Multiresolution Operator Decomposition for Flow Simulation in Fractured Porous Media

Qingfu Zhang a,b, Houman Owhadi b, Jun Yao a*, Florian Schäfer b, Zhaoqin Huang a, Yang Li a

a China University of Petroleum (East China), Qingdao 266580, China

b California Institute of Technology, Pasadena, CA 91125

Abstract

Fractures should be simulated accurately given their significant effects on whole flow patterns in porous media. But such high-resolution simulation imposes severe computational challenges to numerical methods in the applications. Therefore, the demand for accurate and efficient technique is widely increasing. A near-linear complexity multiresolution decomposition is proposed for solving flow problems in fractured porous media. In this work, Discrete Fracture Model (DFM) is used to describe fractures, in which the fractures are explicitly represented as (n-1) dimensional elements. The solution space is decomposed into several subspaces and we then compute the corresponding solutions of DFM in each subspace. The pressure distribution of fractured porous media is obtained by combing the DFM solutions of all subspaces. Numerical results are presented to demonstrate the accuracy and efficiency of the proposed multigrid method. The comparisons with standard method show that the proposed multigrid method is a promising method for flow simulation in fractured porous media.

Keywords

Multigrid method; Discrete fracture model; Flow simulation; Fractured porous media; Multiresolution decomposition
1. Introduction

Subsurface flows in porous media are impacted by heterogeneities in multiscale scales. It is a challenging problem to solve numerically all the scales. Apart from the rough heterogeneous properties, these porous media commonly contain fractures, which will seriously aggravate the heterogeneity. These heterogeneous characteristics should be taken seriously given their significant influence on whole flow patterns. However, the porous flow problems with rough permeability fields impose great difficulties on the direct numerical simulation. The necessary resolution solutions may need several millions of grid cells which are computationally costly.

Among the flow models aimed at representing fractures effectively, dual-porosity model [1-2] supposes that the fractures are highly permeable and interconnected with each other. Although this model is widely applied in the field, researches show that it is only available for the porous media in which the fractures are highly developed. Furthermore, this method could not efficiently model the fluid flow in the large-scale fractures [3]. Large-scale fractures were then described explicitly with single-porosity model in which fractures are regarded as narrow region [4]. Obviously, this model has an expected resolution, but it has to suffer a complicated computation and grid generation process. Equivalent continuum model [36-38] is instead used to form an equivalent permeability field to describe the heterogeneity of the fractured porous media. While ECM is efficient and convenient, it works well only when there exists REV in the fractured media [36]. As an alternative, the discrete fracture model (DFM) is developed [5-9]. The n-D fractures are simplified to (n-1)-D elements using the flux equivalent theory. That is, taking the 2D problems for example, the fracture is treated as a line element and matched grid system is applied. That is the fractures are regarded as inner boundaries of the region. Therefore, DFM is appealing for its ability to avoid the complicated grid generation inside the fractures.

Although DFM is attractive for its simple and efficiency, it is deemed intractable for implement in application as there were no efficient numerical methods could solve DFM fast and accurately. A large amount of CPU time and compute memory are needed which may beyond the controllable level. This issue motivated the development of fast numerical methods.

During the past few years, numerical homogenization was applied as fast methods to model fluid flow in heterogeneous porous media, including upscaling techniques [10-11], multiscale methods [12-15] etc. Recently, multiscale methods have been extended to conduct flow simulation in fractured porous media [16-19] by
using multiscale functions to capture the heterogeneous information on the fine-scale. Although numerical homogenization has obtained satisfying results, and made contributions to the development of fast methods toward fractured porous media, there are difficulties in the identification of interpolation and restriction process [31]. Given multigrid methods are now well known as being one of the fastest for solving elliptic problems [20-23], in this paper, we propose a multigrid formulation for flow simulation in fractured porous media.

While multigrid methods have been effectively extended to solve many kinds of partial differential equations, their convergence rate can be severely affected by the lack of regularity of coefficients [24-25]. Therefore, their ability for flow simulation is limited by the intrinsic strong heterogenous permeability field in porous media. Even though some improved multigrid methods could achieve robustness to some extent [26-29], the design of multigrid methods that are provably robust with respect to rough coefficients was an open problem of practical importance [30] as introduced in [31-32]. Alternative methods such as [33-35] are able to obtain a multiresolution compression of the solution space, however their ability to solve PDEs is impacted by the regularity of coefficients. Hence, it is desirable to develop a method that could lower the computational complexity.

In this paper, we introduce a near-linear complexity multiresolution operator decomposition method (multigrid method) [32] and extend it to simulation of fluid flow in porous media. Then we continue the research and successfully capture the influences of fractures in fractured porous media. Gamblets are derived from a Bayesian and a game theoretic approach to numerical homogenization with a priori rigorous exponential decay estimates [39]. These constructed gamblets (1) are elementary solutions of hierarchical information games associated with the process of computing with partial information and limited resources, (2) have a natural Bayesian interpretation under the mixed strategy emerging from the game theoretic formulation, (3) this method could realize its fast simulation by decomposing the solution space into a direct sum of linear subspace that are orthogonal.

The proposed Gamblets are identified by conditioning Gaussian fields. They are represented as the optimal recovery splines and analyzed through their variational (energy minimizing) characterization. Therefore, Gamblets is a kind of energy minimization methods which can be traced back to optimal recovery splines [43-44]. Many methods can be represented as optimal recovery splines, such as Polyharmonic Splines [55-58] and Rough Polyharmonic Splines [45]. The basis functions
constructed by Variational Multiscale Method [46] and Local Orthogonal Decomposition Method (LOD) [47] can also be represented as optimal recovery splines. The energy minimizing basis functions $\psi_i$ of [48-51] are identified by minimizing the total $\sum_i \| \psi_i \|^2$ subject to the global constraint $\sum_i \psi_i(x) = 1$ can also be identified as approximate optimal recovery splines if the energy minimization is done separately for each spline subject to local (rather than global) constraints.

Gambllets are defined with minimizers of $\| \psi \| = \int_\Omega \nabla \psi^T \nabla \psi \, dx$ subject to $\int_\Omega \phi_j \psi = \delta_{ij}$. Rough Polyharmonic Splines [45] can be recovered as a particular case of gamblets. The basis functions of the LOD method can also be recovered as a particular case of gamblets by selecting the measurement functions $\phi_i = \sum_{j=1}^m M_{ij} \phi_j$, with $\phi_j$ are piecewise linear nodal basis functions and $M_{ij} = \int_\Omega \phi_i \phi_j$. Although both Gamblets and the LOD method lead to an orthogonal decomposition of the solution space the two approaches differ in the following points: (a) by using measurement functions defined by elements $\phi_i$ of the dual space rather than through a correction of classical conforming finite elements $\phi_i$, gamblets avoid the requirement for conforming measurement functions and the optimal $L^2$ projection properties of the Clément interpolation operator (used in [47] to derive exponential decay estimates). (b) The derivation of a corresponding multiresolution method is not based on a hierarchy of corrections of a hierarchy of conforming finite elements but on the pull back of a hierarchy of elements of the dual space.

Constrained energy minimization is also used to construct multiscale basis functions in multiscale finite element method (MsFEM) [53-54] to obtain a mesh-dependent convergence. Both gamblets and MsFEM produce numerical homogenization basis functions but it is not clear how to turn MsFEM into a hierarchy while preserving rigorous a priori near linear complexity versus accuracy estimates.

This paper proceeds as follows: we start by introducing the governing equations of fluid flow in fractured media, including mathematical model of discrete fracture network. Next, we introduce the multiresolution operator decomposition briefly. Then the multiresolution operator decomposition method is applied to solve the heterogenous porous media with rough permeability field and fractured porous media respectively. In the numerical experiment section, several numerical results are
presented to prove the validity and effectiveness of the multigrid method. Finally, the concluding remarks were given in the concluding section.

2. Governing Equations

In this work, the microfracture and matrix are treated as continuum porous media with equivalent heterogeneous permeability field. The large-scale fractures are represented with discrete fracture model (DFM). In DFM, flows in both matrix and fractures obey Darcy’s law. Therefore, different from duel-continuum model [1-2], there are no separate functions designed to describe the interactions between matrix and fractures. The fracture-matrix interactions are also computed with Darcy’s law [5-9]. In the porous flow domain \( \Omega \), the flow of incompressible fluid in porous media is assumed isothermal. The system consists of Darcy’s law, mass balance equation.

\[
-\nabla \cdot v = q \tag{1}
\]

\[
v = \frac{K}{\mu} \nabla p \tag{2}
\]

In the above equation system, \( v \) denotes the fluid velocity; \( q \) is source/sink term; \( \mu \) stands for viscosity of fluid; \( K \) denotes the permeability field; \( p \) denotes fluid pressure; we ignore the influence of gravity. Let \( \lambda = -\frac{K}{\mu} \) be the mobility of fluid, and then the governing equation can be written as

\[
-\nabla \cdot (\lambda \nabla p) = q \tag{3}
\]

The variational formulation of the governing equation is

\[
-\int_{\Omega} \phi \nabla \cdot (\lambda \nabla p) \, d\Omega = \int_{\Omega} \phi \, q \, d\Omega \tag{4}
\]

For \( \forall \phi \in H_0^{dv} \). By using divergence theory, we can get

\[
\int_{\Omega} \nabla \phi \cdot \lambda \nabla p \, d\Omega = \int_{\Omega} \phi \, q \, d\Omega \tag{5}
\]

In each element, \( p \) can be approximated as

\[
p \approx \sum_{i=1}^{m} N_i p_i = \mathbf{N}(x) p \tag{6}
\]

Here \( m \) denotes the number of element nodes; \( \mathbf{N}=[N_1, \cdots, N_m] \) is the basis function; \( p=[p_1, \cdots, p_m] \) is the pressure of fluid at the nodes.
Natural porous media exists with a heterogeneous permeability field in most cases. It is a challenging problem to develop a robust and efficient approach to solve the governing equation with rough permeability. Apart from the rough heterogeneous properties, these porous media commonly contain fractures. Fractures are formed due to the diagenesis or deformation of rock. They widely exist in naturally porous media and their presence will seriously aggravate the heterogeneity. In this work, discrete fracture model is used to represent the fractures where Darcy’s law is used to model fluid flow in fractures and matrix [8]. In DFN model, the variables are assumed to be constant along the width of fracture, that is, the n-D fracture element is simplified to (n-1)-D element as shown in Fig. 1.

Therefore, we need to discrete the governing equation in matrix system and fracture system separately. For 2D problem, 1D line segment is used to represent 2D fracture element. Then the whole domain was

\[
\Omega = \Omega_m + \sum_i d_i \times \Omega_{f,i}
\]

Here the subscript m and f denotes matrix and fracture respectively; \(d_i\) is the aperture of the \(i\)th fracture. Now assume the governing equation is written as \(FEQ\), then the integral form in the whole domain is

\[
\int_\Omega FEQ \text{d}\Omega = \int_{\Omega_m} FEQ \text{d}\Omega_m + \sum_i d_i \times \int_{\Omega_f} FEQ \text{d}\Omega_{f,i}
\]

Then the global stiffness matrix is assembled based on the superposition principle, as shown in Fig 2.
Fig. 2 Schematic of stiffness matrix

That is, we treat fracture as a (n-1)-dimension element. The stiffness matrix of fracture and matrix are assembled together

\[
A' = \sum_{e=1}^{E} \left( \int_{\Omega_{e,m}} \nabla N_{e,m}^T \lambda_m \nabla N_{e,m} \, d\Omega_{e,m} + a_f \int_{\Omega_{e,f}} \nabla N_{e,f}^T \lambda_f \nabla N_{e,f} \, d\Omega_{e,f} \right)
\]

Here \( E \) is the total numerical of elements, \( a_f \) is aperture of fracture, \( N_{e,m} \) is basis function of matrix and \( N_{e,f} \) is basis function of fracture.

Apparently traditional upscaling method must be used with great care since the fine-scale information has essential influence on whole flow patterns. This creates a motivation for the gamblets. The complexity of the method is \( O(N \log^{3d}(N)) \). This is a surprising near-linear complexity which will significantly save computer memory and realize fast simulation. Note that the cost of computing gamblets can be decreased to \( O(N \log^{2d+2}(N)) \) using incomplete Cholesky factorization as in [42].

3. Multiresolution Decomposition

The objective of the multiresolution decomposition [32] is to solve the governing equation of fluid flow as fast as possible. This method is inspired by the suggestion that this link between numerical homogenization and Bayesian inference are not coincidences [59-60], but particular instances of mixed strategies for underlying information games, and that optimal or near-optimal methods could be obtained by identifying such games and their optimal strategies.

First step to identify these games [61] is note that computation can only be done with partial information. That is, the infinite-dimensional operator (3) can only compute with finite-dimensional features of \( p \). However, to obtain an accurate
solution one must fill the information gap between the finite-dimensional features $p_m$ and $p$. That is, one should construct an interpolation operator giving $p$ as a function of $p_m$. Therefore, the identification of the interpolation operator is reformulated as a minmax game where Player $A$ chooses the source/sink term $q$ in an admissible set, Player $B$ is shown $p_m$ and must give the approximation of $p$ from the incomplete measurements. The $p^*$ is Player $A$’s bet, Player $A$’s target is make the error $\| p - p^\star \|$ as big as possible and Player $B$’s target is minimize it. A remarkable result from game theory [40-41] is that optimal strategy is mixed strategy.

To compute rapidly, the game must not be limited to filling the information gap between $p_m$ and $p$. This game must be played over hierarchies of levels of complexity (e.g., one must fill information gaps between $\mathbb{R}^4$ and $\mathbb{R}^{16}$, then $\mathbb{R}^{16}$ and $\mathbb{R}^{64}$, etc). The following hierarchy of labels is introduced.

We define $I^{(q)}$ is the index tree of $q$ if it is the finite set of $q$-tuples of the form $i=(i_1,\ldots,i_q)$. For $1 \leq k \leq q$ and $i=(i_1,\ldots,i_q)\in I^{(q)}$, let $i^{(k)}=(i_1,\ldots,i_k)$ and $I^{(k)}=\{i^{(k)}:i\in I^{(q)}\}$. For $k^i \in \{k,\ldots,q\}$ and $j^{(k)} \in I^{(k)}$, $j^{(k)}$ is defined as $j^{(k)}=(j_1,\ldots,j_k)$. For $i^{(k)} \in I^{(k)}$ and $k^i \in \{k,\ldots,q\}$, we write $j^{(k,i)}$ the set of elements $j \in I^{(k)}$ such that $j^{(k,i)}=i$. For $k \in \{1,\ldots,q-1\}$, $\pi^{(k,k+1)}$ is constructed as a $I^{(k)} \times I^{(k+1)}$ matrix satisfying $\pi^{(k,k+1)}_{i,j}=0$ if $j \in I^{(k+1)}$ and $\pi^{(k,k+1)}(\pi^{(k,k+1)})^T=I^{(k)}$, where $I^{(k)}$ is $I^{(k)} \times I^{(k)}$ identity matrix.
The measurement functions are defined through induction by

\[ \phi_i^{(k)} = \sum_{j \in \Gamma^{(k)}} \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)} \quad (10) \]

As shown in Fig. 3, the \( \Omega \) is divided into several \( 2^{-k} \times 2^{-k} \) subspaces \( (\tau_i^{(k)})_{\in \Gamma^{(k)}} \).

Let \( \phi_i^{(k)} = \frac{1}{v_i^{(k)}} \sqrt{\tau_i^{(k)}} \) where \( v_i^{(k)} \) is the volume of \( \tau_i^{(k)} \), \( 1_{\tau_i^{(k)}} \) is the indicator function of \( \tau_i^{(k)} \). As introduced above, we identify the gamblets by assuming that player A and player B are playing hierarchies of games. Player A chooses the source/sink term \( q \) and player B does not know his choice. Player B is shown \( (\int_\Omega p \phi_i^{(k)})_{\in \Gamma^{(k)}} \) and then gives the approximation of \( p \) and \( (\int_\Omega p \phi_i^{(k+1)})_{\in \Gamma^{(k+1)}} \) from these measurements. Player B will have a loss after he makes the choice, sees \( (\int_\Omega p \phi_i^{(k+1)})_{\in \Gamma^{(k+1)}} \) and give the approximation of \( p \) and \( (\int_\Omega p \phi_i^{(k+2)})_{\in \Gamma^{(k+2)}} \). Player A wants to make the loss of player B as big as possible to win the game, while player B tries to minimize it. The best strategy from game theory [40-41] is mixed strategy, that is player A play at random and player B replaces \( q \) with \( \xi \) in Eq. (3) to form a stochastic partial differential equation system. Player B’s bet at step \( k \) is the expectation of the solution of the stochastic partial differential equation system conditioned on measurements of the solution of Eq. (1).
\[ p^{(k)}(x) = \mathbb{E}\left[ v(x) \right| \int_{\Omega} v(y) \phi^{(k)}(y) \, dy = \int_{\Omega} p(y) \phi^{(k)}(y) \, dy, i \in I^{(k)} \]  

(11)

Here \( v \) is the solution of the stochastic partial differential equation. The optimal strategy \([32]\) of player \( B \) is to choose the \( \xi \) with a centered Gaussian field with covariance function \( \mathcal{L} = -\text{div}(\sigma \nabla) \). In this case \( p^{(k)} \) could be expressed as

\[ p^{(k)}(x) = \sum_{i \in I^{(k)}} \psi_{i}^{(k)}(x) \int_{\Omega} p(y) \phi^{(k)}(y) \, dy \]

(12)

The basis functions \( \psi_{i}^{(k)} \) are referred to gamblets which represent the best bet of player \( B \) on the value of solution of the governing equation, that is

\[ \psi_{i}^{(k)}(x) = \mathbb{E}\left[ v(x) \right| \int_{\Omega} v(y) \phi_{j}^{(k)}(y) \, dy = \delta_{i,j}, i \in I^{(k)} \]  

(13)

However in practical computation, we will work with variational properties inherited from the conditioning of the Gaussian field \( v \). For \( i \in \{1,...,m\} \) consider the quadratic optimization problem

\[
\begin{aligned}
\text{Minimize} & \quad \|\psi\|_d \\
\text{subject to} & \quad \psi \in H^{1}_{0}(\Omega) \text{ and } \int_{\Omega} \phi \psi = \delta_{j} \text{ for } j = 1,...,m
\end{aligned}
\]

(14)

Here the \( \|\psi\|_d \) is the energy norm defined as \( \|\psi\|_d = \int_{\Omega} \nabla \psi^T \lambda \nabla \psi \, dx \).

The following theorem shows that Eq. (14) can be used to identify \( \psi \), and that gamblets are characterized by optimal recovery properties.

**THEOREM.** It holds true that (1) the optimization problem Eq. (14) admits a unique minimizer \( \psi \) defined by Eq. (13), (2) for \( w \in \mathbb{R}^m \), \( \sum_{j=1}^{m} w_j \psi_j \) is the unique minimizer of \( \|\psi\|_d \) subject to \( \int_{\Omega} \psi(x) \phi_j(x) = w_j \) for \( j \in \{1,...,m\} \), and (3) \( \langle \psi, \psi \rangle_d = \Theta^{-1}_{i,j} \), here \( \langle v, w \rangle_d = \int_{\Omega} (\nabla v)^T \lambda \nabla w \), \( \Theta_{i,j} = \int_{\Omega} \phi_i(x) \phi_j(x) \, dx \) and \( G(x,y) \) is the covariance function \([62]\) (where \( G \) is the Green’s function of the governing equation).

**Proof.** Let \( w \in \mathbb{R}^m \) and \( \psi_w = \sum_{j=1}^{m} w_j \psi_j \) with \( \psi_i = \sum_{j=1}^{m} \Theta_{i,j} \int_{\Omega} G(x,y) \phi_j(y) \, dy \).

The definition of \( \Theta \) indicates that \( \int_{\Omega} \psi_w(x) \phi_j(x) = w_j \) for \( j \in \{1,...,m\} \). In addition,
by integration by part we get for all $\varphi \in H^1_0(\Omega)$, $\langle \psi_w \varphi \rangle \lambda = \sum_{i,j=1}^m w_i \Theta^{-1}_{i,j} \int_{\Omega} \phi_j \varphi$.

Hence, if $\psi \in H^1_0(\Omega)$ is such that $\int_{\Omega} \psi(x) \phi_j(x) = w_j$ for $j \in \{1,...,m\}$, then

$$\langle \psi_w, \psi - \psi_w \rangle \lambda = 0 \quad \text{and} \quad \|\psi\|_{H^1}^2 = \|\psi_w\|_{H^1}^2 + \|\psi - \psi_w\|_{H^1}^2.$$ 

Then finish the proof of the optimality of $\psi$ and $\psi_w$.

To make it intuitive, the illustrated $\psi^{(k)}_i$ in flow problems of porous media are shown in Fig. 4. For understandability, we consider a homogeneous porous media in this example. We suppose the incompressible fluid flows from upper boundary to bottom boundary and the left and right boundaries are treated as no-flow boundaries.

Define porous flow domain $\Omega = (0,1)^2$ and let $q=6$.

For $k \in \{1,...,q\}$ we define the spaces

$$\Pi^{(k)} = \text{span}\{\psi^{(k)}_i | i \in I^{(k)}\}$$

(15)

We can derive from Eq. (10) that the spaces are nested. Therefore, for $k \in \{1,...,q-1\}$ there is a restriction matrix $R^{(k+1)}_{i,j}$ such that

$$\psi^{(k)}_i = \sum_{j \in I^{(k+1)}} R^{(k+1)}_{i,j} \psi^{(k+1)}_j$$

(16)
Where \( R_{ij}^{(k,k+1)} \) is a \( I^{(k)} \times I^{(k+1)} \) matrix with the following entry
\[
R_{ij}^{(k,k+1)} = \int_{\Omega} \psi_i^{(k)} \phi_j^{(k+1)} = \mathbb{E} \left[ \int_{\Omega} v(y) \phi_j^{(k+1)}(y) | \int_{\Omega} v(y) \phi_i^{(k)}(y) \right] dy = \delta_{ij}, \quad t \in I^{(k)}
\] (17)

We write \( A_{ij}^{(k)} \) as the stiffness matrix of \( \psi_j^{(k)} \), that is
\[
A_{ij}^{(k)} = \int_{\Omega} \left( \nabla \psi_i^{(k)} \right)^T \lambda \nabla \psi_j^{(k)}
\] (18)

Then the nested computation of stiffness matrix is obtained as
\[
A^{(k)} = R^{(k,k+1)} A^{(k+1)} R^{(k+1,k)}
\] (19)

By this step, the nested computation enables one to model fluid flow in heterogeneous porous media with multigrid method. But we take a further step and introduce a multiresolution decomposition method.

For \( k \in \{2, \ldots, q\} \) let \( J^{(k)} \) be a \( k \)-tuples in the form \( j = (j_1, \ldots, j_k) \) and \( \{ j^{(k-1)} | j \in J^{(k)} \} = I^{(k-1)} \). Let \( W^{(k)} \) be a \( J^{(k)} \times I^{(k)} \) matrix such that for \((j,i) \in J^{(k)} \times I^{(k)} \),
\[
W_{ij}^{(k)} = 0 \quad \text{with} \quad j^{(k-1)} \neq i^{(k-1)} \quad \text{and for} \quad s \in I^{(k-1)}, \quad t \in \{1, \ldots, m_s - 1\} \quad \text{and} \quad i' \in \{1, \ldots, m_s\},
\]
\[
W_{(i,j),(s,t)}^{(k)} = \delta_{i,i'} - \delta_{s+1,t}. \quad \text{For} \quad i \in J^{(k)} \quad \text{let}
\]
\[
\chi_i^{(k)} = \sum_{j \in I^{(k)}} W_{ij}^{(k)} \psi_j^{(k)}
\] (20)

and
\[
\mathbb{W}^{(k)} = \text{span} \{ \chi_i^{(k)} | i \in I^{(k)} \}
\] (21)

Here \( \mathbb{W}^{(k)} \) is the orthogonal complement of \( \mathbb{W}^{(k-1)} \) within \( \mathbb{W}^{(k)} \). Let \( \bigoplus \) denotes the orthogonal direct sum with respect to scalar product \( \langle v, w \rangle = \int_{\Omega} (\nabla v)^T \lambda \nabla w \). Then we can get the multiresolution decomposition of the space
\[
\mathbb{W}^{(k)} = \mathbb{W}^{(1)} \bigoplus \mathbb{W}^{(2)} \bigoplus \cdots \bigoplus \mathbb{W}^{(k)}
\] (22)

and for \( k \in \{1, \ldots, q\} \), \( p^{(k+1)} - p^{(k)} \) belongs to \( \mathbb{W}^{(k)} \) we have
\[
p = p^{(1)} + \left( p^{(2)} - p^{(1)} \right) + \cdots + \left( p^{(q)} - p^{(q-1)} \right) + \left( p - p^{(q)} \right)
\] (23)

Which is orthogonal decomposition of \( p \) and \( p^{(k+1)} - p^{(k)} \) is the solution of the governing equation in \( \mathbb{W}^{(k+1)} \). The illustrated \( \chi_i^{(k)} \) and decomposition of the space are shown in Fig. 5 and Fig. 6.
For fractured porous media, we compute the pressure distribution of fractured porous media $p_f^{(k+1)} - p_f^{(k)}$ in each subspace $H^{(k)}$, here $k \in \{1, ..., q\}$. The influences
of fractures are captured in every level. The final pressure distribution of the whole
media will be obtained by

\[ p_f = p_f^{(1)} + \left( p_f^{(2)} - p_f^{(1)} \right) + \cdots + \left( p_f^{(q)} - p_f^{(q-1)} \right) + \left( p_f - p_f^{(q)} \right) \]  

(24)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Multiresolution Decomposition Solve</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:</td>
<td>( A_{i,j}^{(q)} = A_f )          // Level ( q ), ( \Gamma^{(q)} \times \Gamma^{(q)} ) stiffness matrix</td>
</tr>
<tr>
<td>2:</td>
<td>( \psi_{i}^{(q)} = \varphi_i )</td>
</tr>
<tr>
<td>3:</td>
<td>( g_{i}^{(q)} = g_i )</td>
</tr>
<tr>
<td>4:</td>
<td>for ( k = q ) to 2 do</td>
</tr>
<tr>
<td>5:</td>
<td>( B^{(k)} = W^{(k)} A_f^{(k)} W^{(k),T} )</td>
</tr>
<tr>
<td>6:</td>
<td>For ( i \in J^{(k)} ), ( \chi_{i}^{(k)} = \sum_{i \in \Gamma^{(k)}} W_{i,j}^{(k)} \psi_{j}^{(k)} )</td>
</tr>
<tr>
<td>7:</td>
<td>( w_{i}^{(k)} = B^{(k),-1} W_i^{(k)} g_{i}^{(k)} )</td>
</tr>
<tr>
<td>8:</td>
<td>( v_{i}^{(k)} = \sum_{i \in \Gamma^{(k)}} W_i^{(k)} \chi_{i}^{(k)} )</td>
</tr>
<tr>
<td>9:</td>
<td>( D^{(k,k-1)} = -B^{(k),-1} W^{(k)} A_f^{(k)} \chi^{(k,k-1)} )</td>
</tr>
<tr>
<td>10:</td>
<td>( R^{(k-1,k)} = \pi^{(k-1,k)} + D^{(k-1,k)} W^{(k)} )</td>
</tr>
<tr>
<td>11:</td>
<td>( g_{i}^{(k-1)} = R^{(k-1,k)} g_{i}^{(k)} )</td>
</tr>
<tr>
<td>12:</td>
<td>( A^{(k-1)} = R^{(k-1,k)} A_f^{(k-1,k)} )</td>
</tr>
<tr>
<td>13:</td>
<td>For ( i \in \Gamma^{(k-1)} ), ( \psi_{j}^{(k-1)} = \sum_{i \in \Gamma^{(k-1)}} R_{i,j}^{(k-1,k)} \psi_{j}^{(k)} )</td>
</tr>
<tr>
<td>14:</td>
<td>end for</td>
</tr>
<tr>
<td>15:</td>
<td>( w_{i}^{(1)} = A_{i,j}^{(1),-1} g_{i}^{(1)} )</td>
</tr>
<tr>
<td>16:</td>
<td>( p_{i}^{(1)} = \sum_{i \in \Gamma^{(1)}} W_{i,j}^{(1)} \psi_{j}^{(1)} )</td>
</tr>
<tr>
<td>17:</td>
<td>( p = p_{i}^{(1)} + v_{i}^{(2)} + \cdots + v_{i}^{(q)} )</td>
</tr>
</tbody>
</table>

The algorithm computes the stiffness matrix firstly according to section 2. Then the
level \( q \) gamblets \( \psi_{i}^{(q)} \) and the \( g_{i}^{(q)} \) are introduced. The key step of this algorithm
is the nested computation from line 4-line 14. The \( A^{(k)}, g_{i}^{(k)}, \psi_{i}^{(k)} \) are regarded as
inputs and \( A^{(k-1)} g_{i}^{(k-1)}, \psi_{i}^{(k-1)} \) are regarded as outputs. \( w_{i}^{(1)} \) is computed outside the
nested computation in line 15. The last step of the algorithm is to final solution \( P \)
via addition of the subscale solution.
4. Numerical experiments

In the numerical examples provided here, the simulations of fluid flow in porous media are conducted using the multiresolution decomposition method described in above sections. Firstly, the multigrid method is validated for flow simulation in heterogeneous porous media. Then we extend this method to simulation of fluid flow in long-fractured media. We also consider the co-existence of small-scale fractures and long-scale fracture in this part. The efficiency of the multigrid method fracture network is investigated in the final numerical case. In this section, the discrepancies in pressure are measured using a relative $L^2$ norm

$$ p_e = \frac{\| p_r - p_{mg} \|_2^2}{\| p_r \|_2^2} $$

Here, the reference pressure is denoted by $p_r$. The pressure computed with Multiresolution Operator Decomposition Method is denoted by $p_{mg}$.

4.1 Multiresolution decomposition method for heterogeneous case

The first numerical experiment is set to validate the multiresolution decomposition for heterogeneous porous media. A $1 \text{m} \times 1 \text{m}$ porous model with large number of small-scale fractures is considered, as depicted in Fig. 7(a). This model can be treated as a heterogeneous porous media with equivalent permeability field, as depicted in Fig. 7(b). The fluid flows from upper boundary to bottom boundary. The viscosity of fluid is $\mu = 1 \text{mPa\cdot s}$. The left and right boundaries are treated as no-flow boundaries. The solution is obtained with $q=4$ as shown in Fig. 8.

![Fig.7 Illustration of (a) physical model and (b) permeability field log(K), $\mu$m$^2$](image-url)
The final numerical result of multigrid method is obtained by adding $p^{(k)} - p^{(k-1)}$ up. The Fig. 9 firstly shows the FEM solution and then compares the pressure distribution along $x=0.5m$ with multiresolution decomposition solution. The figures show that this method could reflect the heterogeneity of the pressure distribution. The multiresolution decomposition method could track the reference solution very closely in the heterogeneous case. The pressure error measured with (25)
is $6.3 \times 10^{-15}$. It demonstrates that this method could effectively model fluid flow in porous media with heterogenous permeability field.

### 4.2 Multiresolution decomposition method for fractured porous media

The second numerical experiment is set to test the ability of this method to capture the influence of fracture. A 2D homogeneous porous medium $(1m \times 1m)$ is considered, as shown in Fig. 10(a). The aperture of the fracture is $a=1mm$ and the permeability is $K_f = a^2/12 = 8.33 \times 10^{-7} mD$. The fluid flows from upper boundary to bottom boundary. The viscosity of fluid is $\mu = 1 mPa\cdot s$. Both left boundary and right boundary are set as no-flow boundaries. The DFM is solved on each subspace, and then the pressure distribution in fractured porous media is obtained by

$$p = p^{(1)} + (p^{(2)} - p^{(1)}) + (p^{(3)} - p^{(2)}) + (p - p^{(3)})$$
Fig. 10 Model of fractured porous media (a) and multiresolution decomposition solution (b)

Fig. 11 Comparison of FEM solution (a) and gamblets solution $q=4$ (b) and $q=2$ (c)

Results for pressure distributions $p^{(k)} - p^{(k-1)}$ are shown in Fig. 10(b). It is obvious that the multigrid method could capture the influence of fracture on every level. Then we sum $p^{(k)} - p^{(k-1)}$ up and obtain the final solution, as shown in Fig. 11. Comparison of pressure between FEM result and multiresolution decomposition results is shown in Fig. 12. The pressure error measured with relative $L^2$ norm is $4.2 \times 10^{-5}$. It demonstrates that the multigrid could model the divert effect of fracture effectively.
The decomposition of space may call multiscale methods up. Different from multiscale methods [17-18] that construct basis functions within coarse blocks, the basis functions of multiresolution decomposition method would not cause localized impact on the solution of fractured porous media, see Fig. 13(c). Here the physical model introduced above is studied with an injection well located at the bottom left corner and the production well located at the upper right corner. For the MsMFEM 5×5 coarse grid is used as shown in Fig. 13(a), the multiresolution decomposition solution is obtained with q=2. As shown in Fig. 13(b), the multiscale method could capture the influence of fracture effectively, but there are few errors near by the intersections of fracture and coarse block boundaries. This is caused by the coarse block boundary conditions used to compute multiscale basis functions [12]. In this experiment, no actions (e.g. oversampling technique, iterative procedure, global information and revised coarse boundary condition et al) are applied to improve the accuracy of the multiscale results.
Then we take a further step and embed the long fracture into the heterogeneous porous media, see Fig. 14(a). From Fig. 14(b) we can observe the multigrid result is in close agreement with the fine-scale reference result. The pressure error measured using relative $L^2$ norm is $8.6 \times 10^{-3}$. It confirms that the presented multiresolution decomposition method is very robust for heterogenous permeability field in fractured porous media.

4.3 Multiresolution decomposition method for fracture network

The final experiment was presented to assess the accuracy of our multiresolution decomposition method for fracture network. Fracture network contains several intersected fractures and it will affect the pressure distribution significantly. An injection/production model with length scale $1m \times 1m$ is considered. Five fractures are contained in the porous media, as shown in Fig. 15. The injection well located at the bottom left corner and the production well located at the upper right corner. All boundaries are supposed to be no-flow boundaries. The apertures of the five fractures are $a=1mm$ and the permeability are $K_f = a^2/12 = 8.33 \times 10^7$ mD.

Fig. 15 physical model of fractured network
Fig. 16 physical model of fractured network

(a) 

(b)
In Fig. 16, we show the $p^{(k)} - p^{(k-1)}$ in different subspaces and the comparison of FEM solution and multiresolution decomposition solution is presented in Fig. 17. The pressure distribution shows that the proposed method effectively reflects the pressure distribution of fracture network. The pressure error measured using relative $L^2$ norm is $1.35 \times 10^{-7}$. The comparison implies that this method is able to reflect the interactions between fractures.

Although a fine mesh has been used to facilitate the presentation of the algorithm, this method is a meshless method. A fractured porous media with fracture network is investigated, as shown in Fig. 18(a). The fracture network consists of seven intersected fractures. We consider a triangular mesh as shown in Fig. 18(b) for simplicity. Let $q=2$ and the apertures of the seven fractures are $a=1\text{mm}$ and the permeability are $K_f = a^2/12 \text{mD}$. The injection well located at the bottom left corner and the production well located at the upper right corner. All boundaries are supposed to be no-flow boundaries.
The pressure error measured using relative $L^2$ norm is $2.07 \times 10^{-7}$. We could see that this meshless method is effective for triangular mesh. For a given fractured porous domain, this method could run smoothly and effectively once the stiffness matrix, matrix $W$ and matrix $\pi$ are constructed. It is a robust method for simulation of complex fractured porous media.

5. Concluding remarks

In this work, we proposed an effective Multiresolution Operator Decomposition method for fluid flow in fractured porous media. This method could complete its fast simulation by decomposing the solution space into a direct sum of linear subspace that are orthogonal. Some numerical results presented in the paper demonstrate that the proposed formulation is an accurate and efficient method for flow simulation in fractured porous media. It may difficult to simulate fractured porous media with complicated fracture networks if the measurement functions are not adapted to the high conductivity fractures, especially when fractures are very close to each other, this is an ongoing work. The construction of gamblets can be completed independently with parallel computing method to further reduce computer memory and CPU time requirements. This work leads to the conclusion that the multigrid method is a significant development for the simulation of fluid flow in porous media.

Acknowledgements

The authors gratefully acknowledge support from National Science and Technology Major Project (2016ZX05060-010), the Fundamental Research Funds for the Central Universities (17CX06007), HO and FS gratefully acknowledges this work supported by the Air Force Office of Scientific Research and the DARPA EQUiPS.
Program under award number FA9550-16-1-0054 (Computational Information Games) and the Air Force Office of Scientific Research under award number FA9550-18-1-0271 (Games for Computation and Learning).

References