Information scrambling in chaotic systems with dissipation

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Chaotic dynamics in closed local quantum systems scrambles quantum information, which is manifested quantitatively in the decay of the out-of-time-ordered correlators (OTOC) of local operators. How is information scrambling affected when the system is coupled to the environment and suffers from dissipation? In this paper, we address this question by defining a dissipative version of OTOC and numerically study its behavior in a prototypical chaotic quantum chain in the presence of dissipation. We find that dissipation leads to not only the overall decay of the scrambled information due to leaking but also structural changes so that the 'information light cone' can only reach a finite distance even when the effect of overall decay is removed. Based on this observation we conjecture a modified version of the Lieb-Robinson bound in dissipative systems.

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I. INTRODUCTION

Chaos happens not just in classical systems but in quantum systems as well [1–8]. One characteristic signature of quantum chaos is the scrambling of quantum information which can be quantitatively described by the out-of-time-ordered correlator (OTOC) [9–37]. More specifically, suppose that information is encoded initially in a local operator A. Under the dynamics generated by a local Hamiltonian $H=\sum_i h_i$, $A(t)=e^{iHt}Ae^{-iHt}$ grows in size and becomes nonlocal as t increases. As A grows in size, it starts to overlap with local operators B at other spatial locations and ceases to commute with them. The effect of information scrambling is then manifested as the growth in the norm of the commutator $[A(t),B]$. Correspondingly it is also manifested as the decay of (the real part of) the OTOC $\langle A^\dagger(t)B^\dagger A(t)B\rangle_\beta$ which is related to the commutator as

$$\Re\,\langle A^\dagger(t)B^\dagger A(t)B\rangle_\beta = 1-\tfrac{1}{2}\langle[A(t),B]^\dagger[A(t),B]\rangle_\beta, \quad (1)$$

where local operators A, B are both unitary, $\langle\cdot\rangle_\beta$ represents the thermal average at the inverse temperature $\beta=1/T$, and $\Re$ denotes the real part.

In a chaotic system, the decay of OTOC is usually expected to exhibit the following features: First, after time evolution for a very long time, information initially encoded in A becomes highly nonlocal and cannot be accessed with any individual local operator B. Therefore, all OTOCs at infinite temperature $\beta=0$ decay to zero at late time [15,18,23,25]

$$\lim_{t\to\infty}\Re\,\langle A^\dagger(t)B^\dagger A(t)B\rangle_{\beta=0}=0, \quad (2)$$

where the local operators A and B are traceless.

Secondly, in chaotic zero-dimensional systems, the OTOC starts to decay at early time in an exponential way [18]

$$\Re\,\langle A^\dagger(t)B^\dagger A(t)B\rangle_\beta = f_1-\frac{f_2}{n}e^{\lambda_L t}+O\left(\frac{1}{n^2}\right), \quad (3)$$

where the constants f_1, f_2 depend on the choice of operators A, B, and n is the total number of degrees of freedom. The exponent of the exponential—the Lyapunov exponent—characterizes how chaotic the quantum dynamics is. It is bounded by $\lambda_L\leqslant\frac{2\pi}{\beta}$ [17–20] and is expected to be saturated by quantum systems corresponding to black holes.

Thirdly, in a system with spatial locality, information spreads at a certain speed, giving rise to a delay time before OTOC starts to decay. In some simple cases [17,18,21,38,39], the early-time behavior of OTOC is approximately described by

$$f_1'-f_2'e^{\lambda_L(t-d_{BA}/v_B)}+\text{higher-order terms} \quad (4)$$

with f_1', f_2' that depend on A, B and the local degrees of freedom. d_{BA} is the distance between the local operators A and B. The higher-order terms can be described by $O(\frac{1}{n^2})$ in large-n systems [17] or $O(e^{-2\lambda_L d_{BA}/v_B})$ in spin systems [18]. That is, information spreads with a finite velocity v_B—the butterfly velocity—and forms a 'light cone' [16–18]. In general quantum chaotic spin systems with small local Hilbert space dimensions and short-range interactions, like random circuit models [40–46], the wave front of the light cone becomes wider while propagating out and Refs. [47,48] give an in-depth study of the general form of the early time decay of OTOC. The deep connection between OTOC and quantum chaos generated a lot of interest in the topic, both theoretically and experimentally. Several protocols have been proposed to measure these unconventional correlators in real experimental systems [49–59].

The measurement of OTOC in real experimental systems is complicated by the fact that the system is not exactly closed and suffers from dissipation through coupling to the

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environment. How does dissipation affect the measured signal of OTOC? More generally, we can ask how does dissipation affect information scrambling in a chaotic system? Dissipation leads to leakage of information, and therefore it is natural to expect that any signal of information scrambling would decay. Is it then possible to recover the signatures of information scrambling in a dissipative system and observe the existence of a light cone?

To address these questions, we numerically study a prototypical model of chaotic spin chain [12,16,18,60]—the Ising model with both transverse and longitudinal fields—in the presence of some common types of dissipation: amplitude damping, phase damping, and phase depolarizing. Due to the lack of a "small parameter" as explained in Ref. [48], there is no well-defined Lyapunov exponent in this system. Thus we focus on the structural changes of the light cone, which manifests information scrambling.

The Hamiltonian of the system with open boundary condition is

$$H_s = -J\left[\sum_{i=1}^{N-1}\sigma_i^z\sigma_{i+1}^z + \sum_{i=1}^{N}\left(h_x\sigma_i^x + h_z\sigma_i^z\right)\right], \qquad (5)$$

where N is the number of spins, and we choose the parameters to be $J = 1$, $h_x = -1.05$, and $h_z = 0.5$. This model, far from any integrability limits [60], is believed to have chaotic dynamics. We find that if OTOC is measured using the protocol given in Ref. [49], dissipation leads to the decay of the signal not only due to information leaking into the environment but also information re-structuring. We define a corrected OTOC to remove the effect of leaking, so that the light cone can be recovered to some extent. However, due to the re-structuring, the recovered light cone only persists to a finite distance.

The paper is organized as follows. In Sec. II, we review the dynamics of dissipative systems and define a dissipative version of OTOC based on the measurement protocol given in Ref. [49]. In Sec. III, after observing the fast overall decay of the dissipative OTOC, we define a corrected OTOC to remove the effect of overall information leaking in the hope of recovering the information light cone. However, we see that the corrected light cone still only persists for a finite distance. In Sec. IV, we point out that the corrected light cone is finite due to information re-structuring and investigate the relationship between the width of the partially recovered light cone and the strength of dissipation. In Sec. V, we conjecture a modified Lieb-Robinson bound for dissipative systems based on our observation regarding OTOC in the previous sections.

II. MEASUREMENT OF OTOC IN DISSIPATIVE SYSTEMS

In this section, we provide a brief review of the dynamics of dissipative systems and then generalize the definition of OTOC to dissipative systems based on the measurement protocol in Ref. [49]. A dissipative system is an open quantum system S coupled to its environment E. In this coupled system, the total Hamiltonian is $H = H_s + H_e + H_{\rm int}$, where $H_s(H_e)$ is the Hamiltonian of the system (environment) and $H_{\rm int}$ is the interaction term. The reduced density matrix of the system S changes as a consequence of its internal dynamics and the interaction with the environment E. In most cases, the initial state is assumed to be a product state $\rho_s(0)\otimes\rho_e(0)$. Under the Born, Markov, and secular approximations, the dynamical evolution of a dissipative system $\rho_s(t) = {\rm tr}_e[e^{-iHt}\rho_s(0)\otimes\rho_e(0)e^{iHt}] = \mathcal{V}(t)\cdot\rho_s(0)$ can be described by the Lindblad master equation [61]

$$\frac{d\rho_s(t)}{dt} = \mathcal{L}\cdot\rho_s(t) = -i[H_s,\rho_s(t)] + \sum_k \frac{\Gamma}{2}(2L_k\rho_s(t)L_k^\dagger - \rho_s(t)L_k^\dagger L_k - L_k^\dagger L_k\rho_s(t)), \qquad (6)$$

where $\mathcal{V}(t)$ is the dynamical map that connects $\rho_s(0)$ to $\rho_s(t)$, $\mathcal{L}$ is the Liouvillian super-operator, the first commutator with H_s represents the unitary dynamics, the dissipation rate Γ is a positive number, and the Lindblad operators L_k describe the dissipation. Some common types of dissipation [61,62] act locally on each spin via the Lindblad operators in three different scenarios:

$$\text{amplitude damping:}\ L_k = \sqrt{\frac{1}{2}}(\sigma_k^x - i\sigma_k^y), \qquad (7)$$

$$\text{phase damping:}\ L_k = \sqrt{\frac{1}{2}}\sigma_k^z, \qquad (8)$$

$$\text{phase depolarizing:}\ L_k = \frac{1}{2}\sigma_k^x, \frac{1}{2}\sigma_k^y, \frac{1}{2}\sigma_k^z, \qquad (9)$$

where k denotes the kth spin. Different prefactors are selected to ensure that the Liouvillian superoperator at site k has the same largest nonzero eigenvalue $-\Gamma$ in different dissipative channels.

In the Heisenberg picture, the adjoint dynamical map $\mathcal{V}^\dagger(t)$ acting on the Hermitian operators is defined by ${\rm tr}[O(\mathcal{V}(t)\cdot\rho_s)] = {\rm tr}[(\mathcal{V}^\dagger(t)\cdot O)\rho_s]$ for all states ρ_s. If the Lindblad operators do not depend on time, then the adjoint master equation describing the evolution of the operator $O_H(t) = \mathcal{V}^\dagger(t)\cdot O$ is [61]

$$\frac{dO_H(t)}{dt} = \mathcal{L}^\dagger\cdot O_H(t) = i[H_s, O_H(t)] + \sum_k \frac{\Gamma}{2}(2L_k^\dagger O_H(t)L_k - O_H(t)L_k^\dagger L_k - L_k^\dagger L_k O_H(t)), \qquad (10)$$

where $\mathcal{L}^\dagger$ is the adjoint Liouvillian superoperator.

Given both the dynamical and the adjoint dynamical map, how should we define the OTOC in a dissipative system? Should we just replace $A(t)$ with $\mathcal{V}^\dagger(t)\cdot A$ or do something more complicated? In order to give a meaningful answer to this question, we need to specialize to a particular measurement scheme of OTOC and see how the measured quantity changes due to dissipation. We choose to focus on the measurement scheme given in Ref. [49].

Let us analyze in more detail how the measurement scheme would be affected if dissipation is present. Without dissipation, the protocol involves the system whose unitary dynamics generated by H_s is to be probed and a control qubit c. The system is initialized in a thermal state ρ_s or eigenstate $|\psi\rangle_s$ and the control qubit is initialized in state $|+\rangle_c = \frac{1}{\sqrt{2}}(|0\rangle_c + |1\rangle_c)$. Ignoring dissipation, the measurement scheme involves the following steps of unitary operations:

$$\begin{aligned}
(1):&\quad U_1 = I_s \otimes |0\rangle\langle 0|_c + B_s \otimes |1\rangle\langle 1|_c,\\
(2):&\quad U_2 = e^{-itH_s} \otimes I_c\\
(3):&\quad U_3 = A_s \otimes I_c,\\
(4):&\quad U_4 = e^{itH_s} \otimes I_c,\\
(5):&\quad U_5 = B_s \otimes |0\rangle\langle 0|_c + I_s \otimes |1\rangle\langle 1|_c,
\end{aligned}$$

where A_s and B_s are both local unitary operators in the system. Finally, measurement of σ_c^x is performed to get the real part of OTOC. A nice property of this protocol is that it works for both pure states and mixed states, which allows straightforward generalization to open systems.

Note that the above protocol involves both forward and backward time evolution. With dissipation, we assume that only the Hamiltonian of the system is reversed during the backward time evolution while the effect of the environment is unchanged. The reason is that H_e and H_{int} are usually out of control in experiments. Under this setup, if forward time evolution is governed by $H_f = H_s + H_e + H_{\text{int}}$, then backward time evolution is governed by $H_b = -H_s + H_e + H_{\text{int}}$. Correspondingly, the backward dynamical map $\mathcal{V}_b$ and adjoint dynamical map $\mathcal{V}_b^\dagger$ differ from the forward ones $\mathcal{V}_f = \mathcal{V}$, $\mathcal{V}_f^\dagger = \mathcal{V}^\dagger$ by a minus sign in front of H_s.

This setup is different from the naive time reversal described by $H_b' = -H_f = -(H_s + H_e + H_{\text{int}})$. Under the setup of naive time reverse, if the dynamics of the total system is believed to be chaotic, then the expectation is that OTOCs $F(t, A_s, B_s)$ of local operators in subsystem s have the capability to detect the ballistic butterfly light cone. Reference [46] confirms this expectation via investigating the OTOCs of local operators in subsystem s in the random circuit model of composite spins.

In the presence of dissipation, and assuming that the dissipative part of the dynamics cannot be naively reversed, the full protocol now proceeds as follows. Initially the system is prepared with density matrix $\rho_s(0)$. In addition, a control qubit c is initialized in the state $|+\rangle_c = \frac{1}{\sqrt{2}}(|0\rangle_c + |1\rangle_c)$. The total initial state is $\rho_{\text{init}} = \rho_s(0) \otimes |+\rangle\langle +|_c$. The final state is ρ_f after sequentially applying the following superoperators

$$\begin{aligned}
(1):&\quad \mathcal{S}_1 = \mathcal{C}(I_s \otimes |0\rangle\langle 0|_c + B_s \otimes |1\rangle\langle 1|_c),\\
(2):&\quad \mathcal{S}_2 = \mathcal{V}_f(t) \otimes \mathcal{I}_c,\\
(3):&\quad \mathcal{S}_3 = \mathcal{C}(A_s \otimes I_c),\\
(4):&\quad \mathcal{S}_4 = \mathcal{V}_b(t) \otimes \mathcal{I}_c,\\
(5):&\quad \mathcal{S}_5 = \mathcal{C}(B_s \otimes |0\rangle\langle 0|_c + I_s \otimes |1\rangle\langle 1|_c),\\
&\rho_f = \mathcal{S}_5 \cdot \mathcal{S}_4 \cdot \mathcal{S}_3 \cdot \mathcal{S}_2 \cdot \mathcal{S}_1 \cdot \rho_{\text{init}},
\end{aligned} \tag{11}$$

where $\mathcal{I}$ is the identity superoperator, and the conjugation superoperator is defined by $\mathcal{C}(U) \cdot \rho = U\rho U^\dagger$. Finally we

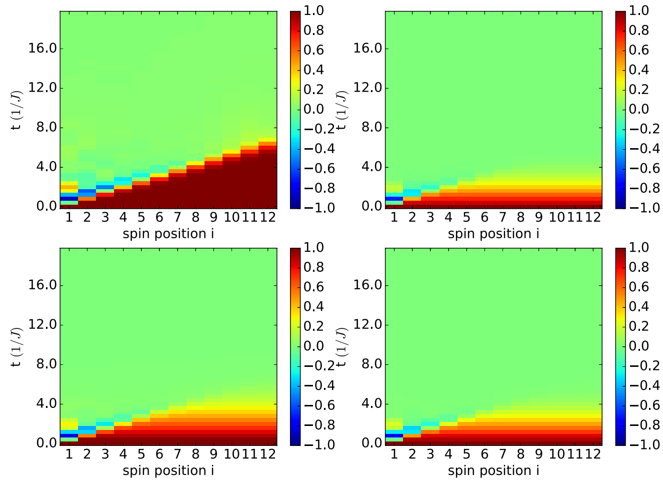

FIG. 1. OTOC $F(t, \sigma_i^z, \sigma_1^z)$ in the chaotic Ising chain (5) with no dissipation (upper left), amplitude damping (upper right), phase damping (lower left), and phase depolarizing (lower right). The dissipation rate is $\Gamma = 0.1$ in these three dissipative channels.

perform the measurement σ_c^x to get the real part of OTOC

$$F(t, A, B) := \text{tr}\left(\sigma_c^x \rho_f\right) = \Re\ \text{tr}\left((\mathcal{V}_b^\dagger(t) \cdot B_s^\dagger) A_s (\mathcal{V}_f(t) \cdot (B_s \rho_s(0))) A_s^\dagger\right). \tag{12}$$

In this paper, we focus on the case where the initial state of the system is prepared in the equilibrium state at infinite temperature, i.e., $\rho_s(0) = I_s/2^N$ and the unitary operators A_s and B_s are selected as local Pauli operators, for example, $B_s = \sigma_1^z$, $A_s = \sigma_i^z$.

III. DISSIPATIVE OTOC CORRECTED FOR OVERALL DECAY

In this section, we observe that the information light cone disappears due to the fast overall decay of OTOC in dissipative systems. In order to recover the light cone as much as possible, we propose a corrected OTOC to remove the effect of overall decay due to the information leaking in dissipative systems.

In a quantum system without dissipation, the OTOC $F(t, A, B) = \Re\langle A^\dagger B_b^\dagger(t) A B_b(t)\rangle_{\beta=0}$ has the same capability to reveal the light cones with different time scaling as the operator norm of the commutator $[B_b^\dagger(t), A^\dagger]$ in the Lieb-Robinson bound [16,17,29], where $B_b^\dagger(t)$ is the operator $e^{itH_b} B^\dagger e^{-itH_b} = e^{-itH_s} B^\dagger e^{itH_s}$ in the Heisenberg picture. When $t < d_{BA}/v_B$, the support of $B_b^\dagger(t)$ and $A^\dagger$ are approximately disjoint, so $F(t, A, B)$ is almost equal to 1, where d_{BA} is the distance between the local operators A and B and v_B is the butterfly velocity. The OTOC begins to decay [15–19] when the support of $B_b^\dagger(t)$ grows to $A^\dagger$. Furthermore, in chaotic systems, OTOC decays to zero at late time in the thermodynamic limit [15,18,23,25]. As shown in the upper left panel of Fig. 1, the OTOC $F(t, A, B)$ is able to reveal the ballistic light cone of information scrambling.

In the presence of dissipation, information is leaking into the environment while being scrambled. Thus $\mathcal{V}_b^\dagger(t) \cdot B^\dagger$ and the OTOC begin to decay when $t > 0$. Intuitively, dissipation

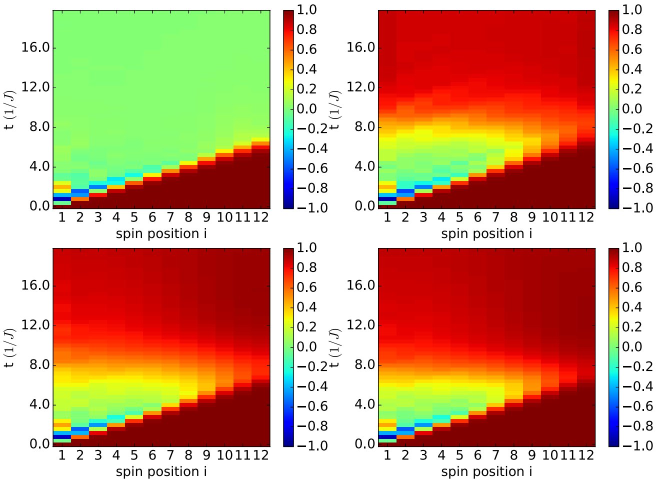

FIG. 2. Corrected OTOC $F(t, \sigma_i^z, \sigma_1^z)/F(t, I, \sigma_1^z)$ in the chaotic Ising chain (5) with no dissipation (upper left), amplitude damping (upper right), phase damping (lower left), and phase depolarizing (lower right). The dissipation rate is $\Gamma = 0.1$ in these three dissipative channels.

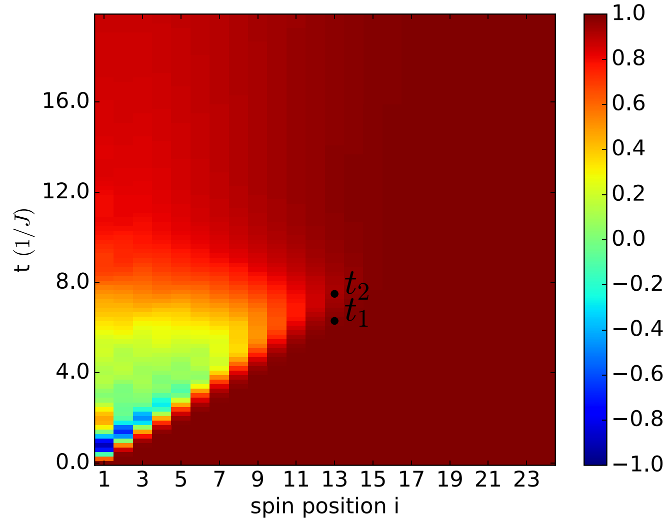

FIG. 3. Corrected OTOC $F(t, \sigma_i^z, \sigma_1^z)/F(t, I, \sigma_1^z)$ recovers some part of the light cone in the channel of phase depolarizing with dissipation rate $\Gamma = 0.1$ and system size $N = 24$.

destroys the light cone revealed by the OTOC $F(t, A, B)$ because the OTOC decays to zero in a short time which is independent of the spatial distance between local operators A and B. In Fig. 1, our numerical calculations confirm that the light cone is destroyed. The OTOC $F(t, \sigma_i^z, \sigma_1^z)$ decays to zero for all i approximately when $t > 4/J$.

In dissipative systems, there are two factors leading to the decay of $F(t, A, B)$: (i) the decay of $\mathcal{V}_b^\dagger(t) \cdot B^\dagger$ related to the information leaking caused by dissipation, (ii) the noncommutativity between $\mathcal{V}_b^\dagger(t) \cdot B^\dagger$ and $A^\dagger$. Information scrambling is manifested only in (ii) but it might be overshadowed by (i). Is it possible to remove the effect of information leaking and recover the destroyed light cone? One natural idea is to divide the OTOC $F(t, A, B)$ by a factor representing the decay related to information leaking. The identity operator I commutes with arbitrary operator, and therefore $F(t, I, B)$ is a factor representing the overall decay of quantum information due to leaking only. Therefore, we propose a corrected OTOC to detect the light cone

$$\frac{F(t, A, B)}{F(t, I, B)}. \tag{13}$$

The numerical results in Fig. 2 show that the corrected OTOC is able to recover the information light cone to some extent in small systems ($N = 12$), with either the dissipation of amplitude damping, phase damping, or phase depolarizing. For small dissipation rate, does the corrected OTOC have the capability to recover the destroyed light cone in the thermodynamic limit? The answer is no. Due to the limited computational resources, we simulate a relatively large system with 24 spins. Figure 3 shows that the boundary of the light cone revealed by the corrected OTOC gradually disappears in space. Based on this result, we expect that the corrected OTOC only has a finite extent in the thermodynamic limit.

Here let us briefly talk about the numerical methods we used. When $N = 12$, quantum toolbox in Python [63,64] is used to numerically solve the master and adjoint master differential equations [Eqs. (6) and (10)]. When $N = 24$, our numerical simulations are based on the time-evolving block decimation (TEBD) algorithm after mapping matrix product operators to matrix product states [65–67], which is able to efficiently simulate the evolution of operators or mixed states. In the singular value decomposition, we ignore the singular values s_k if $s_k/s_1 < 10^{-8}$, where s_1 is the maximal one. The bond dimension is enforced as $\chi \leqslant 500$. Due to the presence of dissipation, the entanglement growth in the matrix product operator is bounded. Therefore, the OTOC can be efficiently calculated using the TEBD algorithm.

IV. THE WIDTH OF THE PARTIALLY RECOVERED LIGHT CONE

The finite extent of the light cone revealed by the corrected OTOC indicates that, besides the overall decay of quantum information, dissipation also leads to structural changes in the scrambled information. In this section, we are going to give a qualitative argument as to why and how the structural change happens. In particular, we find that the re-structuring happens at late time in two aspects: (i) few-body terms dominate when compared with many-body terms, and (ii) at fixed time, the weight of few-body terms decays in space.

Let us define the few-body and many-body terms and their weights. Consider the operator $B_b^\dagger(t) = \mathcal{V}_b^\dagger(t) \cdot B^\dagger$ which can be written in the basis of products of Pauli matrices as

$$B_b^\dagger(t) = \sum_S b_S(t) S = \sum_{i_1 i_2 \cdots i_N} b_{i_1 i_2 \cdots}(t) \sigma_1^{i_1} \sigma_2^{i_2} \cdots \sigma_N^{i_N}, \tag{14}$$

where the Pauli string S is a product of Pauli matrices $\sigma_1^{i_1} \sigma_2^{i_2} \cdots \sigma_N^{i_N}$ with $i_k = 0, x, y,$ or z. In the above decomposition, a few-body (many-body) term is a Pauli string with few (many) nontrivial Pauli matrices. $|b_S(t)|^2 / \sum_{S'} |b_{S'}(t)|^2$ represents the weight of the Pauli string S.

Our qualitative arguments are mainly based on the Suzuki-Trotter expansion of the adjoint propagator in the infinitesimal

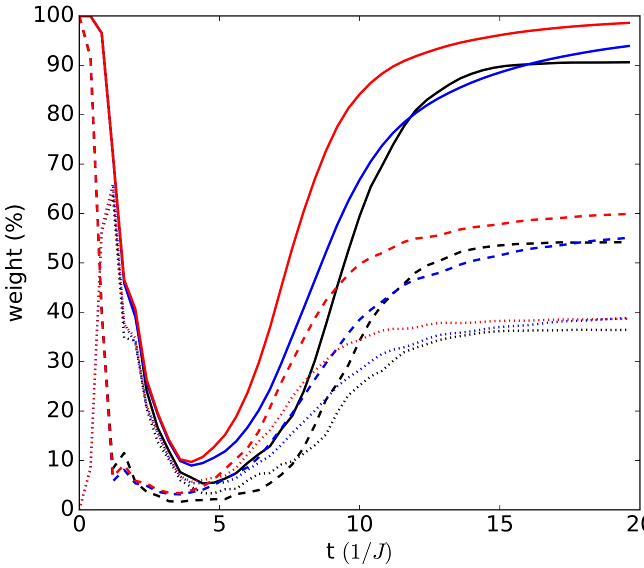

FIG. 4. Color plot of the weights of few-body terms in the chaotic Ising chain (5) with $N = 12$ spins and dissipation rate $\Gamma = 0.1$. Dashed (dotted) line denotes the total weight of one-body (nearest-neighbor two-body) terms in the operator $\mathcal{V}_b^\dagger(t)\cdot\sigma_1^z$, while the solid line is the sum of dotted and dashed lines. Black, blue, and red lines are the results for dissipative channels of amplitude damping, phase damping, and phase depolarizing, respectively.

time steps

$$B_b^\dagger(t+\tau) = \mathcal{V}_b^\dagger(\tau)\cdot B_b^\dagger(t) \approx e^{\mathcal{L}_D^\dagger\tau}\cdot(B_b^\dagger(t) - i\tau[H_s, B_b^\dagger(t)]), \quad (15)$$

where $\mathcal{L}_D^\dagger$ is the adjoint superoperator of the dissipation and τ is the infinitesimal time interval. Based on this expression, we are able to qualitatively discuss the operator spreading in the space of operators during the time evolution.

The nearest-neighbor interactions in H_s lead to operator growth in space. If there is no dissipation, every term inside the light cone is expected to have approximately equal weight at late time [25], so $F(t, A, B)$ is approximately equal to zero inside the light cone.

Intuitively dissipation leads to operator decay. Many-body terms decay at a higher rate than few-body terms, so few-body terms dominate at late time in dissipative systems. In the channel of phase depolarization, $e^{\mathcal{L}_D^\dagger\tau}\cdot\sigma^{k_i} = e^{-\Gamma\tau}\sigma^{k_i}(k_i = x, y, z)$. In one step of evolution, the decaying factors of one-body, two-body, and m-body terms are, respectively, $e^{-\Gamma\tau}$, $e^{-2\Gamma\tau}$, and $e^{-m\Gamma\tau}$. Many-body terms decay faster than few-body terms. Amplitude and phase damping channels have similar behaviors. In the dominating few-body terms, first we need to consider one-body terms. Secondly, the nearest-neighbor two-body terms cannot be ignored because the nearest-neighbor interactions in H_s [Eq. (15)] transform one-body operators into nearest-neighbor two-body operators. Our simulations support these qualitative arguments. Figure 4 shows that the sum of the weights of one-body and nearest-neighbor two-body terms approximately exceeds 90% at late time in the dissipative channels.

Moreover, because of dissipation, the weight of few-body terms decays in space at the same time. In the time-evolved operator $\mathcal{V}_b^\dagger(t)\cdot\sigma_1^z$, few-body terms on the right are sequentially generated from the ones on the left. For example, one-body term $\sigma_{i+1}^{k_{i+1}}$ is generated via the path $\sigma_i^{k_i} \to \sigma_i^{k_i'}\sigma_{i+1}^{k_{i+1}'} \to \sigma_{i+1}^{k_{i+1}}$, where $k_i, k_i', k_{i+1}', k_{i+1}$ are nontrivial indicies x, y, or z. Considering the generating paths and the different decaying rate of few-body terms, we find that extra spatial decaying factor exists when comparing the coefficients of $\sigma_{i+1}^{k_{i+1}}$ and $\sigma_i^{k_i}$. Spacial decaying factors accumulate during the scrambling of information, so the weight of few-body terms decays in space at the same time. In Fig. 3, the corrected OTOC $F(t,\sigma_i^z,\sigma_1^z)/F(t,I,\sigma_1^z)$ approaches 1 from left to right at late time t. In the channel of phase depolarizing, $(1 - F(t,\sigma_i^z,\sigma_1^z)/F(t,I,\sigma_1^z))$ is proportional to the weight of few-body terms S_i near site i at late time, i.e., $|b_{S_i}(t)|^2/\sum_{S'}|b_{S'}(t)|^2$. Thus the numerical result confirms that the weight of few-body terms decays in space at the same time.

Besides the qualitative discussions, we are going to quantitatively study the relationship between the width $d(\Gamma)$ of the partially recovered light cone and the dissipation rate Γ. Appendix provides a lower bound $\sqrt{\epsilon a v_{LR}/\Gamma}$, where a is the distance between two nearest neighbor sites, v_{LR} is the Lieb-Robinson velocity, and ϵ is a small number. This inequality is shown to be satisfied for the width of the light cone revealed by the corrected OTOC in the channel of phase damping or phase depolarizing. In general, we expect that $d(\Gamma)$ obeys a power law c/Γ^α when the dissipation rate Γ is sufficiently small.

Now we discuss how to find the width $d(\Gamma)$ of the partially recovered light cone in the numerical calculations. Our criterion is that if the difference of corrected OTOCs at $(t_1 = (d_{BA} - w/2)/v_B, d_{BA})$ and $(t_2 = (d_{BA} + w/2)/v_B, d_{BA})$ (see Fig. 3) is less than a threshold value δ, for example 0.1, then it is impossible to recognize the boundary of the light cone and we identify the smallest such d_{BA} as the width of the recovered light cone. Here w is the width of the boundary of the light cone in the system without dissipation and v_B is the corresponding butterfly velocity.

Our numerical simulation supports that $d(\Gamma)$ obeys a power law c/Γ^α. In Fig. 5, our fitting results are: $\alpha_2 \approx 0.45$, $\alpha_3 \approx 0.44$ when $0.05 \leqslant \Gamma \leqslant 0.1$, and $\alpha_2 \approx 0.43$, $\alpha_3 \approx 0.40$ when $0.1 \leqslant \Gamma \leqslant 0.16$, where the subscripts 2,3 represent the channels of phase damping and phase depolarizing, respectively. If Γ is sufficiently small, the power-law value c/Γ^α is expected to be greater than or equal to the lower bound $\sqrt{\epsilon a v_{LR}/\Gamma}$. This implies that α should be greater than or equal to 0.5. Here in our simulation, α_2 and α_3 are smaller than 0.5. The reason is that the dissipation rates in the range of [0.05, 0.1] are not small enough. Theoretically, the derivations in Appendix give the condition of sufficiently small Γ via comparing $\sqrt{\epsilon a v_{LR}/\Gamma}$ with ξ. Γ is sufficiently small if it is much less than $\epsilon a v_{LR}/\xi^2$. In this chaotic Ising model, after selecting $\epsilon \sim 0.1$, and estimating the parameters $v_{LR} \sim 2Ja$, $\xi \sim a$, then we obtain that $\Gamma \ll 0.1$ is sufficiently small. Therefore, our numerical result does not contradict the lower bound proved in Appendix. Numerically, we see that α decreases when the range of Γ increases.

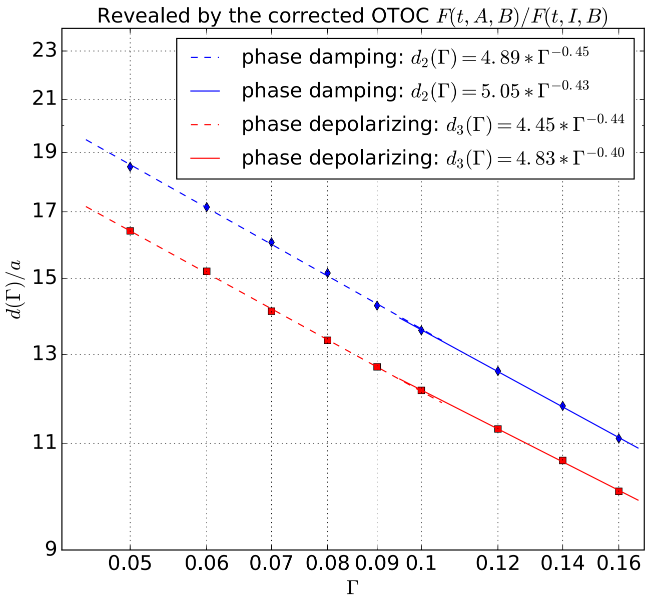

FIG. 5. The log-log plot of $d(\Gamma)$ and Γ.

Even though amplitude damping has different properties when compared with phase damping and phase depolarizing, we numerically verify that $d(\Gamma)$ still scales as a power law of the dissipation rate Γ. In the channel of amplitude damping, the corrected OTOC depends on $\mathcal{V}_b^\dagger(t)$ and $\mathcal{V}_f(t)$ which have different properties. The identity is a fixed point of $\mathcal{V}_b^\dagger(t)$ while $\mathcal{V}_f(t)$ is trace preserving. The proof in Appendix does not apply to amplitude damping, thus the lower bound $\sqrt{\frac{\epsilon a v_{LR}}{\Gamma}}$ does not work for the corrected OTOC in this channel. In the numerical simulation, we confirm that the general expectation of power-law decay is still correct. Figure 6 shows that $d(\Gamma)$ scales as a power law of Γ with the power $\alpha_1 \approx 0.31$ when $0.05 \leqslant \Gamma \leqslant 0.1$, where the subscript 1 represents the channel of amplitude damping.

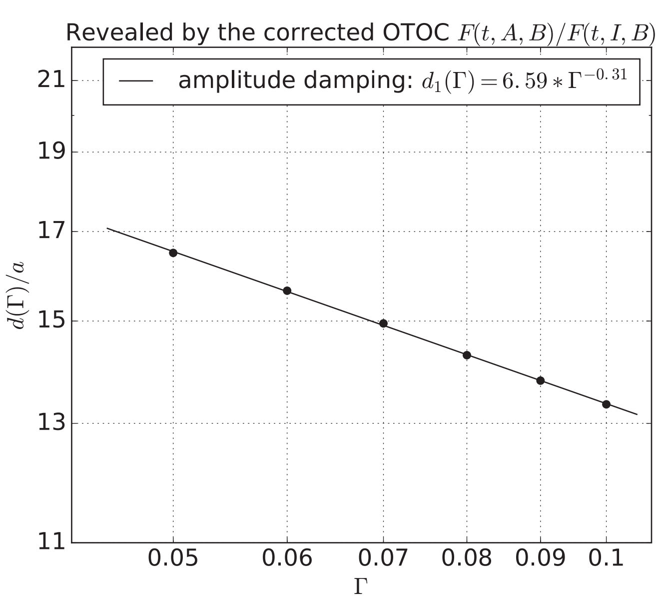

FIG. 6. The log-log plot of $d(\Gamma)$ and Γ.

V. LIEB-ROBINSON BOUND IN DISSIPATIVE SYSTEMS

Now we would like to discuss the Lieb-Robinson bound and its connections with OTOC in open quantum systems. Based on the observation of corrected OTOC, we conjecture a tighter Lieb-Robinson bound for dissipative systems.

The Lieb-Robinson inequality provides an upper bound for the speed of information propagation in quantum systems with local interactions. Let us briefly review the Lieb-Robinson bound.

Two observers, Alice and Bob, have access to the quantum system. The system is initially in the state $\rho(0)$ and its dynamics is governed by the dynamical map $\mathcal{V}_b(t)$ related to the Hamiltonian $H_b = -H_s + H_e + H_{\text{int}}$. The sender Alice has the option to perform some local actions in her region. After some time t, the receiver Bob performs some measurements to detect the signal. No signal is sent to Bob if Alice does nothing. In order to send a signal, Alice performs a small local unitary perturbation $U_A = e^{-i\epsilon O_A}$ in her region, which maps the state $\rho_s(0)$ to $\rho_s'(0) = U_A\rho_s(0)U_A^\dagger \approx \rho_s(0) - i\epsilon[O_A, \rho_s(0)]$, where O_A is a local Hermitian operator. At time t, Bob makes a measurement described by the local Hermitian operator O_B. The difference of outcomes describing the capability to detect the signal is

$$\begin{aligned} &|\operatorname{tr}(O_B\mathcal{V}_b(t)\cdot(\rho_s'(0)-\rho_s(0)))| \\ &\quad = \epsilon|\operatorname{tr}(\rho_s(0)[\mathcal{V}_b^\dagger(t)\cdot O_B, O_A])| \\ &\quad \leqslant \epsilon\|[\mathcal{V}_b^\dagger(t)\cdot O_B, O_A]\|, \end{aligned} \tag{16}$$

where the operator norm is defined by $\|O\| = \sup_{|\psi\rangle}\|O|\psi\rangle\|/\||\psi\rangle\|$. Following the Lieb-Robinson bound in closed systems [68–70], an inequality has been proved in open quantum systems [71–76]

$$\|[\mathcal{V}_b^\dagger(t)\cdot O_B, O_A]\| \leqslant c\|O_A\|\cdot\|O_B\|\, e^{-\frac{d_{BA}-v_{LR}t}{\xi}}, \tag{17}$$

where c, ξ are some constants, v_{LR} is the Lieb-Robinson velocity, and d_{BA} is the distance between the local operators O_A and O_B. The Lieb-Robinson velocity v_{LR} is an upper bound for the speed of information propagation, so it is greater than or equal to the butterfly velocity v_B at $\beta = 0$ in Eq. (4) [16]. References [17,18,26,27] provide more discussions about the relationship between v_B and v_{LR}.

In dissipative systems, the left-hand side of Eq. (17) decays to zero at late time, so Eq. (17) is not tight enough. One reason is that the operator $\mathcal{V}_b^\dagger(t)\cdot O_B$ in the Heisenberg picture is overall decaying because of the dissipation. Reference [73] has proved that the operator norm of $\mathcal{V}_b^\dagger(t)\cdot O_B$ is nonincreasing because of the dissipation, i.e., $\|\mathcal{V}_b^\dagger(t+\mathrm{d}t)\cdot O_B\| \leqslant \|\mathcal{V}_b^\dagger(t)\cdot O_B\|$, where $\mathrm{d}t$ is an infinitesimal time step. This means that the nontrivial elements in the time-evolved operator are decaying during the time evolution. Our numerical simulations (Fig. 7) show that the left-hand side of Eq. (17) decays to zero at late time, and the boundary of the light cone gradually disappears when the distance d_{BA} increases.

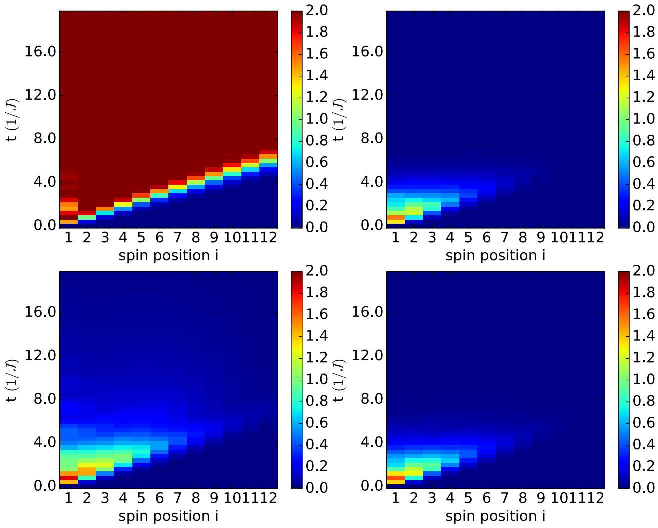

FIG. 7. The operator norm of the commutator $\| [\mathcal{V}_b^\dagger(t)\cdot\sigma_1^z, \sigma_i^z] \|$ in the chaotic Ising chain (5) with no dissipation (upper left), amplitude damping (upper right), phase damping (lower left), and phase depolarizing (lower right). The dissipation rate is $\Gamma = 0.2$ in these three dissipative channels.

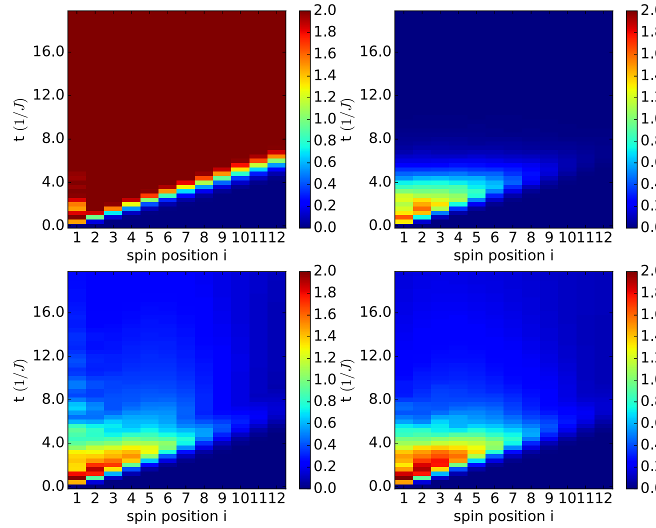

FIG. 8. The corrected operator norm of the commutator $\| [\mathcal{V}_b^\dagger(t)\cdot\sigma_1^z, \sigma_i^z] \| / \| \mathcal{V}_b^\dagger(t)\cdot\sigma_1^z \|$ in the chaotic Ising chain (5) with no dissipation (upper left), amplitude damping (upper right), phase damping (lower left), and phase depolarizing (lower right). The dissipation rate is $\Gamma = 0.2$ in these three dissipative channels.

Inspired by the corrected OTOC, we conjecture a tighter Lieb-Robinson bound in dissipative systems

$$\frac{\| [\mathcal{V}_b^\dagger(t)\cdot O_B, O_A] \|}{\| O_A \| \cdot \| \mathcal{V}_b^\dagger(t)\cdot O_B \|} \leqslant c\, e^{-\frac{d_{AB}-v_{LR}t}{\xi}}, \tag{18}$$

The above tighter bound has deep connections with the corrected OTOC. In the channel of phase damping or phase depolarizing, the adjoint dynamical map $\mathcal{V}_b^\dagger(t)$ is exactly equal to $\mathcal{V}_f(t)$. Then $2(1-\frac{F(t,O_A,O_B)}{F(t,I,O_B)}) = \frac{\| [\mathcal{V}_b^\dagger(t)\cdot O_B, O_A] \|_F^2}{\| \mathcal{V}_b^\dagger(t)\cdot O_B \|_F^2}$ holds when the observables are also unitary, where $\| O \|_F = \sqrt{\mathrm{tr}(O O^\dagger)/2^N}$ is the normalized Frobenius norm of the operator O. We expect that the normalized Frobenius and operator norms exhibit similar behaviors during the time evolution. Based on this expectation, Eq. (18) is conjectured in dissipative systems via changing the normalized Frobenius norm to the operator norm. Similar to the corrected OTOC, the left-hand side of the above modified version of the Lieb-Robinson bound is able to partially recover the destroyed light cone in the chaotic Ising chain with dissipation (see Fig. 8).

In the above tighter Lieb-Robinson bound, the correcting factor $1/\| \mathcal{V}_b^\dagger(t)\cdot O_B \|$ has different behaviors in different dissipative channels. $\| \mathcal{V}_b^\dagger(t)\cdot O_B \|$ decays to zero in the channel of phase damping or phase depolarizing but converges to a positive constant in the channel of amplitude damping (see Fig. 9). In the channel of amplitude damping, the adjoint dynamical map $\mathcal{V}_b^\dagger(t)$ does not preserve the trace of an operator, the identity operator I appears in the decomposition of $\mathcal{V}_b^\dagger(t)\cdot O_B$ in terms of Pauli operators when O_B is traceless. Therefore, the operator norm of $\mathcal{V}_b^\dagger(t)\cdot O_B$ converges to a constant. This can also be observed in the upper right panel of Fig. 8 which is distinct from the lower ones. The operator norm of the commutator is decaying to zero while the denominator converges to a positive constant when $t > 7/J$. In the channel of amplitude damping, the correcting factor $1/\| \mathcal{V}_b^\dagger(t)\cdot O_B \|$ does not play an essential role to remove the effect of overall decay due to the information leaking.

VI. CONCLUSIONS AND DISCUSSIONS

In this paper, we study the effect of dissipation on information scrambling in open chaotic systems. By numerically calculating the measured OTOC signal in a chaotic spin chain in the presence of common types of dissipation, we find that dissipation leads to the decay of the signal not only due to information leaking, but also information re-structuring. We define a corrected OTOC to remove the effect of leaking and partially recover the information light cone. However, due to the re-structuring, the recovered light cone only persists to a finite distance. Based on this understanding of how

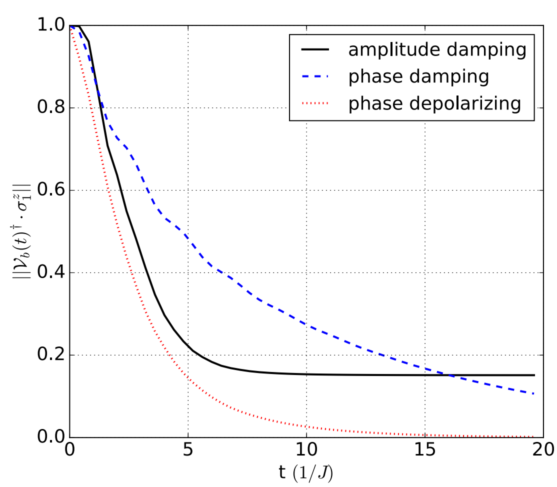

FIG. 9. The decay of the operator norm $\| \mathcal{V}_b^\dagger(t)\cdot\sigma_1^z \|$ in different dissipative channels ($N = 12$, $\Gamma = 0.2$).

dissipation affects information scrambling, we conjecture a tighter version of the Lieb-Robinson bound in open systems, which we support with numerical simulation.

As our study focuses on the overall shape of the light cone in OTOC and how it changes with dissipation, we do not expect the result to depend on model parameters either. The existence of a linear light cone shows up in both integrable and nonintegrable systems [77]. For example, Ref. [37] studied the OTOC in an integrable Ising chain and found a linear shaped light cone just like in the random circuit model case [43,44]. Some details of the light cone might differ, for example, the broadening of the wave front or the late time value of the OTOC. But if we focus on the shape of the light cone, there is no intrinsic difference between the integrable and the nonintegrable case.

Given the observation we made in this paper, several open questions would be interesting to explore in future work. First, we qualitatively discussed the information re-structuring during scrambling. A more accurate estimation of the size of the light cone may be obtained by carefully modeling the dynamics as dissipative quantum walks. Secondly, although we were able to partially recover the light cone numerically, this is not practical experimentally, as the normalization factor we divide out in Eq. (13) decays exponentially in time and quickly becomes too small to be accessible experimentally. Is there a better way to see information scrambling in the presence of dissipation? Are there quantities which are also sensitive to information scrambling as OTOC but more robust to the effect of dissipation? This is an important question to be addressed in future work. Finally, we conjectured the modified version of open system Lieb-Robinson bound based on numerical observation. It would be nice to see if this bound can be analytically proved.

Note. Recently, we learned of the work by Swingle and Yunger Halpern [78] which also studies the problem of extracting OTOCs' early-time dynamics in the presence of error and decoherence.

ACKNOWLEDGMENTS

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APPENDIX: PROOF OF A LOWER BOUND

Here we prove a lower bound $\sqrt{\epsilon a v_{LR}/\Gamma}$ for the width $d(\Gamma)$ of the partially recovered light cone revealed by the corrected OTOC in the channel of phase damping or phase depolarizing. The main ideas in the proof are comparing the difference between the adjoint propagator in the dissipative channel and the unitary one without dissipation and employing the adjoint propagator of spatially truncated adjoint Liouvillians.

Lemma 1. Suppose $\mathcal{L}_1^\dagger(t)$ and $\mathcal{L}_0^\dagger(t)$ are the adjoint Liouvillian superoperators describing Markovian dynamics of the same open quantum system with $\|\mathcal{L}_1^\dagger(t)-\mathcal{L}_0^\dagger(t)\| \leqslant f(t)$, then the difference of adjoint propagators satisfies

$$\|\mathcal{V}_1^\dagger(t,0)-\mathcal{V}_0^\dagger(t,0)\| \leqslant \int_0^t d\tau f(\tau), \tag{A1}$$

where $\mathcal{V}_k^\dagger(t,s)=\mathcal{T}_\rightarrow e^{\int_s^t \mathcal{L}_k^\dagger(\tau)d\tau}(t\geqslant s, k=0,1)$, and $\mathcal{T}_\rightarrow$ or $\mathcal{T}_\leftarrow$ is the time-ordering operator which orders products of time-dependent operators such that their time arguments increase in the direction indicated by the arrow.

Proof.

$$\begin{aligned}
&\|\mathcal{V}_1^\dagger(t,0)-\mathcal{V}_0^\dagger(t,0)\| \\
&\quad= \|\mathcal{V}_0^\dagger(0,0)\mathcal{V}_1^\dagger(t,0)-\mathcal{V}_0^\dagger(t,0)\mathcal{V}_1^\dagger(t,t)\| \\
&\quad= \left\|\int_0^t ds \frac{\partial}{\partial s}(\mathcal{V}_0^\dagger(s,0)\mathcal{V}_1^\dagger(t,s))\right\| \\
&\quad\leqslant \int_0^t ds \|\mathcal{V}_0^\dagger(s,0)(\mathcal{L}_0^\dagger(s)-\mathcal{L}_1^\dagger(s))\mathcal{V}_1^\dagger(t,s)\| \\
&\quad\leqslant \int_0^t ds \|\mathcal{V}_0^\dagger(s,0)\|\cdot\|\mathcal{L}_1^\dagger(s)-\mathcal{L}_0^\dagger(s)\|\cdot\|\mathcal{V}_1^\dagger(t,s)\| \\
&\quad\leqslant \int_0^t ds \|\mathcal{L}_1^\dagger(s)-\mathcal{L}_0^\dagger(s)\| \leqslant \int_0^t ds f(s).
\end{aligned}$$

In the derivation, one uses the fact that the adjoint propagators $\mathcal{V}_k^\dagger(t,s)$ are norm-nonincreasing [61,71,73]. ■

Here, we need to pay attention to the difference between the propagator $\mathcal{V}(t,s)=\mathcal{T}_\leftarrow e^{\int_s^t \mathcal{L}(\tau)d\tau}(t\geqslant s)$ and its adjoint $\mathcal{V}^\dagger(t,s)=\mathcal{T}_\rightarrow e^{\int_s^t \mathcal{L}^\dagger(\tau)d\tau}(t\geqslant s)$. $\mathcal{V}(t,s)$ acts on the density matrix and is trace preserving. $\mathcal{V}^\dagger(t,s)$ acts on the observables and the identity is one of its fixed points. For unitary evolution, $\mathcal{V}^\dagger(t,s)$ and $\mathcal{V}(t,s)$ are the inverse of each other and both norm preserving. When dissipation exists, only $\mathcal{V}^\dagger(t,s)$ is norm nonincreasing for arbitrary observables, i.e., $\|\mathcal{V}^\dagger(t,s)\cdot O\| \leqslant \|O\|$ $(\forall O=O^\dagger)$.

Lemma 2. In a one-dimensional system, $\mathcal{L}_H^\dagger=\sum_i \mathcal{L}_{H_i}^\dagger$ is the sum of local adjoint Liouvillian superoperators, and $\mathcal{L}_D^\dagger=\Gamma\sum_k \mathcal{L}_{D,k}^\dagger$ is the sum of adjoint dissipative superoperators acting on each site, where $\|\mathcal{L}_{D,k}^\dagger\|\leqslant 1$ and Γ is the dissipation rate. During the evolution, the operator difference between the dissipative and unitary channels is upper bounded by

$$\|\mathcal{V}_1^\dagger(t,0)\cdot B-\mathcal{V}_0^\dagger(t,0)\cdot B\| \leqslant \Gamma O(t^2)+\delta \tag{A2}$$

in the limit of small Γ and large t, where $\mathcal{V}_1^\dagger(t,s)=\mathcal{T}_\rightarrow e^{\int_s^t(\mathcal{L}_H^\dagger(\tau)+\mathcal{L}_D^\dagger(\tau))d\tau}(t\geqslant s)$, $\mathcal{V}_0^\dagger(t,s)=\mathcal{T}_\rightarrow e^{\int_s^t \mathcal{L}_H^\dagger(\tau)d\tau}(t\geqslant s)$, B is a local observable at site 0, and δ is an arbitrarily small constant.

Proof. For an open quantum system described by short-range Liouvillians, the Lieb-Robinson bound

$$\|[\mathcal{V}^\dagger(t)\cdot B, A]\| \leqslant c\,\|B\|\cdot\|A\|\,e^{-(d_{AB}-v_{LR}t)/\xi} \tag{A3}$$

implies the existence of an upper limit to the speed of quantum information propagation. The outside signal is exponentially small in the distance from the boundary of the effective light cone. Based on the Lieb-Robinson bound, Ref. [73] obtained the quasilocality of Markovian quantum dynamics: Up to exponentially small error, the evolution of a local observable can be approximately obtained by applying the propagator of a spatially truncated version of the adjoint Liouvillian, provided that the range of the truncated propagator is larger than the support of the time-evolved observable. The truncated propagators we select are

$$\tilde{\mathcal{V}}_0^\dagger(t,0) = \mathcal{T}_\rightarrow e^{\int_0^t d\tau \sum_{i:H_i\subset\Lambda(\tau)} \mathcal{L}_{H_i}^\dagger(\tau)},\ \ \tilde{\mathcal{V}}_1^\dagger(t,0)$$
$$= \mathcal{T}_\rightarrow e^{\int_0^t d\tau (\sum_{i:H_i\subset\Lambda(\tau)} \mathcal{L}_{H_i}^\dagger(\tau) + \sum_{k=-d(\tau)/a}^{d(\tau)/a} \mathcal{L}_{D,k}(\tau))},$$

where $d(t) = v_{LR}t + \xi \log(c''/\delta)$ for some large constant c'', a is the distance between two nearest neighboring sites, and $H_i \subset \Lambda(t)$ means the local term H_i is located in the regime $\Lambda(t) = (-d(t), d(t))$. Let $B_k(t) = \mathcal{V}_k^\dagger(t,0)\cdot B$ and $\tilde{B}_k(t) = \tilde{\mathcal{V}}_k^\dagger(t,0)\cdot B$. Applying the triangle inequality, one obtains

$$\|B_1(t) - B_0(t)\| \leqslant \|\tilde{B}_1(t) - \tilde{B}_0(t)\|$$
$$+\ \|B_1(t) - \tilde{B}_1(t)\| + \|\tilde{B}_0(t) - B_0(t)\|.$$

On the right-hand side, the first term is upper bounded by $\|B\| \int_0^t d\tau \Gamma(\xi/a + v_{LR}\tau/a) = \|B\|\Gamma t(v_{LR}t/2 + \xi)/a$ (Lemma 1). The second and third terms both are less than or equal to $c'\ \|B\|\ e^{(v_{LR}t - d(t))/\xi} = c'\ \|B\|\ \delta/c''$ [73]. Thus we get $\|\mathcal{V}_1^\dagger(t,0)\cdot B - \mathcal{V}_0^\dagger(t,0)\cdot B\| \leqslant \|B\|(v_{LR}\Gamma t^2/2 + \xi\Gamma t + 2c'\delta/c'')/a$. Therefore $\|\mathcal{V}_1^\dagger(t,0)\cdot B - \mathcal{V}_0^\dagger(t,0)\cdot B\| \leqslant \Gamma O(t^2) + \delta$ in the limit of small Γ and large t. ■

Proposition 1. In the chaotic Ising chain with dissipations acting on each site, the light cone within the time range $t \leqslant \sqrt{\frac{\epsilon a}{v_{LR}\Gamma}}$ can be revealed by $\|[\mathcal{V}_1^\dagger(t,0)\cdot B, A]\|$, $\|[\mathcal{V}_1^\dagger(t,0)\cdot B, A]\|/\|\mathcal{V}_1^\dagger(t,0)\cdot B\|$, $\|[\mathcal{V}_1^\dagger(t,0)\cdot B, A]\|_F$ and $\|[\mathcal{V}_1^\dagger(t,0)\cdot B, A]\|_F/\|\mathcal{V}_1^\dagger(t,0)\cdot B\|_F$, where v_{LR} is the Lieb-Robinson velocity, a is the distance between two nearest neighboring sites, ϵ is a small number (for example, $\epsilon \sim 0.1$), the dissipation rate Γ is sufficiently small ($\Gamma \ll \epsilon a v_{LR}/\xi^2$), $\|O\|$ is the operator norm and $\|O\|_F = \lim_{N\to\infty}\sqrt{\mathrm{tr}(OO^\dagger)/2^N}$ is the normalized Frobenius norm of operators in the thermodynamic limit. The width of the light cone is at least $\sqrt{\frac{\epsilon a v_{LR}}{\Gamma}}$.

Proof. According to Lemma 2, the t^2 term plays a dominant role in the inequality for sufficiently small dissipation rate $\Gamma \ll \epsilon a v_{LR}/\xi^2$. If $t \leqslant \sqrt{\frac{\epsilon a}{v_{LR}\Gamma}}$, then one obtains $\|B_1(t) - B_0(t)\| \leqslant \epsilon\|B\|$ when comparing the operators $B_1(t) = \mathcal{V}_1^\dagger(t,0)\cdot B$ in the dissipative channel and $B_0(t) = \mathcal{V}_0^\dagger(t,0)\cdot B$ in the unitary channel. Applying the triangle inequality, one obtains

$$(1-\epsilon)\|B\| \leqslant \|B_1(t)\| \leqslant (1+\epsilon)\|B\|,$$
$$\|[B_0(t),A]\| - \epsilon\|\mathcal{C}_A\|\|B\| \leqslant \|[B_1(t),A]\|$$
$$\leqslant \|[B_0(t),A]\| + \epsilon\|\mathcal{C}_A\|\|B\|,$$
$$(1+\epsilon)^{-1}\left(\frac{\|[B_0(t),A]\|}{\|B\|} - \epsilon\|\mathcal{C}_A\|\right) \leqslant \frac{\|[B_1(t),A]\|}{\|B_1(t)\|}$$
$$\leqslant (1-\epsilon)^{-1}\left(\frac{\|[B_0(t),A]\|}{\|B\|} + \epsilon\|\mathcal{C}_A\|\right),$$

where the superoperator $\mathcal{C}_A$ is defined as $\mathcal{C}_A\cdot O = [O, A]$. The normalized Frobenius norm is less than or equal to the operator norm, i.e., $\|O\|_F \leqslant \|O\|$, so we get

$$\|B_1(t) - B_0(t)\|_F \leqslant \|B_1(t) - B_0(t)\| \leqslant \epsilon\|B\|$$
$$\|[B_0(t),A]\|_F - \epsilon\|\mathcal{C}_A\|\|B\| \leqslant \|[B_1(t),A]\|_F$$
$$\leqslant \|[B_0(t),A]\|_F + \epsilon\|\mathcal{C}_A\|\|B\|,$$
$$\frac{\|[B_0(t),A]\|_F - \epsilon\|\mathcal{C}_A\|\|B\|}{\|B\|_F + \epsilon\|B\|} \leqslant \frac{\|[B_1(t),A]\|_F}{\|B_1(t)\|_F}$$
$$\leqslant \frac{\|[B_0(t),A]\|_F + \epsilon\|\mathcal{C}_A\|\|B\|}{\|B\|_F - \epsilon\|B\|}.$$

In the unitary channel, $\|[B_0(t),A]\|$ and $\|[B_0(t),A]\|_F$ both are able to detect the ballistic light cone. Because ϵ is a small number, it is also small that the difference of the corresponding quantities between the dissipative and unitary channel. Thus, $\|[B_1(t), A]\|$, $\|[B_1(t),A]\|/\|B_1(t)\|$, $\|[B_1(t),A]\|_F$, and $\|[B_1(t), A]\|_F/\|B_1(t)\|_F$ are both able to detect the light cone in the time range $t \leqslant \sqrt{\frac{\epsilon a}{v_{LR}\Gamma}}$. The width of the light cone is at least $\sqrt{\frac{\epsilon a v_{LR}}{\Gamma}}$ for sufficiently small dissipation rate $\Gamma \ll \epsilon a v_{LR}/\xi^2$. ■

Corollary 1. For sufficiently small dissipation rate $\Gamma \ll \epsilon a v_{LR}/\xi^2$, the lower bound $\sqrt{\frac{\epsilon a v_{LR}}{\Gamma}}$ works for the width of the light cone revealed by the corrected OTOC in the chaotic Ising chain with dissipation of phase damping or phase depolarizing.

Proof. In the channel of phase damping or phase depolarizing, the adjoint propagator $\mathcal{V}_b^\dagger(t)$ is exactly equal to the propagator $\mathcal{V}_f(t)$, then

$$2\left(1 - \frac{F(t,A,B)}{F(t,I,B)}\right) = \frac{\|\,[\mathcal{V}_b^\dagger(t)\cdot B, A]\,\|_F^2}{\|\mathcal{V}_b^\dagger(t)\cdot B\|_F^2}. \tag{A4}$$

Based on Proposition 1, the lower bound $\sqrt{\frac{\epsilon a v_{LR}}{\Gamma}}$ works for the width of the light cone revealed by the corrected OTOC in the channel of phase damping or phase depolarizing. ■

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