIC STACKING LAW

Gapped fermionic phases form a commutative monoid under the operation of stacking. The result of stacking fixed point systems corresponding to algebras A_1 and A_2 is the system corresponding to the supertensor product $A_1 \hat{\otimes} A_2$, defined by the multiplication law $(a_1 \hat{\otimes} a_2)(b_1 \hat{\otimes} b_2) = (-1)^{|a_2||b_1|} a_1 b_1 \hat{\otimes} a_2 b_2$ [6,7]. SRE phases are precisely those that are invertible under stacking, and so they form a group. The goal of this section is to derive this group structure on the set of SRE phases in terms of the invariants α , β , and γ . Our plan is to follow the argument presented in Appendix D of Ref. [7], while taking into account that $Q(g)$ is antilinear when $x(g) = 1$. We summarize the results at the end of the section.

The following discussion relies on a result proven in the Appendix: that one can choose a gauge such that the twisted cocycle ω is related to α and β by, for all $g, h \in \mathcal{G}$, where \bar{g} is short for $b(g)$,

$$\omega(g, h) = \alpha(\bar{g}, \bar{h}) + 1/2\beta(\bar{g})\gamma(h). \quad (33)$$

There are three cases to consider: The stacking of (1) two even algebras, (2) an even and an odd algebra, and (3) two odd algebras. When \mathcal{G} does not split, there are no odd algebras so we need only consider the first case.

A. Even-even stacking

Consider the even algebras $\text{End}(U_1)$ and $\text{End}(U_2)$. Their tensor product is $\text{End}(U_1 \widehat{\otimes} U_2)$, where $U_1 \widehat{\otimes} U_2$ carries a projective representation $Q = Q_1 \widehat{\otimes} Q_2$. Then

$$\begin{aligned} Q(g)Q(h) &= [Q_1(g) \widehat{\otimes} Q_2(g)][Q_1(h) \widehat{\otimes} Q_2(h)] \\ &= (-1)^{\beta_2(\bar{g})\beta_1(\bar{h})} Q_1(g)Q_1(h) \widehat{\otimes} Q_2(g)Q_2(h) \\ &= (-1)^{(\beta_2 \cup \beta_1)(\bar{g}, \bar{h})} e^{2\pi i[\alpha_1(\bar{g}, \bar{h}) + 1/2\beta_1(\bar{g})r(h)]} e^{2\pi i[\alpha_2(\bar{g}, \bar{h}) + 1/2\beta_2(\bar{g})r(h)]} Q_1(gh) \widehat{\otimes} Q_2(gh) \\ &= e^{2\pi i(\alpha_1 + \alpha_2 + 1/2\beta_2 \cup \beta_1)(\bar{g}, \bar{h}) + 1/2(\beta_1 + \beta_2)(\bar{g})r(h)} Q(gh). \end{aligned} \quad (34)$$

Thus the invariants of the stacked phase are $\alpha = \alpha_1 + \alpha_2 + 1/2(\beta_1 \cup \beta_2)$ [23] and $\beta = \beta_1 + \beta_2$. Since the stacked algebra is again even, $\gamma = 0$. The presence of antiunitary symmetries does not affect even-even stacking.

B. Even-odd stacking

Now consider the even algebra $A_1 = \text{End}(U_1)$, where U_1 carries a projective representation Q_1 of \mathcal{G} , and the odd algebra $A_2 = \text{End}(U_2) \otimes \mathbb{C}\ell(1)$, where U_2 carries a projective representation Q_2 of G_b . Their tensor product $\text{End}(U_1) \widehat{\otimes} [\text{End}(U_2) \otimes \mathbb{C}\ell(1)]$ is isomorphic as an algebra to the odd algebra $\text{End}(U_1 \otimes U_2) \otimes \mathbb{C}\ell(1)$ by the map

$$JW : M_1 \widehat{\otimes} (M_2 \otimes \Gamma^m) \mapsto M_1 P^m \otimes M_2 \otimes \Gamma^{m+|M_1|}, \quad (35)$$

which has inverse

$$JW^{-1} : M_1 \otimes M_2 \otimes \Gamma^m \mapsto M_1 P^{m+|M_1|} \widehat{\otimes} (M_2 \otimes \Gamma^{m+|M_1|}), \quad (36)$$

where the parity of M_1 is defined by $Q_1 : P_1 M_1 P_1 = (-1)^{|M_1|} M_1$. This isomorphism respects the $\mathbb{Z}/2$ -grading defined by the standard action of fermion parity on even and odd algebras.

It remains to determine the G_b action on the odd algebra. For $g \in \mathcal{G}$ with $t(g) = 0$,

$$\begin{aligned} JW \circ g \circ JW^{-1} \cdot (M_1 \otimes M_2 \otimes \Gamma^m) &= JW \circ g \cdot [M_1 P^{m+|M_1|} \widehat{\otimes} (M_2 \otimes \Gamma^{m+|M_1|})] \\ &= JW \cdot \{Q_1(g)M_1 P^{m+|M_1|} Q_1(g)^{-1} \widehat{\otimes} [Q_2(\bar{g})M_2 Q_2(\bar{g})^{-1} \otimes (-1)^{(m+|M_1|)\beta_2(\bar{g})} \Gamma^{m+|M_1|}]\} \\ &= (-1)^{(m+|M_1|)[\beta_1(\bar{g}) + \beta_2(\bar{g})]} Q_1(g)M_1 Q_1(g)^{-1} \otimes Q_2(\bar{g})M_2 Q_2(\bar{g})^{-1} \otimes \Gamma^m. \end{aligned} \quad (37)$$

To read off the invariants from this group action, we must rewrite it in the standard form by defining $\tilde{Q}_1(g) = Q_1(g)P^{\beta_1(g) + \beta_2(g)}$ [24] and $Q(g) = \tilde{Q}_1(g) \otimes Q_2(\bar{g})$. Then, continuing from (37),

$$g \cdot (M_1 \otimes M_2 \otimes \Gamma^m) = (-1)^{m[\beta_1(\bar{g}) + \beta_2(\bar{g})]} [\tilde{Q}_1(g) \otimes Q_2(\bar{g})] M_1 \otimes M_2 [\tilde{Q}_1(g)^{-1} \otimes Q_2(\bar{g})^{-1}] \otimes \Gamma^m, \quad (38)$$

from which we read off the stacked invariant $\beta = \beta_1 + \beta_2$. And

$$\begin{aligned} Q(g)Q(h) &= [\tilde{Q}_1(g) \otimes Q_2(\bar{g})][\tilde{Q}_1(h) \otimes Q_2(\bar{h})] \\ &= Q_1(g)P^{\beta_1(\bar{g}) + \beta_2(\bar{g})} Q_1(h)P^{\beta_1(\bar{h}) + \beta_2(\bar{h})} \otimes Q_2(\bar{g})Q_2(\bar{h}) \\ &= (-1)^{\beta_1(\bar{h})[\beta_1(\bar{g}) + \beta_2(\bar{g})]} Q_1(g)Q_1(h)P^{\beta_1(\bar{g}\bar{h}) + \beta_2(\bar{g}\bar{h})} \otimes Q_2(\bar{g})Q_2(\bar{h}) \\ &= e^{2\pi i[\alpha_1(g, h) + \alpha_2(g, h) + 1/2(\beta_2 \cup \beta_1)(g, h) + 1/2(\beta_1 \cup \beta_2)(g, h)]} Q(gh), \end{aligned} \quad (39)$$

from which we see $\alpha = \alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2 + 1/2\beta_1 \cup \beta_1$. There is no asymmetry: the $1/2\beta_1 \cup \beta_1$ term always comes from the β of the even algebra [25]. Finally, $\gamma = 1$ since the stacked algebra is odd.

C. Odd-odd stacking

Consider the odd algebras $A_1 = \text{End}(U_1) \otimes \mathbb{C}\ell(1)$, where U_1 carries a projective representation Q_1 of G_b , and $A_2 = \text{End}(U_2) \otimes \mathbb{C}\ell(1)$, where U_2 carries a projective representation Q_2 of G_b . Their tensor product is given by $A_1 \widehat{\otimes} A_2 \simeq \text{End}(U_1 \otimes U_2 \otimes \mathbb{C}^{1|1})$, since $\mathbb{C}\ell(1) \widehat{\otimes} \mathbb{C}\ell(1) \simeq \mathbb{C}\ell(2) \simeq \text{End}(\mathbb{C}^{1|1})$, via an isomorphism

$$(M_1 \otimes \Gamma_1^m) \widehat{\otimes} (M_2 \otimes \Gamma_2^n) \mapsto M_1 \otimes M_2 \otimes \sigma_1^m \sigma_2^n, \quad (40)$$

where σ_1 and σ_2 are any two distinct Pauli matrices. With respect to the action of fermion parity on the $\text{End}(\mathbb{C}^{1|1})$ factor as conjugation by $\sigma_3 = -i\sigma_1\sigma_2$, this map is an isomorphism of $\mathbb{Z}/2$ -graded algebras.

One choice [26] of G_b -action Q on $U_1 \otimes U_2 \otimes \mathbb{C}^2$, with respect to which (40) is equivariant, is

$$g : u_1 \otimes u_2 \otimes v \mapsto Q_1(\bar{g})u_1 \otimes Q_2(\bar{g})u_2 \otimes \sigma_1^{\beta_1(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} v, \quad (41)$$

for $g \in \mathcal{G}$ with $t(g) = 0$, where K denotes complex conjugation in a basis in which σ_1 and σ_2 are real. Then

$$\begin{aligned} Q(\bar{g})Q(\bar{h}) &= [Q_1(\bar{g}) \otimes Q_2(\bar{g}) \otimes \sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})}] [Q_1(\bar{h}) \otimes Q_2(\bar{h}) \otimes \sigma_1^{\beta_2(\bar{h})} \sigma_2^{\beta_1(\bar{h})} K^{x(\bar{h})}] \\ &= e^{2\pi i \alpha_1(\bar{g}, \bar{h})} Q_1(\bar{g}\bar{h}) \otimes e^{2\pi i \alpha_2(\bar{g}, \bar{h})} Q_2(\bar{g}\bar{h}) (-1)^{\beta_1(\bar{g})\beta_2(\bar{h})} \sigma_1^{\beta_1(\bar{g}\bar{h})} \sigma_2^{\beta_2(\bar{g}\bar{h})} K^{x(\bar{g}\bar{h})} \\ &= e^{2\pi i (\alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2)(\bar{g}, \bar{h})} Q(\bar{g}\bar{h}), \end{aligned} \quad (42)$$

from which we see that $\alpha = \alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2$. Since $U_1 \otimes U_2$ is purely even, the parity of Q comes from

$$\begin{aligned} P \sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} P &= (-i\sigma_1\sigma_2) \sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} (-i\sigma_1\sigma_2) \\ &= (-1)^{\beta_2(\bar{g}) + \beta_1(\bar{g})} \sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} (-1)^{x(\bar{g})}. \end{aligned} \quad (43)$$

We read off $\beta = \beta_1 + \beta_2 + x$. Finally, the stacked algebra is even, so $\gamma = 0$.

D. Summary

The stacking law for the invariants (α, β, γ) is given by

$$\begin{aligned} (\alpha_1, \beta_1, 0) \cdot (\alpha_2, \beta_2, 0) &= (\alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2, \beta_1 + \beta_2, 0), \\ (\alpha_1, \beta_1, 0) \cdot (\alpha_2, \beta_2, 1) &= (\alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2 + 1/2\beta_1 \cup \beta_1, \beta_1 + \beta_2, 1), \\ (\alpha_1, \beta_1, 1) \cdot (\alpha_2, \beta_2, 1) &= (\alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2, \beta_1 + \beta_2 + x, 0). \end{aligned} \quad (44)$$

This group law inherits the properties of commutativity and associativity from the tensor product of algebras. When \mathcal{G} does not split, γ is not present, and the stacking law is simply

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2, \beta_1 + \beta_2). \quad (45)$$

We emphasize that while data $[\alpha, \beta]$ are equivalent to $[\omega] \in H^2[\mathcal{G}, U(1)_T]$, the group structure on $H^2[\mathcal{G}, U(1)_T]$ differs from (44). On the other hand, the stacking of bosonic SRE phases, which are also characterized by classes $[\omega]$, is described by the usual group structure on $H^2[\mathcal{G}, U(1)_T]$.

IV. EXAMPLES

A. Class BDI fermions: $\mathcal{G} = \mathbb{Z}_2^F \times \mathbb{Z}_2^T$

Let us consider SRE phases with symmetry $\mathcal{G} = \mathbb{Z}_2^F \times \mathbb{Z}_2^T$. The two classes $\alpha \in H^2[\mathbb{Z}_2^T, U(1)_T] = \mathbb{Z}/2$, two classes $\beta \in H^1[\mathbb{Z}_2^T, \mathbb{Z}/2] = \mathbb{Z}/2$, and two classes $\gamma \in \mathbb{Z}/2$ make for a total of eight phases. A straightforward application of the general stacking law (44) reveals that these phases stack like the cyclic group $\mathbb{Z}/8$. In this section, we will reproduce this group law by exploiting the relationship between \mathcal{G} -equivariant algebras, real superdivision algebras, and Clifford algebras, which have Bott periodicity $\mathbb{Z}/8$.

We begin by describing simple \mathcal{G} -equivariant algebras. The matrix algebra $M_2\mathbb{C}$ represents the sole Morita class of simple complex algebras. This algebra has a unitary structure $*$ given by conjugate transposition. Its action fixes a basis $\{\mathbb{1}, X, Y, Z = -iXY\}$. On $C\ell_2\mathbb{C} \simeq M_2\mathbb{C}$, $*$ acts by Clifford transposition and complex conjugation of coefficients with respect to a pair of generators that square to $+1$.

There are two distinct real structures on $M_2\mathbb{C}$ given by complex conjugation T on the second component of $M_2\mathbb{C} \simeq M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ and $M_2\mathbb{C} \simeq \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. The unitary structure $*$ of $M_2\mathbb{C}$ acts by transposition on $M_2\mathbb{R}$, complex conjugation on \mathbb{C} , and inversion of the generators \hat{i} and \hat{j} of \mathbb{H} ; that is, its fixed bases are

$$\begin{aligned} \{\mathbb{1} \otimes 1, X \otimes 1, iY \otimes i, Z \otimes 1\} &\in M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}, \\ \{\mathbb{1} \otimes 1, \hat{i} \otimes i, \hat{j} \otimes i, \hat{k} \otimes i\} &\in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned} \quad (46)$$

These bases have the same T -eigenvalues as they do $*T$ -eigenvalues, where $*T$ acts as transposition on $M_2\mathbb{R}$ and

inversion of generators on \mathbb{H} . Under the algebra isomorphisms $M_2\mathbb{R} \simeq C\ell_{1,1}\mathbb{R} \simeq C\ell_{2,0}\mathbb{R}$ and $\mathbb{H} \simeq C\ell_{0,2}\mathbb{R}$, $*T$ acts by inverting the generators and products of generators that square to -1 .

Let us derive the invariants α of these real structures. Pulled back from $M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ to $M_2\mathbb{C}$, T acts like complex conjugation and $*T$ like transposition; that is $M(t) = \mathbb{1}$. Then

$$M(t)M(t)^{-1T} = \mathbb{1}, \quad (47)$$

which means $\alpha(t, t) = \omega(t, t) = 0$. Pulled back from $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, T acts like complex conjugation and conjugation by Y , while $*T$ acts like transposition and conjugation by Y ; that is $M(t) = Y$. Then $\alpha(t, t) = 1/2$ since

$$M(t)M(t)^{-1T} = e^{i\pi} \mathbb{1}. \quad (48)$$

By the Skolem-Noether theorem, a superalgebra structure on $M_2\mathbb{C}$ is given by conjugation by an element that squares to 1. If this element is $\mathbb{1}$, the $\mathbb{Z}/2$ -grading is purely even; otherwise, it has two even dimensions and two odd. All structures of the last type are isomorphic in the absence of a real structure.

In the presence of the real structure $M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, there are three distinct gradings. First, there is the purely even grading, given by $P = \mathbb{1}$. This structure has $1/2\beta(t) = \omega(t, p) = 0$. Second, there is conjugation by Z (or X), which gives $M_2\mathbb{R}$ the superalgebra structure of $C\ell_{1,1}\mathbb{R}$. Again, $\beta(t) = 0$ since $P = Z$ means

$$PM(t)P^T = Z\mathbb{1}Z^T = \mathbb{1}. \quad (49)$$

The matching of the invariants alludes to the fact that the real superalgebra structures $M_2\mathbb{R}$ and $C\ell_{1,1}\mathbb{R}$ are graded Morita equivalent. Third, there is conjugation by Y ; that is, $P = Y$. Then $\beta(t) = 1$ since

$$PM(t)P^T = Y\mathbb{1}Y^T = e^{i\pi}\mathbb{1}. \quad (50)$$

The corresponding real Clifford algebra is $C\ell_{2,0}\mathbb{R}$ and represents a distinct Morita class.

On the real structure $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, there are two distinct gradings. First, there is the purely even grading $P = \mathbb{1}$, which has $\beta(t) = 0$. The second grading is given by conjugation by Z (or X or Y) on $M_2\mathbb{C}$ and gives \mathbb{H} the superalgebra structure of $C\ell_{0,2}\mathbb{R}$. Then $\beta(t) = 1$ since

$$PM(t)P^T = ZYZ^T = e^{i\pi}Y. \quad (51)$$

Now consider algebras of the form $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$. The second component $C\ell_1\mathbb{C}$ has a unitary structure $*$ given by complex conjugation of coefficients of the generator Γ that squares to $+1$. There are two distinct real structures on $C\ell_1\mathbb{C}$ given by complex conjugation T on the second component of $C\ell_{1,0}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ and $C\ell_{0,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$. The unitary structure $*$ of $C\ell_1\mathbb{C}$ acts by complex conjugation on \mathbb{C} and inversion of generators that square to -1 ; that is, the fixed bases are $\{1 \otimes 1, \gamma \otimes 1\}$ for $C\ell_{1,0}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ and $\{1 \otimes 1, \gamma \otimes i\}$ for $C\ell_{0,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$. The map $*T$ is trivial on $C\ell_{1,0}\mathbb{R}$ and inversion of the generator on $C\ell_{0,1}\mathbb{R}$. Therefore, pulled back from $C\ell_{1,0}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ to $C\ell_1\mathbb{C}$, T is complex conjugation and $*T$ is trivial. From $C\ell_{0,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, $*T$ is inversion of Γ .

As discussed in Sec. II and in Ref. [7], we need only to consider a single $\mathbb{Z}/2$ -grading on $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$ —the one where $M_2\mathbb{C}$ is purely even and the generator of $C\ell_1\mathbb{C}$ is odd. The algebra $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$ has four real structures: a choice of $M_2\mathbb{R}$ or \mathbb{H} for the first component and $C\ell_{1,0}\mathbb{R}$ or $C\ell_{0,1}\mathbb{R}$ for the second. As was true for even algebras, the first choice determines whether $M(t)$ is $\mathbb{1}$ or Y ; that is, whether $\alpha(t, t)$ is 0 or $1/2$. The second choice determines whether $*T$ inverts the odd generator; this is $\beta(t)$.

Due to the Morita equivalence $M_2\mathbb{R} \sim \mathbb{R}$, several of the eight Morita classes are represented by algebras of lower dimension; for example, $C\ell_{1,0}\mathbb{R}$ instead of $M_2\mathbb{R} \otimes_{\mathbb{R}} C\ell_{1,0}\mathbb{R}$. Up to this substitution, the eight real-structured superalgebras we found are complexifications of the eight *central real superdivision algebras*—real superalgebras with center \mathbb{R} that are invertible under a supertensor product up to graded Morita equivalence [27,28]. They constitute a set of representatives of the eight graded Morita classes of real superalgebras. These algebras appear in the second column of Table I, next to their invariants in the third column.

Another set of Morita class representatives is the Clifford algebras $C\ell_{n,0}\mathbb{R}$. In terms of these algebras, stacking is simple, as

$$C\ell_{n,0}\mathbb{R} \widehat{\otimes} C\ell_{m,0}\mathbb{R} \simeq C\ell_{n+m,0}\mathbb{R} \quad (52)$$

and

$$C\ell_{n,0}\mathbb{R} \sim C\ell_{m,0}\mathbb{R} \quad \text{for } n \equiv m \pmod{8}. \quad (53)$$

Each central superdivision algebra can be matched with the Clifford algebra $C\ell_{n,0}\mathbb{R}$, $n < 8$ in its Morita class [28], as in the first column of Table I. This determines a $\mathbb{Z}/8$ stacking

TABLE I. The 10-fold way of $\mathbb{Z}_2^F \times \mathbb{Z}_2^T$ -symmetric fermionic phases.

$C\ell_{n,0}$	A_{div}	α, β, γ	Fermionic	Bosonic	(H, ω)
0	\mathbb{R}	0,0,0	trivial	trivial	$(\mathcal{G}, 0)$
1	$C\ell_{1,0}$	0,0,1	SRE	SB	$(\mathbb{Z}_2^T, 0)$
2	$C\ell_{2,0}$	0,1,0	SRE	SPT	(\mathcal{G}, ω_1)
3	$\mathbb{H} \otimes C\ell_{0,1}$	1,1,1	SRE	mixed	$(\mathbb{Z}_2^{\text{diag}}, \alpha)$
4	\mathbb{H}	1,0,0	SRE	SPT	(\mathcal{G}, ω_2)
5	$\mathbb{H} \otimes C\ell_{1,0}$	1,0,1	SRE	mixed	(\mathbb{Z}_2^T, α)
6	$C\ell_{0,2}$	1,1,0	SRE	SPT	$(\mathcal{G}, \omega_1 + \omega_2)$
7	$C\ell_{0,1}$	0,1,1	SRE	SB	$(\mathbb{Z}_2^{\text{diag}}, 0)$
-	\mathbb{C}	-	SB	SB	$(\mathbb{Z}_2^F, 0)$
-	$\mathbb{C}\ell_1$	-	SB	SB	$(1,0)$

law on central superdivision algebras and their invariants that agrees with the more general law (44).

Physically speaking, the $\mathbb{Z}/8$ classification is generated by the time-reversal-invariant Majorana chain [4,29]. While the symmetry protects pairs of dangling Majorana zero modes from being gapped out, turning on interactions can gap out these modes in groups of eight. Fidkowski and Kitaev formulated their stacking law in terms of three invariants that are equivalent to α, β , and γ . Their results match ours.

For contrast, we list the invariants of the corresponding bosonic phases in the rightmost column of Table I. There, H denotes the subgroup of unbroken symmetries and ω denotes 2-cocycle characterizing the SPT order. These invariants can be obtained from the fermionic invariants [7]. We observe that invertibility is not preserved by bosonization; in particular, only the fermionic SREs with $\gamma = 0$ become bosonic SREs. The four bosonic SRE phases have a $\mathbb{Z}/2 \times \mathbb{Z}/2$ stacking law. We also include the two *noncentral* superdivision algebras \mathbb{C} and $\mathbb{C}\ell_1$ at the bottom of the table. These correspond to symmetry-breaking (SB) phases.

B. Class DIII fermions: $\mathcal{G} = \mathbb{Z}_4^{FT}$

In the following, $\mathcal{G} = \mathbb{Z}_4^{FT}$ denotes the nontrivial extension of $G_b = \mathbb{Z}_2^T$ by fermion parity. Let us consider fermionic SRE phases with this symmetry. There are two distinct classes $\beta \in H^1[\mathbb{Z}_2^T, \mathbb{Z}/2]$, determined by $\beta(t) = 0$ and $\beta(t) = 1$. The trivial β has a single α , the trivial one, that satisfies $\delta_T \alpha = 1/2\beta \cup \rho$, up to the proper equivalence [30]. The nontrivial β also has a single compatible α , up to equivalence $\alpha(t, t) = 1/4$.

The trivial phase is represented by the algebra \mathbb{C} with trivial actions of p and t , as always. For the nontrivial phase, consider $A = \text{End}(U)$, where P and T act on U as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad M(t) = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}. \quad (54)$$

Then the invariants can be recovered as follows:

$$M(t)M(t)^{-1T} = e^{2\pi i/4}P \quad \Rightarrow \quad \alpha(t, t) = 1/4, \quad (55)$$

$$PM(t)P^T = e^{i\pi}M(t) \quad \Rightarrow \quad \beta(t) = 1, \quad (56)$$

TABLE II. Fermionic phases with unitary and antiunitary symmetries.

Symmetry class	Fermionic	Bosonic
$\mathbb{Z}_2 \times \mathbb{Z}_2^T \times \mathbb{Z}_2^F$	$\mathbb{Z}_4 \times \mathbb{Z}_8^\gamma$	$(\mathbb{Z}_2)^4$
$\mathbb{Z}_2 \times \mathbb{Z}_4^{FT}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_2^T \times \mathbb{Z}_4^F$	\mathbb{Z}_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_2^F \times \mathbb{Z}_4^T$	$\mathbb{Z}_2 \times \mathbb{Z}_4^\gamma$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
\mathbb{Z}_8^{FT}	\mathbb{Z}_2	\mathbb{Z}_2

According to the rule (44), stacking two copies of this phase results in the trivial phase

$$(1/4, 1) \cdot (1/4, 1) = (1/4 + 1/4 + 1/2 \cdot 1 \cdot 1, 1 + 1) = (0, 0). \quad (57)$$

We find that fermionic SRE phases with symmetry \mathbb{Z}_4^{FT} have a $\mathbb{Z}/2$ classification, in agreement with the condensed matter literature [31,32]. The nontrivial phase appears as a Majorana chain with two dangling modes protected by the symmetry.

C. Unitary $\mathbb{Z}/2$ symmetry

As a last set of examples, let us consider systems with a unitary bosonic symmetry group $G_0 = \mathbb{Z}/2$, in addition to time-reversal and fermion parity. There are many ways to organize these symmetries into a full symmetry class (\mathcal{G}, p, x) . Here we consider the five abelian possibilities, which are listed with their fermionic and bosonic phase classifications in Table II. The first three have $G_b = \mathbb{Z}_2 \times \mathbb{Z}_2^T$, the last two $G_b = \mathbb{Z}_4^T$. In the two cases where the central extension of G_b by \mathbb{Z}_2^F splits, we use a superscript γ to denote the subgroup of the fermionic classification that contains the odd phases.

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APPENDIX: RELATIONS BETWEEN BOSONIC AND FERMIONIC INVARIANTS

Lemma A.1. For a twisted cocycle $\omega \in Z^2[\mathcal{G}, U(1)_T]$, the 1-cochain defined by

$$\begin{aligned} 1/2\beta(g) &:= \omega(g, p) - \omega(p, g) + x(g)\omega(p, p) \\ &= \begin{cases} \omega(g, p) - \omega(p, g), & g \in G_0, \\ \omega(g, p) - \omega(p, g) + \omega(p, p), & g \notin G_0 \end{cases} \end{aligned} \quad (A1)$$

is gauge-invariant, satisfies $\beta(gp) = \beta(g)$, takes values in $\{0, 1/2\}$, and defines a G_b -cocycle.

Proof. First, $1/2\beta(g)$ picks up a factor of

$$\begin{aligned} [L(g) + (-1)^{x(g)}L(p) - L(gp)] - [L(p) + L(g) - L(gp)] + x(g)[L(p) + L(p) - L(1)] \\ = -2x(g)L(p) + 2x(g)L(p) - x(g)L(1) = 0 \end{aligned} \quad (A2)$$

under a transformation $\omega \mapsto \omega + \delta_T L$ for some 1-cochain L of \mathcal{G} satisfying $L(1) = 0$ [33].

Second,

$$\begin{aligned} 1/2\beta(gp) &= \omega(gp, p) - \omega(p, gp) + x(gp)\omega(p, p) \\ &= \omega(g, p) - \omega(p, g) + x(g)\omega(p, p) - \delta_T \omega(p, g, p) \\ &= 1/2\beta(g). \end{aligned} \quad (A3)$$

Third,

$$\begin{aligned} \omega(g, p) - \omega(p, g) &= (-1)^{x(g)}\omega(p, p) - \omega(g, p) - \omega(p, gp) - (\delta_T \omega)(g, p, p) + (\delta_T \omega)(p, g, p) \\ &= (-1)^{x(g)}\omega(p, p) - \omega(g, p) - \omega(p, p) + \omega(p, g) + (\delta_T \omega)(p, p, g) \\ &= -\omega(g, p) + \omega(p, g) - [1 - (-1)^{x(g)}]\omega(p, p) \end{aligned} \quad (A4)$$

means that $1/2\beta$ takes values in the $\mathbb{Z}/2$ subgroup of $U(1)$.

Therefore $1/2\beta$ defines a $\beta \in C^1[G_b, \mathbb{Z}/2]$. Let $g_b, h_b \in G_b$ and choose any lifts g, h to \mathcal{G} . Fourth,

$$\begin{aligned} (\delta\beta)(g_b, h_b) &= 1/2[\beta(g) + \beta(h) - \beta(ghp^{\rho(\bar{g}, \bar{h})})] \\ &= 1/2[\beta(g) + \beta(h) - \beta(gh)] \\ &= \omega(g, p) - \omega(p, g) + x(g)\omega(p, p) + \omega(h, p) - \omega(p, h) + x(h)\omega(p, p) - \omega(gh, p) + \omega(p, gh) - x(gh)\omega(p, p) \\ &= \omega(g, p) - \omega(p, g) + \omega(h, p) - \omega(p, h) + \omega(g, h) - \omega(g, hp) - (-1)^{x(g)}\omega(h, p) \\ &\quad - \omega(g, h) + \omega(p, g) + \omega(pg, h) + 2x(g)x(h)\omega(p, p) \end{aligned}$$

$$\begin{aligned}
&= \omega(g, p) + 2x(g)\omega(h, p) - \omega(p, h) + (-1)^{x(g)}\omega(p, h) - \omega(g, p) + 2x(g)x(h)\omega(p, p) \\
&= 2x(g)[\omega(h, p) - \omega(p, h) + x(h)\omega(p, p)] \\
&= 2x(g) \cdot 1/2\beta(h) \\
&= 0.
\end{aligned} \tag{A5}$$

Lemma A.2. Each cohomology class $H^2[\mathcal{G}, U(1)_T]$ contains an element ω that satisfies, for all $g, h \in \mathcal{G}$,

$$\omega(pg, h) = \omega(g, h), \tag{A6}$$

$$\omega(g, ph) = \omega(g, h) + \omega(g, p). \tag{A7}$$

Proof. For an arbitrary 2-cocycle $W \in Z^2[\mathcal{G}, U(1)_T]$, define

$$\omega = W - \delta_T L, \tag{A8}$$

where $L \in C^1[\mathcal{G}, U(1)_T]$ satisfies

$$L(1) = 0,$$

$$L(p) = 1/2W(p, p) \text{ or } 1/2W(p, p) + 1/2, \tag{A9}$$

$$L(p\bar{g}) = L(\bar{g}) - W(p, \bar{g}) + L(p).$$

Here we abuse notation by letting \bar{g} denote a $g \in \mathcal{G}$ with $t(g) = 0$. This implies $L(p) = 1/2W(p, p)$. We fixed $L(p\bar{g})$ in terms of $L(\bar{g})$ and $L(p)$, but left $L(\bar{g})$ undetermined, while $L(p)$ is fixed up to a $1/2$.

We see that

$$\begin{aligned}
\omega(p, p) &= W(p, p) - (-1)^{x(p)}L(p) - L(p) + L(1) \\
&= W(p, p) - 2 \cdot 1/2W(p, p) = 0
\end{aligned} \tag{A10}$$

and

$$\begin{aligned}
\omega(p, \bar{g}) &= W(p, \bar{g}) - [(-1)^{x(p)}L(\bar{g}) + L(p) - L(p\bar{g})] \\
&= W(p, \bar{g}) - W(p, \bar{g}) + 1/2W(p, p) - 1/2W(p, p) \\
&= 0.
\end{aligned} \tag{A11}$$

Next we show that any ω satisfying (A10) and (A11) must also satisfy the gauge conditions (A6) and (A7). First,

$$\begin{aligned}
\omega(p, p\bar{g}) &= -\delta_T \omega(p, p, \bar{g}) - (-1)^x(p)\omega(p, \bar{g}) \\
&\quad + \omega(p, p) + \omega(1, \bar{g}) = 0.
\end{aligned} \tag{A12}$$

Similarly, computing $0 = \delta_T \omega(p, \bar{g}, \bar{h})$ shows that $\omega(\bar{g}p, \bar{h}) = \omega(\bar{g}, \bar{h})$ and computing $0 = \delta_T \omega(p, \bar{g}, \bar{h}p)$ shows that $\omega(\bar{g}p, \bar{h}p) = \omega(\bar{g}, \bar{h}p)$. Putting these together, we see that (A6) is satisfied.

Now we compute $0 = \delta_T \omega(\bar{g}, p, \bar{h})$ which shows that $\omega(\bar{g}, p\bar{h}) = \omega(\bar{g}, \bar{h}) + \omega(\bar{g}, p)$ and $0 = \delta_T \omega(\bar{g}, p, p\bar{h})$ which shows that $\omega(\bar{g}, \bar{h}) = \omega(\bar{g}, p\bar{h}) + \omega(\bar{g}, p)$. Putting these together, we see that (A7) is satisfied.

Lemma A.3. Given a trivialization t , the map

$$\omega(g, h) = \alpha(\bar{g}, \bar{h}) + 1/2\beta(\bar{g})t(h) \tag{A13}$$

defines a bijection from pairs $(\alpha, \beta) \in C^2[G_b, U(1)_T] \times C^1[G_b, \mathbb{Z}/2]$ that satisfy $\delta_T \alpha = 1/2\beta \cup \rho$ and $\delta\beta = 0$ [where $1/2\beta$ is regarded as a $U(1)_T$ -valued cocycle] to twisted cocycles $\omega \in Z^2[\mathcal{G}, U(1)_T]$ that satisfy (A6) and (A7) for all

$g, h \in \mathcal{G}$. In particular, for all $g_b, h_b \in G_b$, this map has an inverse

$$\begin{aligned}
\alpha(g_b, h_b) &= \omega[s(g_b), s(h_b)], \\
1/2\beta(g_b) &= \omega[s(g_b), p].
\end{aligned} \tag{A14}$$

Proof. First we show that ω is a twisted cocycle:

$$\begin{aligned}
(\delta_T \omega)(g, h, k) &= (-1)^{x(g)}\omega(h, k) + \omega(g, hk) - \omega(g, h) - \omega(gh, k) \\
&= (-1)^{x(g)}\alpha(\bar{h}, \bar{k}) + \alpha(\bar{g}, \bar{h}\bar{k}) - \alpha(\bar{g}, \bar{h}) - \alpha(\bar{g}\bar{h}, \bar{k}) \\
&\quad + 1/2(-1)^{x(g)}\beta(\bar{h})t(k) + 1/2\beta(\bar{g})t(hk) \\
&\quad - 1/2\beta(\bar{g})t(h) - 1/2\beta(\bar{g}\bar{h})t(k) \\
&= (\delta_T \alpha)(\bar{g}, \bar{h}, \bar{k}) + 1/2(\delta_T \beta)(\bar{g}, \bar{h})t(k) \\
&\quad - 1/2\beta(\bar{g})(\delta t)(h, k) \\
&= 0.
\end{aligned} \tag{A15}$$

Next we verify that ω satisfies the gauge conditions

$$\begin{aligned}
\omega(pg, h) &= \alpha(\bar{p}\bar{g}, \bar{h}) + 1/2\beta(\bar{p}\bar{g})t(h) \\
&= \alpha(\bar{g}, \bar{h}) + 1/2\beta(\bar{g})t(h) \\
&= \omega(g, h),
\end{aligned} \tag{A16}$$

$$\begin{aligned}
\omega(g, ph) &= \alpha(\bar{g}, \bar{p}\bar{h}) + 1/2\beta(\bar{g})t(ph) \\
&= \alpha(\bar{g}, \bar{h}) + 1/2\beta(\bar{g})[t(h) + 1] \\
&= \omega(g, h) + \omega(g, p).
\end{aligned} \tag{A17}$$

Then we check the conditions for α and β . For these two calculations, let \bar{g} denote g_b and g denote $s(g_b)$. Note that $s(\bar{g}h) = p^{\rho(\bar{g}, \bar{h})}gh$. Then

$$\begin{aligned}
(\delta_T \alpha)(\bar{g}, \bar{h}, \bar{k}) &= (-1)^{x(g)}\alpha(\bar{h}, \bar{k}) + \alpha(\bar{g}, \bar{h}\bar{k}) - \alpha(\bar{g}, \bar{h}) - \alpha(\bar{g}\bar{h}, \bar{k}) \\
&= (-1)^{x(g)}\omega(h, k) + \omega(g, p^{\rho(\bar{h}, \bar{k})}hk) \\
&\quad - \omega(g, h) - \omega(p^{\rho(\bar{g}, \bar{h})}gh, k) \\
&= (\delta_T \omega)(g, h, k) + \{\text{terms of the form } \omega(p^-, -)\} \\
&\quad + \omega(g, p^{\rho(\bar{h}, \bar{k})}) + (\delta_T \omega)(p^{\rho(\bar{g}, \bar{h})}, gh, k) \\
&\quad - (\delta_T \omega)(g, p^{\rho(\bar{h}, \bar{k})}, hk) + (\delta_T \omega)(p^{\rho(\bar{h}, \bar{k})}, g, hk) \\
&= 1/2\beta(\bar{g})\rho(\bar{h}, \bar{k}).
\end{aligned} \tag{A18}$$

The object $1/2\beta$ defined in (A14) is the gauge-fixed form of (A1). Then, by A.1, it defines a $\beta \in Z^1[G_b, \mathbb{Z}/2]$.

It remains to show that these maps are indeed inverses. Since $1/2\beta$ is the image of a $\mathbb{Z}/2$ -valued cocycle β , ω can be written with a minus sign like $\omega = \alpha - 1/2\beta \cup t$. Note also

that $s(\bar{g}) = p^{t(g)}g$. Then

$$\begin{aligned}\omega(g, h) &= \alpha(\bar{g}, \bar{h}) - \frac{1}{2}\beta(\bar{g})t(h) \\ &= \omega(p^{t(g)}g, p^{t(h)}h) - \omega(p^{t(g)}g, p)t(h) \\ &= \omega(g, h),\end{aligned}\tag{A19}$$

$$\begin{aligned}\alpha(g_b, h_b) &= \omega[s(g_b), s(h_b)] \\ &= \alpha(g_b, h_b) + \frac{1}{2}\beta(g_b)t[s(g_b)] \\ &= \alpha(g_b, h_b),\end{aligned}\tag{A20}$$

$$\begin{aligned}\frac{1}{2}\beta(g_b) &= \omega[s(g_b), p] \\ &= \alpha(g_b, 1) + \frac{1}{2}\beta(g_b)t(p) \\ &= \frac{1}{2}\beta(g_b).\end{aligned}\tag{A21}$$

Theorem A.4. $H^2[\mathcal{G}, U(1)_T]$ equals, as a set, the set of pairs (α, β) (see A.3) modulo the equivalence

$(\alpha', \beta) \sim (\alpha, \beta)$ if $\alpha' = \alpha + \delta_T \lambda$ with λ a cochain in $C^1[\mathcal{G}, U(1)_T]$ satisfying $\lambda[s(g_b)p] = \lambda[s(g_b)] + \lambda(p)$ [34].

Proof. The preceding lemmas show that the set of twisted cocycles ω satisfying the gauge conditions (A6) and (A7) is equivalent to the set of pairs (α, β) . After transforming ω into this gauge, there remains freedom to choose $L(g)$ for each $g \in \mathcal{G}$ such that $t(g) = 0$, and to shift $L(p)$ by $\frac{1}{2}$. We have already seen that β is invariant under an arbitrary gauge transformation. However, there is some residual gauge freedom for α .

Let $\omega' = \omega + \delta_T \lambda$ be another 2-cocycle satisfying the gauge conditions. It takes the form $W - \delta_T L'$, with L' possibly differing from L in its values on $s(g_b)$ and p . We see from $\delta_T \lambda = \omega' - \omega = \delta_T(L' - L)$ that $\lambda = L' - L + \kappa$ where κ is a twisted 1-cocycle. Then, by (A9), $\lambda[s(g_b)p] = \lambda[s(g_b)] + \lambda(p)$. The quantities $L(p)$, $L'(p)$, $\kappa(p)$, and therefore $\lambda(p)$, can each be chosen to be 0 or $\frac{1}{2}$. Finally, by (A14), this freedom in gauge-fixed ω translates into the desired freedom in α .

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- [21] If g is antilinear, the expression (8) is not invariant under the change of gauge $\omega \mapsto \omega + \delta\Lambda$.
- [22] In the Lagrangian picture, we expect α to be related to trivalent junctions of possibly orientation-reversing domain walls.
- [23] We used the fact that $\beta_2 \cup \beta_1$ is cohomologous to $\beta_1 \cup \beta_2$ in $\mathbb{Z}/2$.
- [24] Adding a phase factor to \tilde{Q} would have shifted the resulting 2-cocycle α by an irrelevant coboundary. For example, if we had chosen a factor $i^{\beta_1(g)}$ as in Ref. [7], we would have gotten $\beta_1 \cup x$ instead of $\beta_1 \cup \beta_1$ in the final answer.
- [25] Note that while $\beta_1 \cup \beta_1$ is an ordinary coboundary and hence could be ignored for phases without time-reversal, it is not a twisted coboundary and so cannot be ignored when time-reversing symmetries are present. By adding a twisted coboundary, we can put it in the form $\beta_1 \cup x$, which makes the dependence on time-reversal symmetry manifest.

- [26] Again, had we chosen a different G_b -action on $\mathbb{C}^{1|1}$ compatible with the action $g_b : \Gamma_i \mapsto (-1)^{\beta_i(g_b)} \Gamma_i$ on the $\mathbb{C}\ell(1)$ factors, the 2-cocycle α would be shifted by a twisted coboundary.
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