STABILITY THEORY FOR CROSS HATCHING

Part 1. Linear Stability theory

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Lester Lees, Toshi Kubota and Denny R-S Ko
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Contract No. F04701-68-C-0151

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FOREWORD

This report was prepared by the California Institute of Technology, Pasadena, California under USAF Contract F04701-68-C-0151, Project "Research on Fluid Mechanics of Striation Ablation." The work was administered under the direction of the Space and Missile Systems Organization, Air Force Systems Command.

This report covers work performed between 1 September 1968 to 15 August 1971.

Professors Lester Lees and Toshi Kubota were the Co-Principal Investigators, Dr. Denny R-S Ko was a Post-Doctoral Research Fellow and Mr. Asher Sigal was a Graduate Research Assistant.

The manuscript of this report was released by the authors in January 1972.

This technical report has been reviewed and is approved.

ABSTRACT

A linear instability theory was developed which couples small perturbations in surface shape of a subliming surface and resulting perturbations in aerodynamic heat-transfer rates. Equations governing compressible turbulent boundary layers were linearized for small perturbations whose streamwise scale lengths are comparable with the undisturbed boundary layer thickness. For turbulent shear stress, the mixing-length approximation was employed, and the turbulent Prandtl number was assumed to be unity for turbulent heat flux. Heat conduction within the ablating solid was analyzed by linearizing the boundary condition for small amplitude. Sample computations were made for a teflon surface ablating under turbulent boundary layer at Mach 2.6, and the results indicate that the small surface perturbations are unstable within certain regions in the wave number - wave angle space.

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1. Introduction

The cross-hatching on ablating surfaces is treated as a stability problem of perturbations on ablating surfaces, coupled with resulting perturbations in the aerodynamic forces and heating from the gas boundary layer and the internal heat conduction and deformation within the ablating material, fed back into the surface perturbation. The initial phase of the stability can be treated by a linearized theory for small perturbations. In this boundary layer-ablation feedback model, then, the necessary ingredients before we can treat the stability problem are

1) small perturbations in boundary layer heat transfer rates;
2) heat transfer and deformation of the wall material;
3) phenomena occurring at the interface.

For subliming ablators without appreciable softening at ablating temperature the interior problem is that of heat conduction. At the crests of surface perturbations the heat conduction away from the interface to the interior is decreased, and in the valleys the heat conduction is increased, which has the stabilizing effect. If the material becomes soft and deformable, we may have to include visco-elastic motion of a layer next to the interface. For melting ablators, we have to include a liquid layer between the gas boundary layer and the solid material. With charring ablators, the analysis becomes more complex, since it must include the ablation of charred material by mechanical stress and chemical reaction at the surface, the heat and mass transfers in the charred layer, the pyrolysis at the bottom of the charred layer, and the heat conduction in the solid ablator, all of which are perturbed by the changes in the aerodynamic stresses and heat transfer rate from the gas boundary layer.

Regardless of which mechanism we assume for the behavior of ablating material, then, the perturbations in the aerodynamic stresses and heat transfer rate must be known accurately for given perturbations in the boundary conditions at the interface.

The small perturbations in boundary layers have been treated extensively in the boundary layer stability theory for both incompressible and compressible laminar boundary layers. The stability theory, however, are concerned with eigen-value problems of small perturbations that vanish at the boundaries. On the other hand, we are interested in small perturbations forced by the wall perturbation. Another limitation is that the stability theory deals almost exclusively with laminar boundary layers, while the cross-hatch was observed, thus far, only with supersonic turbulent boundary layers.

The forced small perturbation was treated by Lighthill in the shock wave-boundary layer interaction, by Miles and Benjamin in connection with wind-generated waves on the surface of the water. But none of these works treats the perturbations in shear stress and heat transfer in a satisfactory manner.

At the present state of knowledge of turbulent shear flows it is not possible for us to analyze the problem from the first principles. We must employ certain simplified models for turbulent shear stresses and heat flux. There are some models available which are proven (by comparing with measurements) to be applicable to conventional boundary layers, mixing layers, wakes and jets for obtaining solutions satisfactory for engineering purposes. The solutions derived from such models, however, have to be compared with measurements when they are applied to those flows that are outside the class of flows for which these models were invented.

Since no experimental verification of the turbulent-shear-stress models are available for boundary layers over wavy surfaces with the wave length comparable to the boundary layer thickness, the second part of the present study was devoted to a detailed investigation of low-speed turbulent boundary layers over solid stationary wavy surface in which distributions of turbulent shear stress and energy as well as the mean flow field are measured.

Low-speed boundary layer rather than supersonic turbulent boundary layer was chosen for the experiment since 1) it is relatively easy to obtain thick boundary layers, which facilitate detailed measurements of distributions of flow properties across the layer, without use of large expensive wind tunnels; 2) the data reduction of hot-wire anemometry is well established in low-speed turbulent flows, whereas at supersonic speeds the complexity of data acquisition is magnified several folds and some drastic approximations have to be made in data reduction of turbulence components; 3) the significant frequency spectra extend only to a few kilo-Hertz in low-speed thick boundary layers but to hundreds of kilo-Hertz in supersonic turbulent layers which is beyond the frequency range of conventional hot-wire anemometry; 4) the primary aim of the experiment is to examine the applicability of currently available models for turbulent shear flow, which are developed mainly for conventional turbulent boundary layers, to the flow over wavy surfaces whose wave lengths are comparable to the boundary layer, and for this purpose even the low-speed data are virtually nonexistent.

The first part of the study covers a linear analysis of small perturbations in incompressible turbulent boundary layer utilizing an eddy-viscosity model which has been proven to yield satisfactory results for conventional boundary layers. The results from the analysis are compared with the experimental results.

2. Formulation of Stability Problem for Subliming Wall

If we denote the perturbation of a surface undergoing steady subliming ablation under a turbulent boundary layer by $\epsilon\,\bar{y}(x,t)$, the perturbation in the ablation rate by $\epsilon\,\dot{m}'$, the density of the solid by ρ_s, the heat of sublimation by L_E, the convective heat-transfer perturbation by $\epsilon\,\dot{q}_a'$, the heat-conduction perturbation in the solid by $\epsilon\,\dot{q}_s'$ (positive away from the interface), then from the heat-energy balance at the interface we obtain

$$\epsilon\,\rho_s \frac{\partial \bar{y}}{\partial t} = -\epsilon\,\dot{m}' = \frac{\epsilon}{L_E}(\dot{q}_s' - \dot{q}_a')$$

By dividing the convective and conductive heat-flux perturbations by the values of these fluxes under a steady ablation rate $\dot{m} = \rho_s v_s$

$$\bar{q}_s = \dot{m}\,L_T, \quad \bar{q}_a = \dot{m}(L_E + L_T)$$

$$L_T \equiv C_s\,[T_w - T(-\infty)] \ ,$$

we can rewrite the above equation in the form

$$\frac{1}{v_s}\frac{\partial \bar{y}}{\partial t} = \frac{L_T}{L_E}\frac{q_s'}{\bar{q}_s} - \left(1 + \frac{L_T}{L_E}\right)\frac{q_a'}{\bar{q}_a} \tag{2.1}$$ *

If the surface perturbation is very small and given by

$$\bar{y}(x,t) = e^{+i\alpha(x - ct)} \tag{2.2}$$

then the heat-flux perturbations will be of the form

* All variables in this section are dimensionless: the reference dimensions are the boundary-layer thickness and the freestream gas properties.

$$\left.\begin{aligned} \epsilon\,\frac{q_a'}{\bar{q}_a} &= \epsilon\,\tilde{Q}_a(\alpha, c; M_\infty, \beta)e^{i\alpha(x-ct)} \\ \epsilon\,\frac{q_s'}{\bar{q}_s} &= \epsilon\,\tilde{Q}_s(\alpha, c; v_s/\kappa_s)e^{i\alpha(x-ct)} \end{aligned}\right\} \tag{2.3}$$

where κ_s is the thermal diffusivity of the solid, and β is the angle between the wave crest and the freestream velocity. By substituting Eqs. (2.2) and (2.3) into Eq. (2.1), we obtain

$$\begin{aligned} i\frac{\alpha c}{v_s} + \lambda_L\,\tilde{Q}_s(\alpha, c) - (1+\lambda_L)\tilde{Q}_a(\alpha, c) &= 0 \\ \lambda_L &\equiv L_T/L_E \end{aligned} \tag{2.4}$$

This is the eigenvalue equation which determines the value of c. The real part of c is the wave speed of the surface pattern. The imaginary part of c is related to the amplification rate of the pattern; namely, the time rate of amplification σ is given by

$$\sigma = \alpha\, c_i \tag{2.5}$$

Thus, if $c_i > 0$, the small perturbations are amplified, and if $c_i < 0$ they are damped.

2.1. Heat Conduction in Solid

We take the undisturbed surface to be at $y = 0$ with solid material in $y < 0$. Thus the solid is moving toward the plane $y = 0$ at a constant speed v_s. If the perturbed surface is described by $y = \epsilon\,\bar{y}(x, t)$, the average value of $\bar{y}$ is zero when nonlinear effects are neglected.

If we assume a constant thermal diffusivity, the temperature in the solid is governed by the equation

$$\frac{\partial T}{\partial t} + v_s \frac{\partial T}{\partial y} = \kappa_s \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \qquad (2.6)$$

If T_{in} denotes the temperature at large distance from the interface, then

$$T = T_{in} \quad \text{as} \quad y \to -\infty \qquad (2.7a)$$

If we neglect small changes in ablation temperature with pressure, then

$$T = T_w, \text{ constant at } y = \epsilon \bar{y}(x, t) \quad . \qquad (2.7b)$$

When ϵ is small, we expand T in a series*

$$T = T_o(y) + \epsilon\, T_1(x, y, t) + \ldots \qquad (2.8)$$

Then from Eq. (2.6) we obtain

$$v_s \frac{T_o}{dy} = \kappa_s \frac{d^2 T_o}{dy^2} \qquad (2.9)$$

$$\frac{\partial T_1}{\partial t} + v_s \frac{\partial T_1}{\partial y} = \kappa_s \left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} \right) \qquad (2.10)$$

The boundary condition (2.7a) becomes

$$\left.\begin{aligned} T_o(y) &= T_{in} \\ T_1(x, y, t) &= 0 \end{aligned}\right\} \text{ as } y \to -\infty \qquad (2.11)$$

From the boundary condition (2.7b) we get

$$\begin{aligned} T_w &= T_o(\epsilon \bar{y}) + \epsilon\, T_1(x, \epsilon \bar{y}, t) + \ldots \\ &= T_o(0) + \epsilon \left[T_1(x, 0, t) + T_o'(0) \bar{y} \right] + \ldots \end{aligned}$$

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* For validity of this expansion it is required that $\epsilon\, v_s / \kappa_s \ll 1$.

Therefore

$$\left.\begin{aligned} T_o(0) &= T_w \\ T_1(x, 0, t) &= -T_o'(0)\,\overline{y}(x, t) \end{aligned}\right\} \qquad (2.12)$$

The heat flux at the interface is given by

$$\begin{aligned} q_s &= k_s \frac{\partial T}{\partial y} \\ &= k_s\,[T_o'(\epsilon y^*) + \epsilon\, T_{1y}(x, \epsilon y^*, t) + \ldots] \\ &= k_s\,[T_o'(0) + \epsilon\{T_{1y}(x, 0, t) + T_o''(0) y^*\} + \ldots] \end{aligned}$$

Hence

$$\left.\begin{aligned} \overline{q}_s &= k_s\, T_o'(0) \\ \epsilon\, q_s' &= \epsilon\, k_s [T_{1y}(x, 0, t) + T_o''(0) y^*(x, t)] \end{aligned}\right\} \qquad (2.13)$$

The solution for T_o is easily obtained and

$$\left.\begin{aligned} T_o(y) &= T_{in} + (T_w - T_{in}) \exp(v_s y/\kappa_s) \\ T_o'(0) &= (T_w - T_{in}) v_s/\kappa_s \\ T_o''(0) &= (T_w - T_{in})(v_s/\kappa_s)^2 \end{aligned}\right\} \qquad (2.14)$$

For the linear perturbation, let

$$\left.\begin{aligned} y^* &= e^{i\alpha(x-ct)} \\ T_1(x, y, t) &= -(T_w - T_{in}) \frac{v_s}{\kappa_s} f(y) e^{i\alpha(x-ct)} \end{aligned}\right\} \qquad (2.15)$$

Substitution of Eq. (2.15) into Eqs. (2.10), (2.11) and (2.12) yields

$$f'' - \frac{v_s}{\kappa_s} f' - (\alpha^2 - i \frac{\alpha c}{\kappa_s})f = 0$$

$$f(0) = 1, \quad f(-\infty) = 0$$

The solution for f is given by

$$f(y) = e^{\Lambda v_s y/\kappa_s}$$

with

$$\Lambda = \frac{1}{2}\left\{1+\left[1+4\left(\frac{\alpha \kappa_s}{v_s}\right)^2 - 4i \frac{\kappa_s}{v_s} \frac{\alpha c}{v_s}\right]^{\frac{1}{2}}\right\} \tag{2.16}$$

Hence

$$T_1(x, y, t) = -(T_w - T_{in}) \frac{v_s}{\kappa_s} e^{\Lambda v_s y/\kappa_s} e^{i\alpha(x-ct)}$$

and

$$T_{1y}(x, 0, t) = -(T_w - T_{in}) \left(\frac{v_s}{\kappa_s}\right)^2 \Lambda e^{i\alpha(x-ct)} \tag{2.17}$$

Therefore from (2.13), (2.14) and (2.17)

$$\left.\begin{aligned} \overline{q}_s &= k_s(T_w - T_{in}) \left(\frac{v_s}{\kappa_s}\right) \\ \epsilon \overline{q}_s' &= \epsilon i_s(T_w - T_{in}) \left(\frac{v_s}{\kappa_s}\right)^2 (1-\Lambda)e^{i\alpha(x-ct)} \end{aligned}\right\} \tag{2.18}$$

Thus

$$\widetilde{Q}_s(\alpha, c; \frac{v_s}{\kappa_s}) = \frac{v_s}{\kappa_s} (1-\Lambda) \tag{2.19}$$

3. Small Perturbations in Turbulent Boundary Layer

One of the basic difficulties of problems involving turbulent boundary layers is our lack of fundamental knowledge in turbulence. Even today we cannot compute the turbulent shear stress and heat flux in boundary layers from the first principles of physics. Therefore, we have to rely heavily on empirical description of turbulent shear flows. Classical examples are the mixing-length and eddy-viscosity approximations for boundary-layer like turbulent shear flows. With the progress of high-speed digital computors and better understanding of the "physics" of turbulent flow, more sophisticated models (with accompanying complexity in computation) has been proposed in the last few years. These models, nevertheless, do not show clear advantages over simpler models when applied to boundary-layer flows. In the present investigation, therefore, we adopted the eddy viscosity model to relate the turbulent shear stress and heat transfer to the mean flow gradients. Specifically, we use the eddy viscosity, μ_R, used by Smith et al in their finite-difference computation of compressible turbulent boundary layer equations.

$$\mu_R = \begin{cases} \rho(\kappa y E)^2 \left|\frac{du}{dy}\right| & 0 \leq y \leq y_1 \\ K \rho u_e \delta^* \Gamma(y) & y_1 \leq y \end{cases} \tag{3.1}$$

$$E \equiv 1 - \exp(-A \rho u_\tau y/\mu)$$

$$u_\tau \equiv (\tau_w/\rho_w)^{\frac{1}{2}}$$

$$\kappa = 0.40$$

$$K = 0.016$$

$$A = 0.0583$$

$\Gamma(y)$ = turbulence intermittency factor

Following Benjamin[3] we adopt an orthogonal curvilinear coordinate system which conforms with wavy surfaces; namely, we choose as independent variables ξ, η and ζ defined by

$$\begin{cases} x = [\xi + a\, e^{-\alpha\eta}\, \mathrm{Re}(i\, e^{i\alpha\xi})]\sin\beta - \zeta\cos\beta \\ y = \eta + a\, e^{-\alpha\eta}\, \mathrm{Re}(e^{i\alpha\xi}) \\ z = [\xi + a\, e^{-\alpha\eta}\, \mathrm{Re}(i\, e^{i\alpha\xi})]\cos\beta + \zeta\sin\beta \end{cases} \tag{3.2}$$

Here x, y and z are cartesian coordinates, x taken in the direction of undisturbed flow, y in the direction perpendicular to the undisturbed plane wall, z in the undisturbed plane wall perpendicular to x and y. For $\eta = 0$ we obtain

$$\begin{cases} x = [\xi + a\, \mathrm{Re}(ie^{i\alpha\xi})]\sin\beta - \zeta\sin\beta \\ y = a\, \mathrm{Re}(e^{i\alpha\xi}) \\ z = [\xi + a\, \mathrm{Re}(ie^{i\alpha\xi})]\cos\beta + \zeta\sin\beta \end{cases} \tag{3.3}$$

which represents, for small values of a, a sinusoidal wave pattern with wave crests making an angle β with the x-axis.

The elements of length at ξ, η, ζ in the directions of increasing ξ, η, ζ are $h\,d\xi$, $h\,d\eta$, $d\zeta$ with

$$h = 1 - a\,\alpha\, e^{-\alpha\eta}\, \mathrm{Re}(e^{i\alpha\xi}) \tag{3.4}$$

If ρ is the density, p the pressure, T the temperature, v_1, v_2, v_3 the velocity components in the direction of increasing ξ, η, ζ respectively, and σ_{ij} the components of the stress tensor, then

$$\frac{\partial}{\partial\xi}(h\rho v_1) + \frac{\partial}{\partial\eta}(h\rho v_2) = 0 \tag{3.5a}$$

$$\rho\left(\frac{v_1}{h}\frac{\partial v_1}{\partial \xi} + \frac{v_2}{h}\frac{\partial v_2}{\partial \eta} + \frac{v_1 v_2}{h^2}\frac{\partial h}{\partial \eta} - \frac{v_2^2}{h^2}\frac{\partial h}{\partial \xi}\right) + \frac{1}{h}\frac{\partial p}{\partial \xi}$$

$$= \frac{1}{h^2}\left[\frac{\partial}{\partial \xi}(\sigma_{11}) + \frac{\partial}{\partial \eta}(h\,\sigma_{12})\right] + \frac{\sigma_{12}}{h^2}\frac{\partial h}{\partial \eta} - \frac{\sigma_{22}}{h^2}\frac{\partial h}{\partial \xi} \tag{3.5b}$$

$$\rho\left(\frac{v_1}{h}\frac{\partial v_2}{\partial \xi} + \frac{v_2}{h}\frac{\partial v_2}{\partial \eta} + \frac{v_1 v_2}{h^2}\frac{\partial h}{\partial \xi} - \frac{v_1^2}{h^2}\frac{\partial h}{\partial \eta}\right) + \frac{1}{h}\frac{\partial p}{\partial \eta}$$

$$= \frac{1}{h^2}\left[\frac{\partial}{\partial \xi}(h\,\sigma_{12}) + \frac{\partial}{\partial \eta}(h\sigma_{22}) + \sigma_{12}\frac{\partial h}{\partial \xi} - \sigma_{11}\frac{\partial h}{\partial \eta}\right] \tag{3.5c}$$

$$\rho\left(\frac{v_1}{h}\frac{\partial v_2}{\partial \xi} + \frac{v_2}{h}\frac{\partial v_3}{\partial \eta}\right)$$

$$= \frac{1}{h^2}\left[\frac{\partial}{\partial \xi}(h\,\sigma_{13}) + \frac{\partial}{\partial \eta}(h\,\sigma_{23})\right] \tag{3.5d}$$

$$\rho\, c_p\left(\frac{v_1}{h}\frac{\partial T}{\partial \xi} + \frac{v_2}{h}\frac{\partial T}{\partial \eta}\right) - \left(\frac{v_1}{h}\frac{\partial p}{\partial \xi} + \frac{v_2}{h}\frac{\partial p}{\partial \eta}\right)$$

$$= \frac{1}{h^2}\left[\frac{\partial}{\partial \xi}(h\,q_1) + \frac{\partial}{\partial \eta}(h\,q_2)\right] + \Phi \tag{3.5e}$$

$$\Phi = \sum_{i,j} \sigma_{ij}\, e_{ij}$$

q_1, q_2 = components of heat-conduction vector

e_{ij} = components of strain tensor

The stress tensor and heat-conduction vector include the turbulence contributions - Reynolds stress and heat flux - as well as the molecular stress and heat conduction.

In the above equations the ζ-derivatives are omitted by assuming that the variation of flow properties in the direction parallel to the wave crest is negligible. When β is small, this may not be a good approximation.

When the amplitude a of the surface wave is small, we assume that the flow properties in the boundary layer are of the form

$$v = \bar{v} + a\,v'$$

where $\bar{v}$ is the value for a = 0 and av' is a small perturbation. If we substitute these into the governing equations and collect terms multiplied by a, we obtain

$$\frac{\partial v_1'}{\partial \xi} + \frac{\partial v_2'}{\partial \eta} + \bar{v}_1 \frac{\partial h'}{\partial \xi} + \frac{1}{\bar{\zeta}}\left(\bar{v}_1 \frac{\partial \rho'}{\partial \xi} + v_2' \frac{\partial \bar{\rho}}{\partial \eta}\right) = 0 \qquad (3.6a)$$

$$\bar{\zeta}\left(\bar{v}_1 \frac{\partial v_1'}{\partial \xi} + v_2' \frac{\partial \bar{v}_1}{\partial \eta}\right) + \frac{\partial p'}{\partial \xi} = \frac{\partial \sigma_{12}'}{\partial \eta} \qquad (3.6b)$$

$$\bar{\rho}\left(\bar{v}_1 \frac{\partial v_2'}{\partial \xi} - \bar{v}_1^2 \frac{\partial h'}{\partial \eta}\right) + \frac{\partial p'}{\partial \eta} = 0 \qquad (3.6c)$$

$$\bar{\rho}\left(\bar{v}_1 \frac{\partial v_3'}{\partial \xi} + v_2' \frac{\partial \bar{v}_3}{\partial \eta}\right) = \frac{\partial \sigma_{23}'}{\partial \eta} \qquad (3.6d)$$

$$\bar{\rho}\, C_p\left(\bar{v}_1 \frac{\partial T'}{\partial \xi} + v_2' \frac{\partial \bar{T}}{\partial \eta}\right) - \bar{v}_1 \frac{\partial p'}{\partial \xi} = \frac{\partial q_2'}{\partial \eta} + \Phi' + 2h'\,\bar{\Phi} \qquad (3.6e)$$

Here terms multiplied by $\partial\bar{v}_1/\partial\xi$, $\partial\bar{v}_2/\partial\eta$ or $\partial\bar{T}/\partial\xi$ are neglected, since we regard partial derivatives of the primed variables as quantities of the order $1/\delta$, where δ is the boundary layer thickness, whereas the former quantities are considered as $O(1)$. Also, of the viscous terms, only those which become significant in a thin sublayer next to the wall are retained, since in the outer part of the boundary layer the viscous terms are inversely proportional to the Reynolds number based on the boundary-layer thickness compared to the convective terms.

Outside the boundary layer the viscous the turbulent stresses are negligible, and the undisturbed flow is uniform flow, $\bar{v}_1 = u_\infty \sin\beta$, $\bar{\rho} = \rho_\infty$, $\bar{T} = T_\infty$. When $M_n = M_\infty \sin\beta < 1$, the perturbation flow is subsonic in character and decays exponentially at large distances from the wall. On the other hand, when $M_n > 1$, the perturbation flow is supersonic in character, and then the condition is that the disturbances propagate away from the wall, not toward the wall.

At the surface $\eta = 0$, $v_1' = v_3' = 0$ and also $T' = 0$ since we assume that the ablation temperature is unchanged.

From the mass-conservation at the interface

$$\rho_s \frac{\partial \bar{y}}{\partial t} = -\dot{m}'$$

we obtain

$$\frac{v_2'}{\bar{v}_2} + \frac{p'}{p_\infty} = -\frac{i\beta c\delta}{v_s} \quad \text{at } \eta = 0 \quad .$$

Since

$$\bar{\rho}_w \bar{v}_2 = \rho_s v_s$$

this can be written as

$$\frac{v_2'}{u_\infty} + \frac{\rho_\infty}{\bar{\rho}_w} \lambda_{in} \frac{p'}{\bar{\rho}_\infty} = -\frac{\rho_\infty}{\bar{\rho}_w} \lambda_{in} \frac{i\alpha c\delta}{v_s} \tag{3.7}$$

where

$$\lambda_{in} = \frac{\rho_s v_s}{\rho_\infty u_\infty} \quad .$$

Reduction to Ordinary Differential Equations:

Let

$$\bar{v}_1 = (u_\infty \sin\beta)\, w_1(\eta)$$

$$\bar{v}_3 = -(u_\infty \mathrm{cis}\beta) w_3(\eta)$$

$$v_1' = (u_\infty \sin\beta) Z_1(\eta)\, e^{i\alpha\xi}$$

$$v_2' = i\alpha (u_\infty \sin\beta) Z_2(\eta)\, e^{i\alpha\xi}$$

$$v_3' = -(u_\infty \cos\beta) Z_5(\eta) e^{i\alpha\xi} \tag{3.8}$$

$$p' = p_\infty Z_3(\eta) e^{i\alpha\xi}$$

$$T' = T_\infty Z_4(\eta) e^{i\alpha\xi}$$

$$\sigma_{12}' = (\mu+\mu_R)\frac{\partial v_1'}{\partial\eta} = \frac{1}{\delta}\mu_\infty u_\infty \sin\beta.\, \bar{\mu}_t \frac{dZ_1}{d\eta}$$

$$\Phi' + 2h'\bar{\Phi} = \frac{1}{\delta^2}\mu_\infty u_\infty^2 \sin^2 B.\, 2\bar{\mu}_t \left(\frac{dw_1}{d\eta}\frac{dZ_1}{d\eta} + \frac{dw_3}{d\eta}\frac{dZ_5}{d\eta}\cot^2\beta\right)$$

Here μ_R is the turbulent eddy viscosity and η, ξ are made dimensionless by dividing by the boundary layer thickness δ.

Then the differential equations become

$$\frac{dZ_2}{d\eta} + Z_1 + w_1 Z_3 - \frac{1}{\bar{T}}\left(w_1 Z_4 + \frac{d\bar{T}}{d\eta} Z_2\right) = \alpha\, w_1\, e^{-\alpha\eta} \tag{3.9a}$$

$$\frac{1}{\bar{T}}\left(w_1 Z_1 + \frac{dw_1}{d\eta} Z_2\right) + \frac{1}{\gamma M_n^2} Z_3 = \frac{1}{i\alpha R_n}\frac{d}{d\eta}\left(\bar{\mu}_t \frac{dZ_1}{d\eta}\right) \tag{3.9b}$$

$$-\alpha^2 \frac{w_1}{\bar{T}} Z_2 + \frac{1}{\gamma M_n^2}\frac{dZ_3}{d\eta} = \frac{w_1^2}{\bar{T}}\alpha^2 e^{-\alpha\eta} \tag{3.9c}$$

$$\frac{1}{\bar{T}}\left(w_1 Z_5 + \frac{dw_3}{d\eta} Z_2\right) = \frac{1}{i\alpha R_n}\frac{d}{d\eta}\left(\bar{\mu}_t \frac{dZ_5}{d\eta}\right) \tag{3.9d}$$

$$\frac{1}{\bar{T}}\left(w_1 Z_4 + \frac{d\bar{T}}{d\eta} Z_2\right) - \frac{\gamma-1}{\gamma} Z_3$$
$$= \frac{1}{i\alpha R_n Pr}\frac{d}{d\eta}\left(\bar{k}\frac{dZ_4}{d\eta}\right) + \frac{2(\gamma-1)M_n^2}{i\,\alpha\, R_n}\left(\frac{dw_1}{d\eta}\frac{dZ_1}{d\eta} + \frac{dw_3}{d\eta}\frac{dZ_5}{d\eta}\cot^2\beta\right) \tag{3.9e}$$

This system of five ordinary differential equations is identical to the system which appears in the theory of hydrodynamical stability of compressible laminar boundary layer except for the forcing terms due to the coordinate curvature.

Here

$$M_n = M_\infty \sin\beta$$

$$R_n = \mathrm{Re}\sin\beta = \frac{\rho_\infty u_\infty \delta}{\mu_\infty}\sin\beta \qquad (3.10)$$

and, except for $dz_5/d\eta$ term appearing in the dissipation term of the energy equation, the equations for Z_1 through Z_4 are uncoupled from Z_5 and are identical to the equations for two-dimensional (unyawed) disturbances for Mach number and Reynolds number corresponding to M_n and R_n.

Equations (3.9) are a linear system of order 8, and the general solution can be expressed as a linear combination of a particular solution and eight complementary solutions, i.e.

$$Z_i = \varphi_{pi} + \sum_{n=1}^{8}{}' C_n \varphi_{ni}, \quad i = 1, 2, \ldots, 8.$$

Four of the undetermined constants, C_4, $C_5, \ldots, C_8$, say, are eliminated by the boundary condition at $\eta = \infty$. $C_1, \ldots, C_4$ are chosen to satisfy the boundary condition at $\eta = 0$,

$$Z_1 = 0: \quad \varphi_{p1} + \sum_{n=1}^{4} C_n \varphi_{n1} = 0 \qquad (3.11a)$$

$$Z_4 = 0: \quad \varphi_{p4} + \sum_{n=1}^{4}{}' C_n \varphi_{n4} = 0 \qquad (3.11b)$$

$$Z_5 = 0: \quad \varphi_{p5} + \sum_{n=1}^{4}{}' C_n \varphi_{n5} = 0 \qquad (3.11c)$$

From Eq. (3.7)

$$(\varphi_{p2}+B\varphi_{p3}) + \sum_{n=1}^{4} C_n(\varphi_{n2}+B\varphi_{n3}) = -B\frac{i\alpha c\,\delta}{v_s} \qquad (3.11d)$$

$$B \equiv \frac{\rho_\infty}{\overline{\rho}_w}\,\frac{\lambda_{in}}{\sin\beta}$$

and from Eq. (2.4)

$$\varphi'_{p4} + \sum_{n=1}^{4} C_n\,\varphi'_{n4} = \frac{\overline{q}_a\,\delta}{(1+\lambda_+)k_w\,T_\infty}\left(\frac{i\alpha c\delta}{v_s} + \lambda_L\,\tilde{Q}_s\right) \qquad (3.11e)$$

If we eliminate C_n's from (3.11), we obtain the eigenvalue equation for C;

$$\begin{vmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} & \varphi_{41} & \varphi_{p1} \\ \varphi_{14} & \varphi_{24} & \varphi_{34} & \varphi_{44} & \varphi_{p4} \\ \varphi_{15} & \varphi_{25} & \varphi_{35} & \varphi_{45} & \varphi_{p5} \\ \varphi_{12}+B\varphi_{13} & \varphi_{22}+B\varphi_{23} & \varphi_{32}+B\varphi_{33} & \varphi_{42}+B\varphi_{43} & B\frac{i\alpha c\delta}{v_s} \\ \varphi'_{14} & \varphi'_{24} & \varphi'_{34} & \varphi'_{44} & \Psi \end{vmatrix} = 0 \qquad (3.12)$$

where

$$\Psi = \varphi'_{p4} - \frac{\overline{q}_a\,\delta}{(1+\lambda_L)k_w T_\infty}\left(\frac{i\alpha c\delta}{v_s} + \lambda_L\,\tilde{Q}_s\right)$$

3.1. Small-Perturbation Solution for Incompressible Flow

As stated in the Introduction, an experiment was conducted in a low-speed wind tunnel in order to obtain data on the turbulent boundary-layer properties over wavy surfaces. The purpose of the experiment is to provide a testing ground for various models for turbulent transport properties. In order to test the turbulent eddy viscosity model, which will be employed in compressible flow computation, the linear perturbation equations (3.9) are integrated for $M_\infty = 0$, $\sin\beta = 1$, $\bar{T} = 1$, $Z_4 = Z_5 = 0$. For incompressible flows Z_3 in Eq. (3.9) has to be replaced by $\gamma M_\infty^2 Z_3$, which is equivalent to normalizing the pressure perturbation by $\rho_\infty u_\infty^2$ rather than by p_∞. With these changes the governing equations are

$$\frac{dZ_2}{d\eta} + Z_1 = \alpha w_1 e^{-\alpha\eta} \tag{3.13a}$$

$$w_1 Z_1 + \frac{dw_1}{d\eta} Z_2 + Z_3 = \frac{1}{i\alpha Re} \frac{d}{d\eta}\left(\bar{\mu}_t \frac{dZ_1}{d\eta}\right) \tag{3.13b}$$

$$-\alpha^2 w_1 Z_2 + \frac{dZ_3}{d\eta} = \alpha^2 w_1^2 e^{-\alpha\eta} \tag{3.13c}$$

From (3.1)

$$\bar{\mu}_t = 1 + \begin{cases} Re(\kappa\eta E)^2 \left|\frac{dw_1}{d\eta}\right| & , \quad 0 \le \eta \le \eta \\ Re\, K \Delta^* \Gamma & , \quad \eta_1 \le \eta \end{cases} \tag{3.14}$$

$$\Delta^* \equiv \delta^*/\delta$$

The boundary conditions are

$$\begin{aligned} Z_1 = Z_2 = 0 \quad & \text{at} \quad \eta = 0 \\ Z_1, Z_2, Z_3 \to 0 \quad & \text{as} \quad \eta \to \infty \end{aligned} \tag{3.15}$$

The undisturbed velocity profile $w_1(\eta)$ and eddy viscosity $\bar{\mu}_t$ are computed by a program described in Appendix I.

Outside the boundary layer $w_1 = 1$ and Eq. (3.13) reduces to

$$\frac{dZ_2}{d\eta} + Z_1 = \alpha e^{-\alpha\eta}$$

$$Z_1 + Z_3 = \frac{1}{i\alpha Re} \frac{d^2 Z_1}{d\eta^2}$$

$$\frac{dZ_3}{d\eta} - \alpha^2 Z_2 = \alpha^2 e^{-\alpha\eta}$$

The solutions that vanish at large values of η are given by

$$Z_1 = C_1 m_1^2 e^{m_1\eta} + C_2 m_2^2 e^{m_2\eta}$$

$$Z_2 = -C_1 m_1 e^{m_1\eta} - C_2 m_2 e^{m_2\eta} - e^{-\alpha\eta}$$

$$Z_3 = -\alpha^2 (C_1 e^{m_1\eta} + C_2 e^{m_2\eta})$$

where

$$m_1 = -\alpha/F$$

$$m_2 = -(i\alpha Re)^{\frac{1}{2}} F \tag{3.16}$$

$$F^2 = \tfrac{1}{2}\left[1 + (1 + \frac{i\alpha}{Re})^{\frac{1}{2}}\right]$$

Let $\underset{\sim}{Z}$ be a vector (mathematical, not physical) with components Z_1, Z_2, Z_3; $\underset{\sim}{\varphi}_n$ be a vector with components $\varphi_{n1}, \varphi_{n2}, \varphi_{n3}$, and then the above solution can be written in the form

$$\underset{\sim}{Z} = C_1 \underset{\sim}{\varphi}_1 + C_2 \underset{\sim}{\varphi}_2 + \underset{\sim}{\varphi}_p \tag{3.17}$$

where

$$\underset{\sim}{\varphi}_1 = \begin{bmatrix} m_1^2 \\ -m_1 \\ -\alpha^2 \end{bmatrix} e^{m_1\eta}, \quad \underset{\sim}{\varphi}_2 = \begin{bmatrix} m_2^2 \\ -m_2 \\ -\alpha^2 \end{bmatrix} e^{m_2\eta}, \quad \underset{\sim}{\varphi}_p = \begin{bmatrix} 0 \\ -e^{-\alpha\eta} \\ 0 \end{bmatrix} \tag{3.18}$$

$\underset{\sim}{\varphi}_n$ has a property

$$\frac{d\underset{\sim}{\varphi}_n}{d\eta} = +m_n \underset{\sim}{\varphi}_n \tag{3.19}$$

$$(\underset{\sim}{\varphi}_3 \equiv \underset{\sim}{\varphi}_p, \; m_3 = -\alpha)$$

Using (3.18) and (3.19) at a suitably large value of η as initial conditions, we integrate Eq. (3.13) toward $\eta = 0$ twice without forcing terms to obtain $\underset{\sim}{\varphi}_1(\eta)$

and $\varphi_2(\eta)$ and once with forcing terms to generate $\varphi_p(\eta)$. The constants C_1 and C_2 in Eq. (3.17) are then determined by satisfying the boundary condition $Z_1(0) = Z_2(0) = 0$.

One complication in carrying out the numerical computation indicated above is that, since $|m_2|$ is very large compared to $|m_1|$ or α, numerical solutions for φ_1 and φ_p will be very quickly contaminated by φ_2. Then C_1 and C_2 as determined by the condition at $\eta = 0$ will contain significant errors, or it becomes impossible to solve for C_1 and C_2. In order to reduce the magnitude of the contamination, the parallel-shooting method (Ref. 4) was programmed for numerical solution of present boundary-value problem. Also a solution based on the matched-asymptotic-expansion method was worked out for the purpose of comparison with the numerical solution (Appendix II).

The results from the numerical solution are compared with the experimental results in Fig. 2-7. Even though there are some significant disagreement in some regions, especially for shear-stress distributions, a fair agreement is obtained in overall distribution, and it was concluded that the eddy-viscosity model for turbulent shear stress could be used in the cross-hatch problem without incurring too great an error.

4. Stability Calculation for Teflon Ablation

A stability calculation was carried out for ablating teflon surface. The flow condition at the boundary-layer edge was:

$$M_\infty = 2.5$$

$$Re_\delta = 1.17 \times 10^5$$

$$T_o = 4700^\circ R$$

$$T_w = 1210^\circ R$$

The undisturbed flat-plate boundary layer for this condition was computed from the program described in Appendix III.

In order to facilitate the computation, an approximation was made in the stability-problem formulation. In Eq. (3.7) the pressure and ablation terms are multiplied by the injection parameter λ_{in}. In many cases λ_{in} is of the same order of magnitude as the skin friction coefficient, which is very small compared with one. Then Eq. (3.7) reduces to

$$\frac{v_2'}{u_\infty} = 0$$

With this approximation in addition to the approximation $c \ll u_\infty$ already made previously, the boundary-layer computation can be decoupled from the ablation perturbation; namely, Eq. (3.11d) reduces to

$$\varphi_{p2} + \sum_{n=1}^{4} C_n \varphi_{n2} = 0 ,$$

and this equation together with (3.11 a, b, c) determine C_n's independent of c.

Once the boundary-layer solution is determined as a function of α, the amplification rate may be determined from Eq. (2.4).

Another ad hoc approximation introduced to simplify the computation is that the $dZ_5/d\eta$-term in Eq. (3.9e) is neglected in order to decouple the energy equation from Z_5. With this approximation, Z_1 through Z_4 can be computed without Z_5, reducing the order of the system by two. Since Z_5 is induced through $Z_2\, dw_3/d\eta$, it is reasonable to expect that Z_5 is much less than Z_1.

For the thermal properties of Teflon, we have used the following values:

$$\rho_s = 130 \text{ lb/ft}^3$$

$$k_s = 3.4\text{x}10^{-5} \frac{\text{Btu}}{\text{ft } {}^\circ\text{R Sec}}$$

$$C_s = 0.25 \frac{\text{Btu}}{\text{lb } {}^\circ\text{R}}$$

$$L_E = 750 \text{ Btu/lb}$$

$$T_{ab} = 800^\circ F$$

The results of the computation are summarized in Fig. 8, which shows the instability domain in the $\alpha\beta$-plane.

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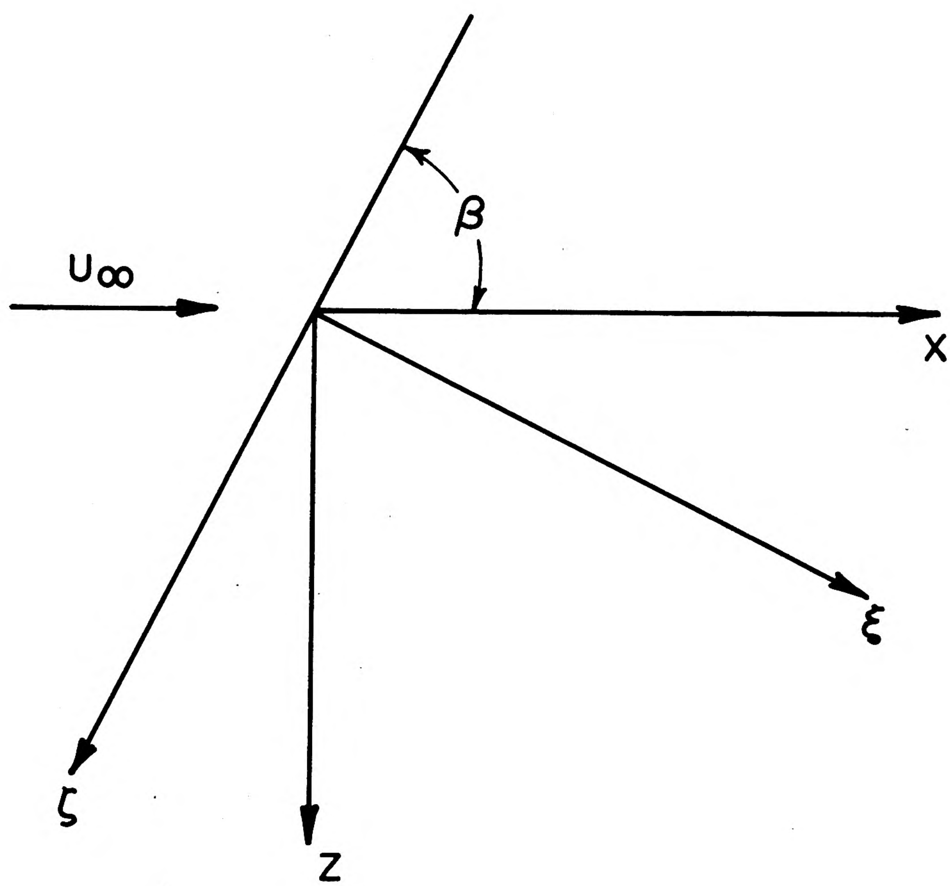

FIG. 1 COORDINATES ON SURFACE

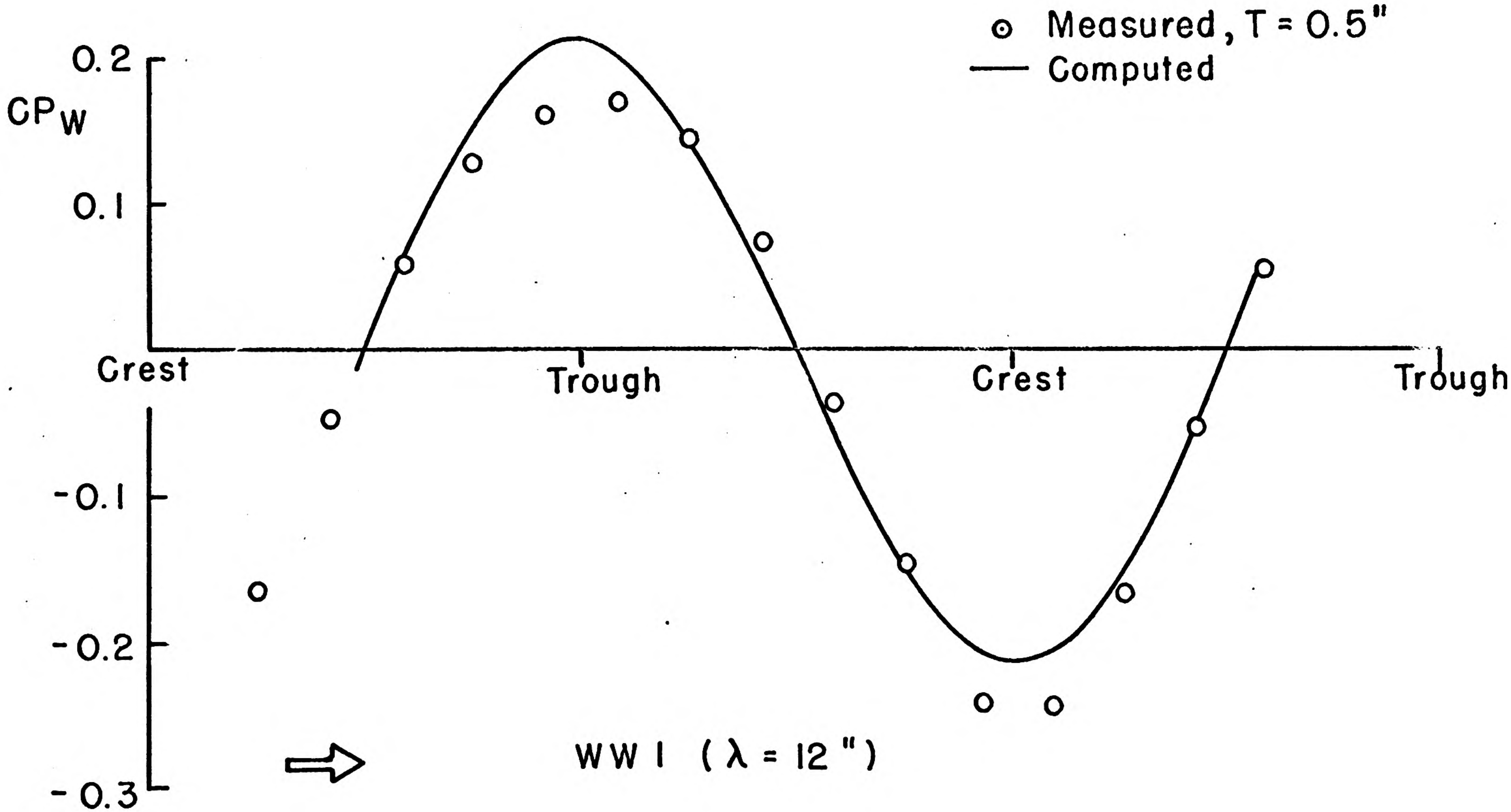

FIG. 2 COMPARISON BETWEEN COMPUTED AND MEASURED WALL PRESSURE DISTRIBUTION

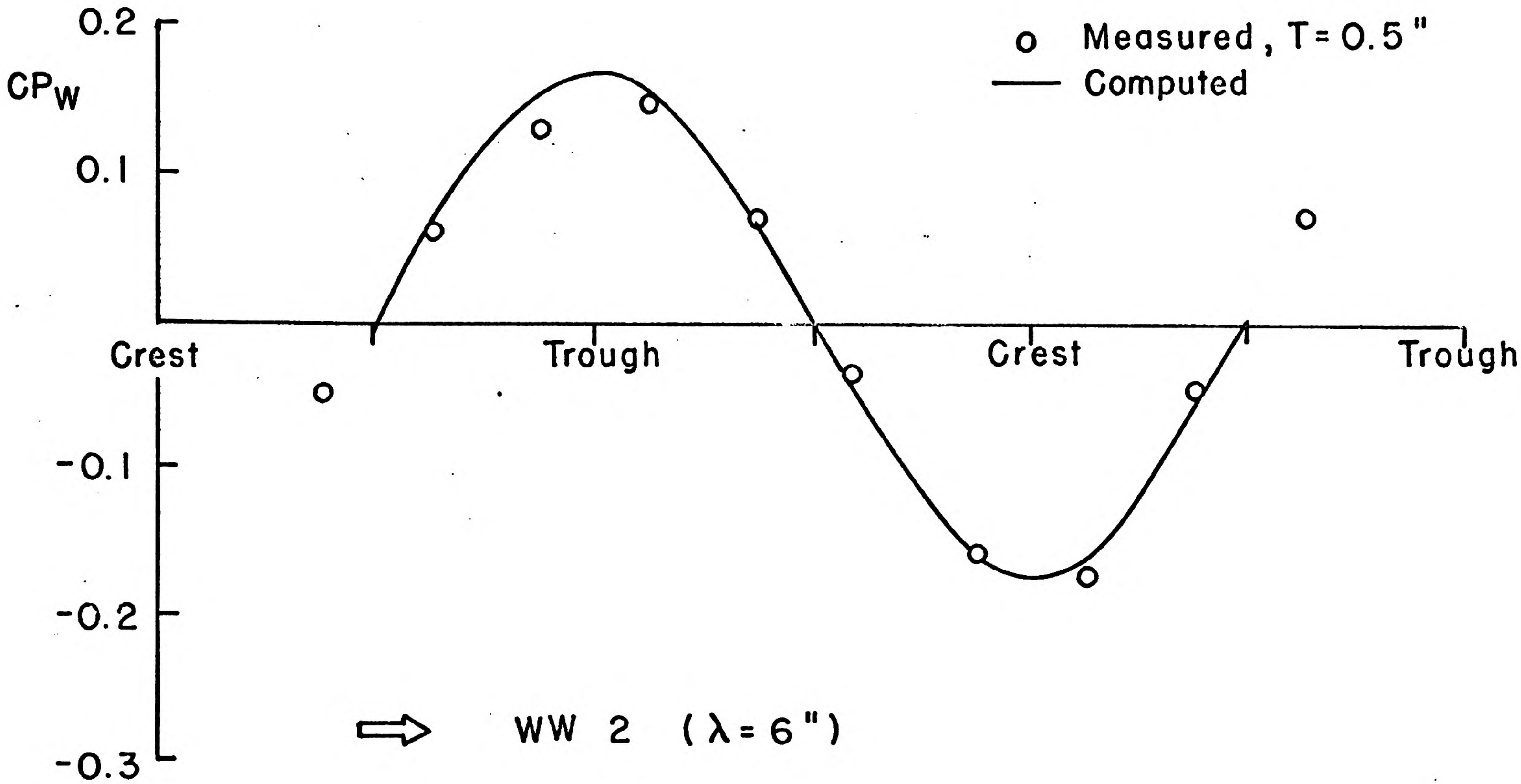

FIG. 3 COMPARISON BETWEEN COMPUTED AND MEASURED WALL PRESSURE DISTRIBUTIONS

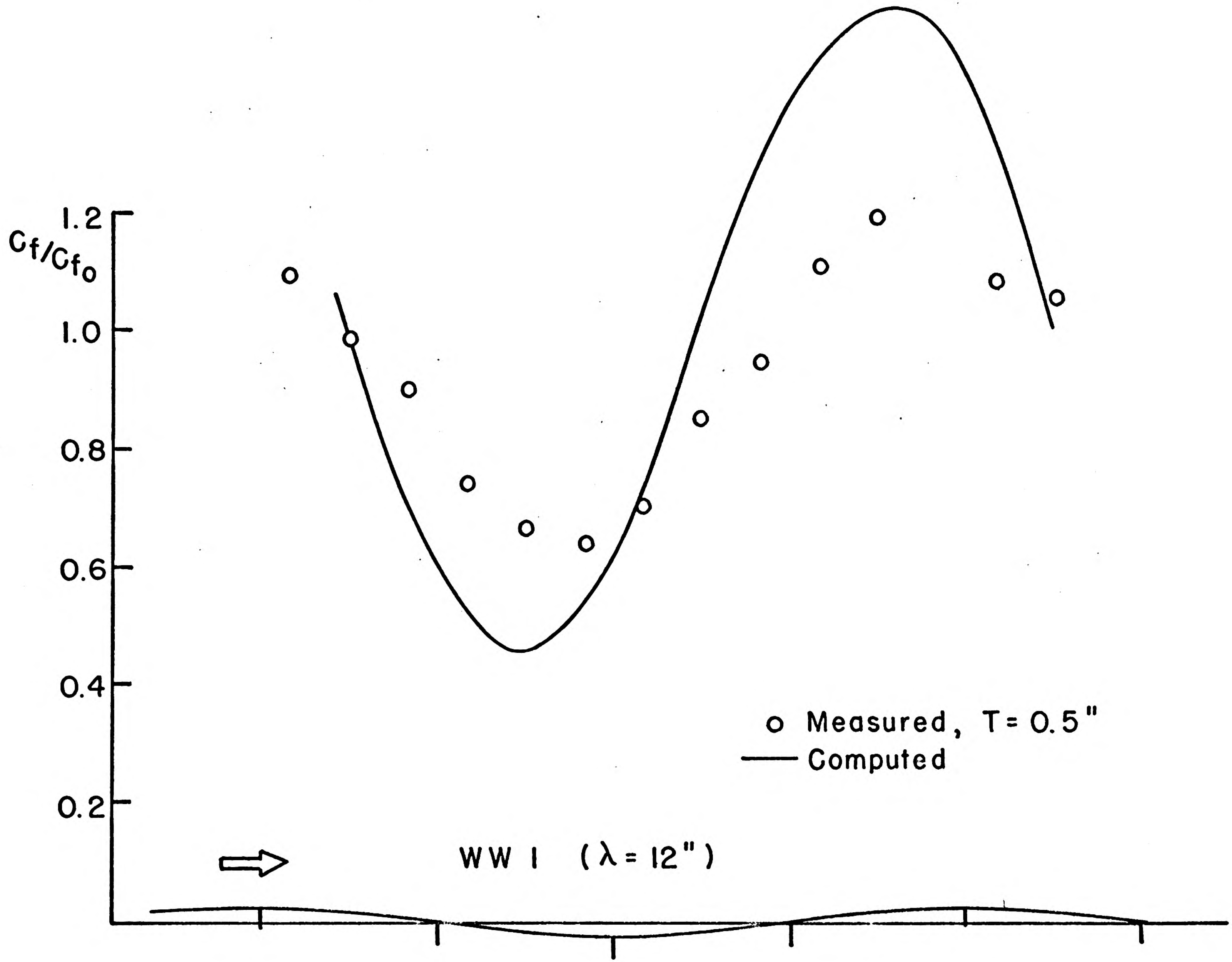

FIG. 4 COMPARISON BETWEEN COMPUTED AND MEASURED WALL SHEAR DISTRIBUTION

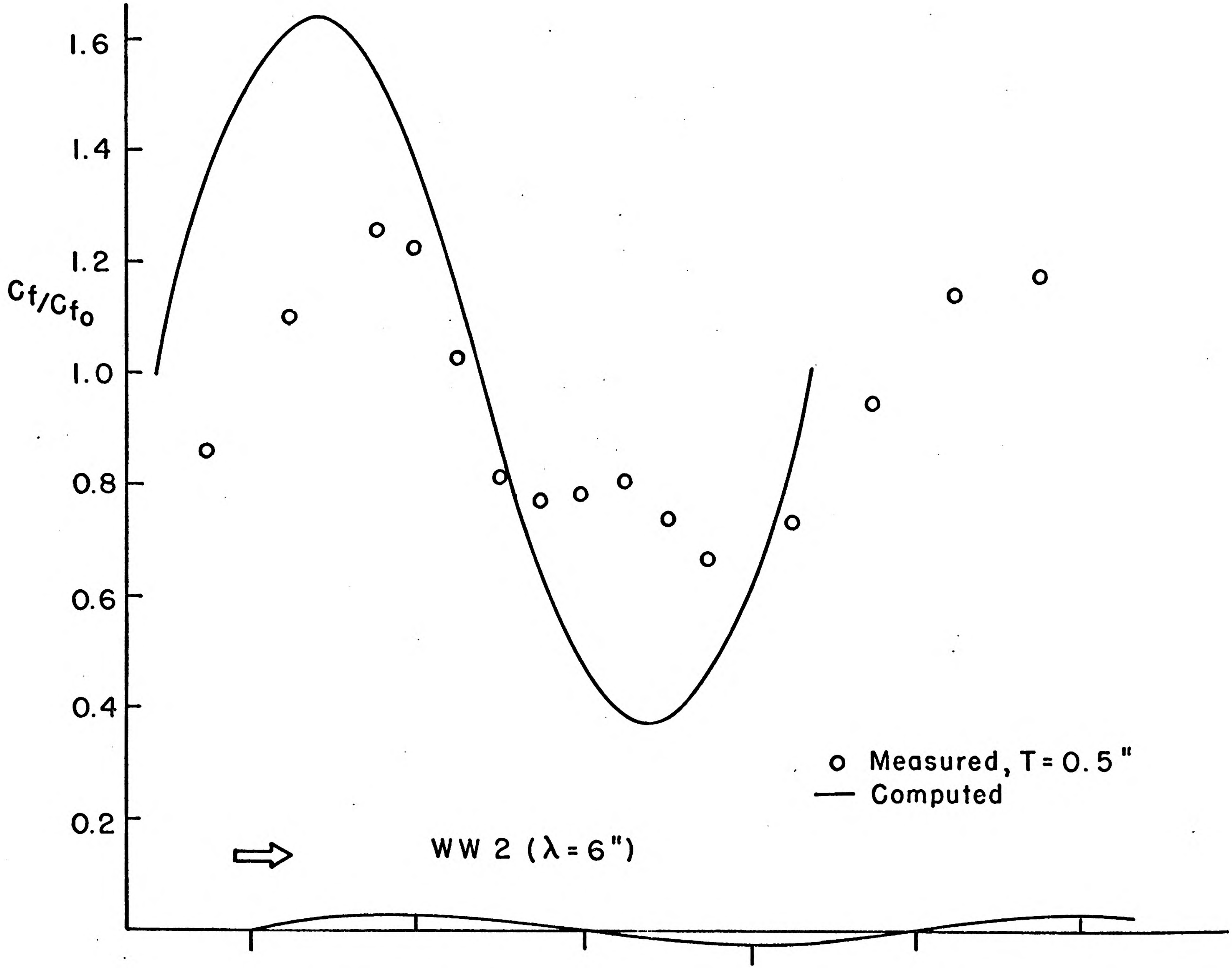

FIG. 5 COMPARISON BETWEEN COMPUTED AND MEASURED WALL SHEAR DISTRIBUTION

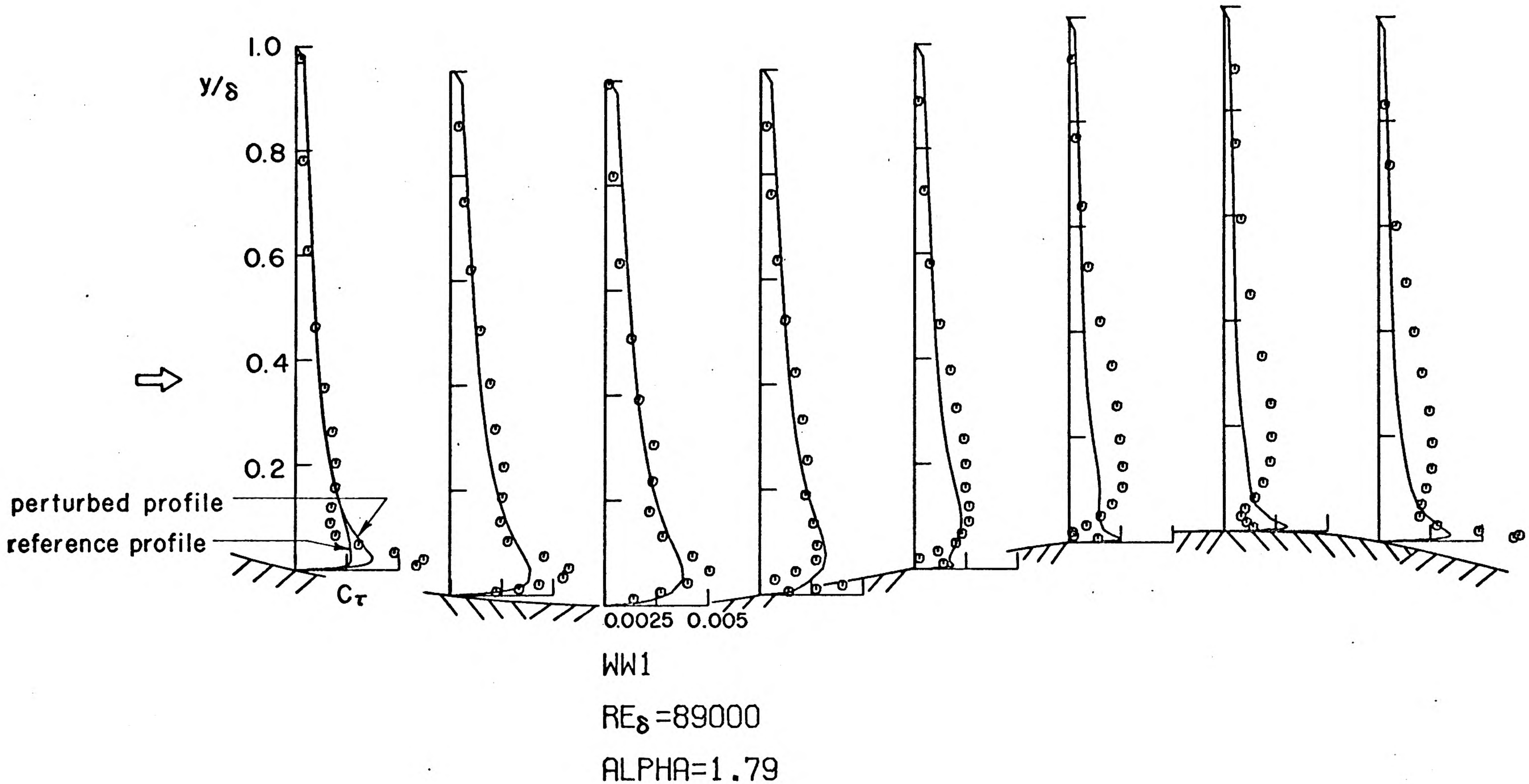

FIG. 6 COMPARISON BETWEEN COMPUTED AND MEASURED REYNOLDS' SHEAR STRESS

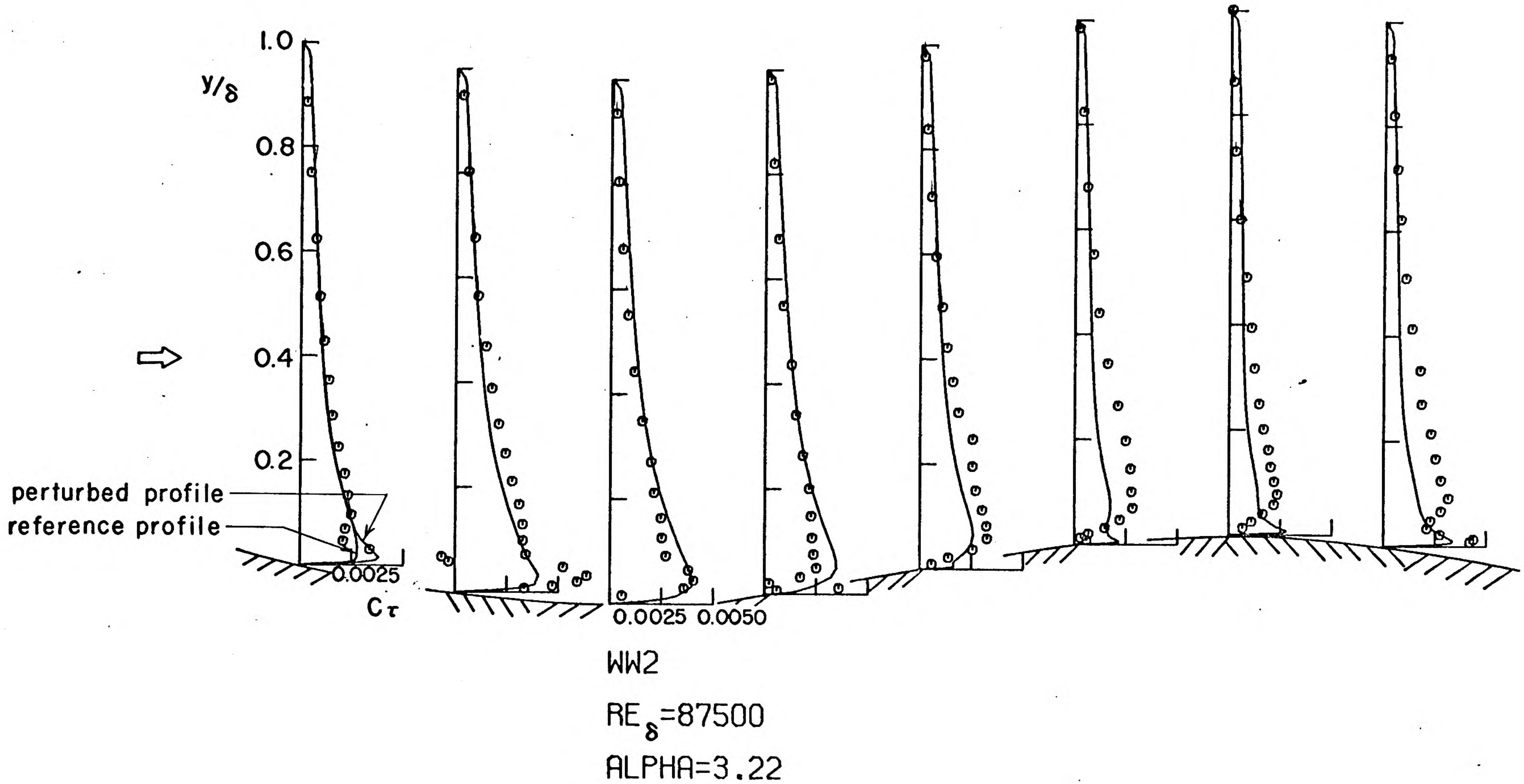

FIG. 7 COMPARISON BETWEEN COMPUTED AND MEASURED REYNOLDS' SHEAR STRESS

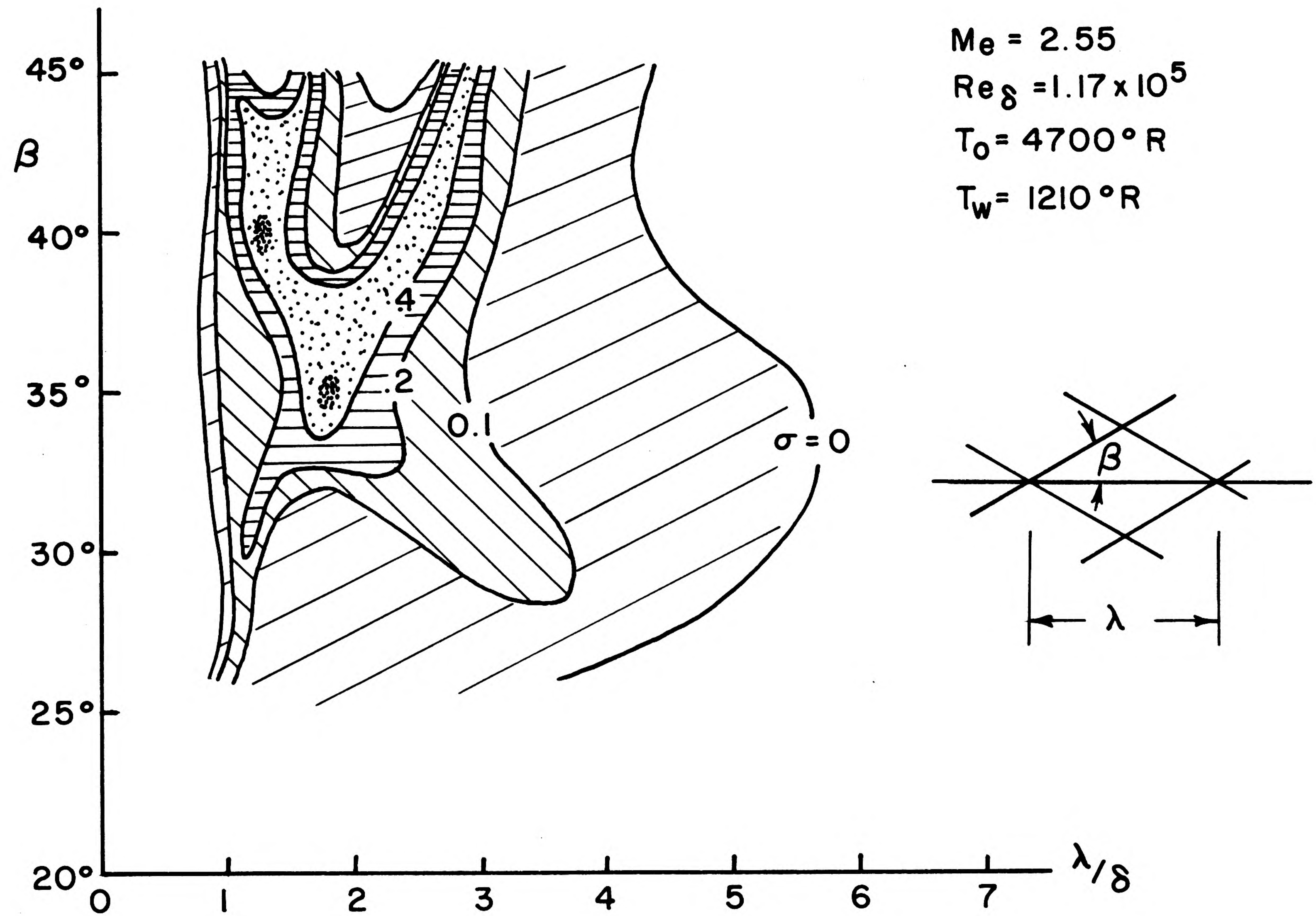

FIG. 8 INSTABILITY DOMAIN - TEFLON ABLATION

Appendix I

Constant-Pressure Incompressible Turbulent Boundary Layer

For undisturbed velocity profile in a flat-plate turbulent boundary layer we use Coles' empirical velocity distribution

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\left\{F\left(\frac{u_\tau y}{\nu}\right) + \tilde{\Pi} . w\left(\frac{y}{\delta}\right)\right\}$$

$$u_\tau = (\tau_w/\rho)^{\frac{1}{2}}$$

$$\kappa = 0.40$$

$$\Pi = 0.55 \text{ for constant-pressure layer .}$$

For large values of its argument the law-of-the-wall function $\frac{1}{\kappa}F(\zeta)$ behaves as follows:

$$\frac{1}{\kappa}F(\zeta) \simeq \frac{1}{\kappa}\log(\zeta) + 5.1$$

as $\zeta \to 0$ in the viscous sublayer

$$\frac{1}{\kappa}F(\zeta) \to \zeta .$$

An analytical approximation for this function was obtained to fit experimental data by Reichardt and by Laufer in the following form:

$$\frac{1}{\kappa}F(\zeta) = 2.5\log(1+0.4\zeta) + 7.39(1-e^{-0.14\zeta})^2$$

The wake function w is approximated by

$$w(\eta) = 1 - \cos(\pi\eta)$$

The eddy-viscosity distribution is given by the following expression.

$$\frac{\mu_R}{\mu} = K\frac{u_\infty\delta^*}{\nu}\left[1-\exp\left(-0.0583\frac{u_\tau y}{\nu}\right)\right]^2 \tanh\frac{\kappa u_\tau y}{K u_\infty \delta^*}$$

with

$$\kappa = 0.40$$

$$K = 0.016$$

APPENDIX II

TURBULENT BOUNDARY LAYER OVER WAVY SURFACE

1. Small Perturbation Equations

The governing differential equations for the incompressible turbulent shear layers are

$$\text{div } \underline{u} = 0$$

$$(\underline{u} \cdot \nabla)\underline{u} = -\frac{1}{\rho} \text{ grad } p + \frac{1}{\rho} \text{ div } \underline{\underline{\tau}}$$

In dealing with the boundary layer over wavy wall of small amplitude, it is convenient to introduce the following orthogonal vurvilinear coordinates (Benjamin, 1959)

$$x = \xi - a e^{-\alpha\eta} \sin \alpha\xi$$
$$y = \eta + a e^{-\alpha\eta} \cos \alpha\xi \qquad \text{(II. 1. 1)}$$

For $a \ll 1$, $y = a \cos\alpha x + O(a^2)$ at $\eta = 0$.

If $a = 0$, we obtain the boundary layer over a flat plate, in which the velocity distribution is $U(\xi, \eta)$. For small nonzero a, we assume all perturbations are of the order of a, for example

$$u = U + a u' \qquad \text{(II. 1. 2)}$$

Furthermore, if we consider the wave lengths comparable to the boundary layer thickness, we may assume that the undisturbed flow is parallel, i. e. $U = U(\eta)$ and $V = 0$ in the first approximation.

Then we obtain the following linearized equations for perturbations, by letting

$$u' = \tilde{u}(\eta)e^{i\alpha\xi}, \text{ etc.}$$

$$i\alpha\tilde{u} + \frac{d\tilde{v}}{d\eta} = i\,\alpha^2\, U\, e^{-\alpha\eta}$$

$$i\alpha U\tilde{u} + \frac{dU}{d\eta}\tilde{v} + i\alpha\tilde{p} = \frac{1}{R}\left\{\frac{d}{d\eta}\left[\bar{\mu}\left(\frac{d\tilde{u}}{d\eta} + i\alpha\tilde{v}\right) + \alpha\bar{\mu}\left(\frac{dU}{d\eta} - \alpha U\right)e^{-\alpha\eta} + \tilde{\mu}\frac{dU}{d\eta}\right]\right.$$

$$\left. - 2\bar{\mu}\alpha^2\tilde{u} + 2\alpha^2\bar{\mu}\frac{dU}{d\eta}e^{-\alpha\eta}\frac{d}{d\eta}\left(\bar{\mu}\frac{dU}{d\eta}\right)\right\}$$

$$U(i\alpha v^2 - \alpha^2 U_e^{-\alpha\eta}) + \frac{d\tilde{p}}{d\eta} = \frac{1}{R}\left\{i\alpha\bar{\mu}\left[\frac{d\tilde{u}}{d\eta} + i\alpha\tilde{v} + \alpha\frac{d}{d\eta}(U_e^{-\alpha\eta})\right] + i\alpha\tilde{\mu}\frac{dU}{d\eta} + \ldots\right\}$$

Here variables are made dimensionless by dividing the lengths by undisturbed boundary-layer thickness, the velocities by u_∞, the pressure by ρu_∞^2, and the viscosity coefficient by μ_∞. The viscosity coefficient includes the turbulent eddy viscosity as well as the molecular viscosity, and it is assumed to be a scalar quantity as in the laminar flows. This may not be a valid assumption in turbulent shear flow, but it is used here because of the lack of knowledge and for the sake of simplicity. A further justification for this assumption is that the viscous stresses are significant only in a thin layer near the wall where the usual boundary layer approximation is again valid except for extremely large α. In view of the last remark, the momentum equations are simplified further, and we are going to use the following reduced equations in our analysis:

$$\begin{aligned} i\alpha\tilde{u} + \frac{d\tilde{v}}{d\eta} &= i\alpha^2 U e^{-\alpha\eta} \\ i\alpha(U\tilde{u}+\tilde{p}) + \frac{dU}{d\eta}\tilde{v} &= \frac{1}{R}\frac{d}{d\eta}\left(\bar{\mu}\frac{d\tilde{u}}{d\eta}\right) \qquad \text{(II. 1. 3)} \\ i\alpha U\tilde{v} + \frac{d\tilde{p}}{d\eta} &= \alpha^2 U^2 e^{-\alpha\eta} \end{aligned}$$

2. Mean Flow

Following Coles,( ) we assume that the mean flow profile has the following representation

$$U = \frac{\epsilon}{\varkappa}\left[F(\zeta) + \tilde{\pi}\, w(\eta)\right] \qquad \text{(II. 2. 1)}$$

where

$$\epsilon = u_\tau/u_\infty = \sqrt{C_f/2} \qquad \text{(II. 2. 2)}$$

$$\varkappa = 0.4$$

$$\zeta = R\epsilon\eta \qquad \text{(II. 2. 3)}$$

Here $F(\zeta)$ is the law-of-the-wall profile, and

$$F(0) = 0; \text{ and } F \sim \log \zeta + B, \; B = 2.04 \text{ for large } \zeta$$

A good approximation for $w(\eta)$ is

$$w(\eta) = 1 - \cos \pi\eta$$

At $\eta = 1$, then, (II. 2. 1) becomes

$$\frac{\epsilon}{\varkappa}(\log R\epsilon + B + 2\tilde{\pi}) = 1, \quad B + 2\tilde{\pi} = 3.16 \qquad \text{(II. 2. 4)}$$

which is the skin friction law for flat plates. From (II. 2. 1) and (II. 2. 4), we can write the velocity distribution in the defect form

$$U = 1 - \frac{\epsilon}{\varkappa}\overline{W}(\eta; R\epsilon) \qquad \text{(II. 2. 5)}$$

where

$$\overline{W}(\eta; R\epsilon) = -F(\zeta) + \log R\epsilon + B + \tilde{\pi}[2 - w(\eta)] \qquad \text{(II. 2. 5a)}$$

As $R \to \infty$, we obtain $\epsilon \to 0$, but the rate at which $\epsilon \to 0$ is so small that $R\epsilon \to \infty$. Then in the outer region where $\eta = O(1)$

$$\overline{W} \sim W(\eta) = -\log \eta + \tilde{\pi}(2 - w)$$

W is the velocity-defect profile. On the other hand in the inner region where $\zeta = O(1)$, (II. 2. 1) reduces to

$$U = \frac{\epsilon}{\varkappa} F(\zeta) \qquad \text{(II. 2. 6)}$$

For the turbulent eddy viscosity we assume

$$\overline{\mu}_e \sim [K R \delta^*/\delta] C(\eta) \qquad \text{(II. 2. 7)}$$

in the outer region, and

$$\overline{\mu}_e \sim \varkappa\zeta(1 - e^{-\alpha^*\zeta})^2$$

in the inner region. Here K and α^* are constants and

$$K = 0.0135$$

$$\alpha^* = 0.0583$$

The function $C(\eta)$ must be linear in η as $\eta \to 0$ in order to match with the inner region viscosity and taken to be

$$C(\eta) = \begin{cases} 1 & \text{for } \eta > K\,\Delta^*/\varkappa \\ \varkappa\eta/K\Delta^* & \text{for } \eta > K\,\Delta^*/\varkappa \end{cases} \tag{II. 2. 8}$$

where

$$\Delta^* = \delta^*/\epsilon\delta$$

From the assumed mean flow profile

$$\Delta^* = (1 + \widetilde{\pi})/\varkappa = 3.9$$

and $K\Delta^*/\varkappa = 0.132$

3. <u>Outer Expansion</u>

Using the defect form (II. 2. 5) for U, we rewrite (II. 1. 3) as

$$\begin{aligned} i\,\alpha\,\widetilde{u} + \frac{d\widetilde{v}}{d\eta} &= i\,\alpha^2\, e^{-\alpha\eta} - \frac{\epsilon}{\varkappa} i\,\alpha^2\, \overline{W}\, e^{-\alpha\eta} \\ i\,\alpha(\widetilde{u} + \widetilde{p}) &= \frac{\epsilon}{\varkappa}\left[i\,\alpha\,\overline{W}\,\widetilde{u} + \frac{d\overline{W}}{d\eta}\,\widetilde{v} + \varkappa\, K\,\Delta^*\, \frac{d}{d\eta}\left(C\,\frac{d\widetilde{u}}{d\eta}\right)\right] \\ \frac{d\widetilde{p}}{d\eta} + i\,\alpha\,\widetilde{v} &= \alpha^2\, e^{-\alpha\eta} + \frac{\epsilon}{\varkappa}(i\,\alpha\,\widetilde{v} - 2\alpha^2\, e^{-\alpha\eta})\overline{W} \\ &\quad + \frac{\epsilon^2}{\varkappa^2}\,\alpha^2\,\overline{W}^2\, e^{-\alpha\eta} \end{aligned} \tag{II. 3. 1}$$

In the outer layer where $\eta = O(1)$ as $\epsilon \to 0$, let

$$\begin{aligned} \widetilde{u}(\eta;\epsilon) &\sim \widetilde{u}_o(\eta) + \frac{\epsilon}{\varkappa}\widetilde{u}_1(\eta) + \ldots \\ \widetilde{v}(\eta;\epsilon) &\sim i\,\alpha[\widetilde{f}_o(\eta) + \frac{\epsilon}{\varkappa}\widetilde{f}_1(\eta) + \ldots] \\ \widetilde{p}(\eta;\epsilon) &\sim \widetilde{p}_o(\eta) + \frac{\epsilon}{\varkappa}\,\widetilde{p}_1(\eta) + \ldots \end{aligned} \tag{II. 3. 2}$$

then

$$\tilde{u}_o + \frac{d\tilde{f}_o}{d\eta} = \alpha\, e^{-\alpha\eta}$$

$$\tilde{u}_o + \tilde{p}_o = 0 \qquad \text{(II. 3. 3)}$$

$$\frac{d\tilde{p}_o}{d\eta} - \alpha^2 \tilde{f}_o = \alpha^2\, e^{-\alpha\eta} \qquad \text{(II. 3. 4)}$$

The solution of (II. 3. 3) which vanishes as $\eta \to \infty$ is

$$\tilde{u}_o = -\tilde{p}_o = (1 + A_o)\alpha\, e^{-\alpha\eta}$$

$$\tilde{f}_o = A_o\, e^{-\alpha\eta} \qquad \text{(II. 3. 5)}$$

Substituting (II. 3. 5) for $\tilde{u}_o$, $\tilde{p}_o$ and $\tilde{f}_o$ in (II. 3. 4), we obtain

$$\frac{d\tilde{f}_1}{d\eta} + \tilde{u}_1 = -\alpha\, W\, e^{-\alpha\eta}$$

$$u_1 + p_1 = [(1+A_o)\alpha W + A_o \frac{dW}{d\eta}]e^{-\alpha\eta} - (1+A_o)\varkappa K\Delta^* i\alpha \frac{d}{d\eta}(Ce^{-\alpha\eta}) \qquad \text{(II. 3. 6)}$$

$$\frac{d\tilde{p}_1}{d\eta} - \alpha^2 \tilde{f}_1 = -(2-A_o)\alpha^2\, W\, e^{-\alpha\eta}$$

The solution of (II. 3. 6) which vanishes as $\eta \to \infty$ is given by

$$\tilde{u}_1 = \alpha\Big\{A_1 e^{-\alpha\eta} + (1+A_o)W\, e^{-\alpha\eta} + A_o \frac{dW}{d\eta}\, e^{-\alpha\eta} - 2(1+A_o)de^{+d\eta}\int_\eta^\infty We^{-2\alpha\eta}\, d\eta_1$$

$$- i(1+A_o)\varkappa\, K\, \Delta^*\Big[\frac{d}{d\eta}(Ce^{-\alpha\eta}) + \tfrac{1}{2}\alpha^2\, e^{\alpha\eta}\int_\eta^\infty Ce^{-2\alpha\eta_1}\, d\eta_1 - \tfrac{1}{2}\alpha^2 e^{-\alpha\eta}\int_o^\eta C\, d\eta_1\Big]\Big\}$$

$$\text{(II. 3. 7a)}$$

$$\tilde{f}_1 = A_1\, e^{-\alpha\eta} - A_o\, We^{-\alpha\eta} + (1+A_o)\Big\{2\,\alpha\, e^{\alpha\eta}\int_\eta^\infty W\, e^{-2d\eta_1}\, d\eta_1$$

$$+ \tfrac{1}{2}\varkappa\, K\, \Delta^*\, i\alpha\, (-2\, Ce^{-\alpha\eta} + \alpha\, e^{\alpha\eta}\int_\eta^\infty C\, e^{-2\alpha\eta_1}\, d\eta_1 + \alpha e^{-\alpha\eta}\int_o^\eta C\, d\eta_1)\Big\}$$

$$\text{(II. 3. 7b)}$$

$$\tilde{p}_1 = \alpha\Big\{-A_1\, e^{\alpha\eta} + 2\alpha(1+A_o)e^{\alpha\eta}\int_\eta^\infty W\, e^{-2\alpha\eta_1}\, d\eta_1$$

$$+ \tfrac{1}{2}(1+A_o)\varkappa\, K\Delta^*\, i\,\alpha^2\, (e^{\alpha\eta}\int_\eta^\infty C\, e^{-2\alpha\eta_1}\, d\eta_1 - e^{-\alpha\eta}\int_o^{\eta_1} C\, d\eta_1)\Big\} \qquad \text{(II. 3. 7c)}$$

Thus to O(ϵ) the outer solution becomes as $\eta \to 0$

$$\tilde{u} \sim \alpha(1+A_o)(1-\alpha\eta + \ldots) + \frac{\epsilon}{\kappa}\alpha\left\{A_1+(1+A_o)(-\log\eta+2\tilde{\pi})-A_o\frac{1}{\eta} - 2(1+A_o)\alpha\beta_1 - i(1+A_o)\kappa K\Delta^*\left(\frac{\kappa}{K\Delta^*} + \frac{1}{2}\alpha^2\beta_2\right)\right\} \tag{II. 3. 8a}$$

$$\tilde{v} \sim i\alpha\left\{A_o(1-\alpha\eta + \ldots) + \frac{\epsilon}{\kappa}[A_1-A_o(-\log\eta+2\tilde{\pi}) + 2(1+A_o)\alpha\beta_1 + \tfrac{1}{2}(1+A_o)\kappa K\Delta^* i\alpha^2\beta_2] + \ldots\right\} \tag{II. 3. 8b}$$

$$\tilde{p} \sim -\alpha(1+A_o)(1-\alpha\eta) + \frac{\epsilon}{\kappa}\alpha\,[-A_1+2(1+A_o)\alpha\beta_1+\tfrac{1}{2}(1+A_o)\kappa K\Delta^*\, i\,\alpha^2\beta_2] \tag{II. 3. 8c}$$

where

$$\beta_1 \equiv \int_0^\infty W\, e^{-2\alpha\eta_1}\, d\eta_1$$

$$\beta_2 \equiv \int_0^\infty C\, e^{-2\alpha\eta_1}\, d\eta_1$$

4. Intermediate Expansion

The outer region is convection dominated, and the inner region is viscosity dominated. We expect the convection and the viscous stress balance each other in the intermediate region.

If we let

$$\eta = g(\epsilon)\hat{\eta} \ ; \quad g(\epsilon) \to 0, \ \ Reg(\epsilon) \to \infty \ \text{ as } \ \epsilon \to 0 \tag{II. 4. 1}$$

Eq. (II. 1. 3) becomes

$$i\alpha\tilde{u} + \frac{1}{g}\frac{d\tilde{v}}{d\eta} = i\alpha^2[1 - \frac{\epsilon}{\kappa}W(g\hat{\eta})]e^{-\alpha g\hat{\eta}}$$

$$i\alpha[1 - \frac{\epsilon}{\kappa}W(g\hat{\eta})]\tilde{u} - \frac{\epsilon}{\kappa}W'(g\hat{\eta})\tilde{v} + i\alpha\tilde{p} = \frac{1}{Rg^2}\frac{d}{d\hat{\eta}}(\bar{\mu}\frac{d\tilde{u}}{d\hat{\eta}})$$

$$\frac{1}{g}\frac{d\tilde{p}}{d\hat{\eta}} + i\alpha[1 - \frac{\epsilon}{\kappa}W(g\hat{\eta})\tilde{v}] = \alpha^2[1 - \frac{\epsilon}{\kappa}W(g\hat{\eta})]^2\, e^{-\alpha g\hat{\eta}}$$

Now in this region we obtain

$$W(g\hat{\eta}) = -\log g - \log\hat{\eta} + \tilde{\pi}[2-w(g\hat{\eta})]$$

$$W'(g\hat{\eta}) = -\frac{1}{g\hat{\eta}} - \tilde{\pi}\, w'(g\hat{\eta})$$

$$\bar{\mu} = R\,\epsilon\, u\, g\,\hat{\eta}$$

Therefore if we choose

$$g = \epsilon$$

the convection term and the viscous term balance in the $\tilde{u}$-momentum equation, provided $\alpha = O(1)$.

Thus, in the intermediate region, we let

$$\begin{aligned} \eta &= \epsilon\,\hat{\eta} \\ \tilde{u} &= u(\hat{\eta};\epsilon) \\ \tilde{v} &= i\,\alpha\,\epsilon\,\hat{f}(\hat{\eta};\epsilon) \\ \tilde{p} &= \hat{p}(\hat{\eta};\epsilon) \end{aligned} \tag{II. 4. 2}$$

Then the governing equations become

$$\hat{u} + \frac{d\hat{f}}{d\hat{\eta}} = \alpha[1 + \frac{\epsilon}{\kappa}\log\epsilon + \frac{\epsilon}{\kappa}(\log\hat{\eta} - 2\tilde{\pi} - \kappa d\hat{\eta}) + \ldots]$$

$$\frac{\kappa}{i\alpha}\frac{d}{d\hat{\eta}}(\hat{\eta}\frac{d\hat{u}}{d\hat{\eta}}) - (\hat{u}+\hat{p}) = (\frac{\epsilon}{\kappa}\log\epsilon)\hat{u} + \frac{\epsilon}{\kappa}[(\log\hat{\eta} - 2\tilde{\pi})\hat{u} + \frac{1}{\hat{\eta}}\hat{f}] + \ldots$$

$$\frac{d\hat{p}}{d\hat{\eta}} = \epsilon\,\alpha^2 + \ldots$$

If we assume

$$\hat{u} = \hat{u}_o + (\frac{\epsilon}{\kappa}\log\epsilon)\hat{u}_1 + \frac{\epsilon}{\kappa}\hat{u}_2 + \ldots, \text{ etc.} \tag{II. 4. 3}$$

we obtain

$$\begin{cases} \dfrac{\varkappa}{i\alpha}\dfrac{d}{d\hat{\eta}}\left(\hat{\eta}\dfrac{d\hat{u}_n}{d\hat{\eta}}\right) - \hat{u}_n - \hat{p}_n = \mathcal{F}_n \\ \dfrac{d\hat{p}_r}{d\hat{\eta}} = \mathcal{G}_n \\ \dfrac{d\hat{f}_n}{d\hat{\eta}} + \hat{u}_n = \mathcal{K}_n \end{cases} \qquad \text{(II. 4. 4)}$$

$$\mathcal{F}_1 = 0, \quad \mathcal{F}_2 = \hat{u}_o, \quad \mathcal{F}_3 = (\log\hat{\eta} - 2\tilde{\pi})\hat{u}_o + \frac{\hat{f}_o}{\hat{\eta}}$$

$$\mathcal{G}_1 = \mathcal{G}_2 = 0, \quad \mathcal{G}_3 = \varkappa\,\alpha^2 \qquad \text{(II. 4. 5)}$$

$$\mathcal{K}_1 = \alpha, \quad \mathcal{K}_2 = \alpha, \quad \mathcal{K}_3 = \alpha(-2\tilde{\pi} + \log\hat{\eta} - \alpha\,\varkappa\,\hat{\eta})$$

We immediately get

$$\hat{p}_o = \text{const.}, \quad \hat{p}_1 = \text{const.}, \quad \hat{p}_2 = \hat{p}_2(0) + \varkappa\,\alpha^2\,\hat{\eta} \qquad \text{(II. 4. 6)}$$

Then the solutions for $\hat{u}_o$ and $\hat{f}_o$ are

$$\hat{u}_o = -\hat{p}_o + a_o\,K_o(x)$$

$$\hat{f}_o = \hat{f}_o(0) + \frac{\varkappa\,u_o}{2i\alpha} \times K_1(x) \qquad \text{(II. 4. 7)}$$

where

$$x = 2(i\,\alpha\,\hat{\eta}/\varkappa)^{\frac{1}{2}} \qquad \text{(II. 4. 8)}$$

and K_0, K_1 are the modified Bessel functions of the second kind.

The intermediate expansions of the two-term outer solution are, to $O(\varepsilon)$

$$\tilde{u}_{outer} \sim \alpha\Big\{(1+A_o)[1 - \frac{\varepsilon}{\varkappa}\log\varepsilon - \frac{\varepsilon}{\varkappa}(\log\hat{\eta} - 2\tilde{\pi} + \varkappa\,d\hat{\eta})] - A_o(1-\varepsilon\,\alpha\,\hat{\eta})/\varkappa\hat{\eta}$$

$$+ \frac{\varepsilon}{\varkappa}[-2(1+A_o)\alpha\beta_1 + A_1 + i(1+A_o)\varkappa(\varkappa - \tfrac{1}{2}K\Delta^*\beta_2\alpha^2)]\Big\} \qquad \text{(II. 4. 9a)}$$

$$\tilde{f}_{outer} \sim A_o[1 + \frac{\varepsilon}{\varkappa}\log\varepsilon + \frac{\varepsilon}{\varkappa}(\log\hat{\eta} - 2\tilde{\pi} - \varkappa\,\alpha\,\hat{\eta})]$$

$$+ \varepsilon[2(1+A_o)\beta_1\alpha + A_1 + \tfrac{1}{2}\,i(1+A_o)K\,\Delta^*\beta_2\alpha^2] \qquad \text{(II. 4. 9b)}$$

$$\tilde{p}_{outer} \sim \alpha\Big\{-(1+A_o)(1-\varepsilon\,\alpha\,\hat{\eta}) + \frac{\varepsilon}{\varkappa}[2(1+A_o)\beta_1\alpha - A_1 + \tfrac{1}{2}i(1+A_o)\varkappa\,K\Delta^*\beta_2\alpha^2]\Big\} \qquad \text{(II. 4. 9c)}$$

For large values of $\hat{\eta}$ we obtain

$$\tilde{f}_{inter} \sim \varepsilon \, \hat{f}_o(0)$$

Therefore

$$\begin{aligned} A_o &= 0, \quad A_1 + 2\beta_1\alpha + \tfrac{1}{2} i\, K\Delta^* \beta_2 \alpha^2 = \hat{f}_o(0) \\ \hat{p}_o &= -\alpha, \quad \hat{p}_1 = 0, \quad \hat{p}_2(0) = \alpha[4\beta_1\alpha + i\varkappa K\Delta^* \beta_2\alpha^2 - \hat{f}_o(0) \end{aligned} \qquad \text{(II. 4. 10)}$$

The solution for $\hat{u}_1$ and $\hat{f}_1$ are, then,

$$\begin{aligned} \hat{u}_1 &= -\alpha - \tfrac{1}{2} a_o \, x \, K_1(x) + a_1 \, K_o(x) \\ \hat{f}_1 &= \hat{f}_1(0) + 2\alpha\,\hat{\eta} + \frac{i\varkappa}{2\alpha}[\tfrac{1}{2} a_o \, x^2 \, K_2(x) - a_1 \, x \, K_1(x)] \end{aligned} \qquad \text{(II. 4. 11)}$$

The solution for $\hat{u}_2$ is given by

$$\begin{aligned} \hat{u}_2 = {} & -i\,\varkappa^2\alpha - \varkappa\alpha^2\hat{\eta} - \alpha(\log\hat{\eta} - 2\hat{\pi}) - \hat{p}_2(0) - a_o[1 + \log x)K_o(x) + (1 - \log x)x\, K_1(x)] \\ & + 2i\frac{\alpha}{\varkappa}\,\hat{f}_o(0)\,[I_o^2(x)K_o^2(x) - 6\,K_o(x)\int_o^x I_o I_1 K_o\, dx_1 - 3\, I_o \int_x^\infty I_1\, K_o^2\, dx_1] \\ & + a_2\, K_o(x) \end{aligned} \qquad \text{(II. 4. 12)}$$

5. Inner Expansion

In the region near the wall where

$$\zeta = R\,\varepsilon\,\eta = O(1)$$

the mean velocity distribution is given by

$$U = \frac{\varepsilon}{\varkappa} F(\zeta)$$

and the viscosity coefficient is given by

$$\bar{\mu} = 1 + \varkappa\,\zeta\, E(\zeta), \quad E(\zeta) \equiv (1 - e^{-\alpha^* \zeta})^2$$

The governing equations become

$$\tilde{u} + R\varepsilon\frac{d\tilde{f}}{d\zeta} = \alpha\,\frac{\varepsilon}{\varkappa}F(\zeta)e^{-\frac{\alpha}{R\varepsilon}\zeta}$$

$$\frac{\varepsilon}{\varkappa}\,F(\zeta)\tilde{u} + R\,\varepsilon^2\,F'(\zeta)\,f + p = \frac{R\,\varepsilon^2}{i\alpha}\,\frac{d}{d\zeta}(\bar{\mu}\frac{d\tilde{u}}{d\zeta})$$

$$R\varepsilon\,\frac{dp}{d\zeta} - \alpha^2\,\frac{\varepsilon}{\varkappa}F\,f = \alpha^2\,\frac{\varepsilon^2}{\varkappa}\,F^2\,e^{-\frac{\alpha}{R\varepsilon}\zeta}$$

Let

$$\tilde{u} \sim \epsilon\, u^*(\zeta;\epsilon)$$
$$\tilde{f} \sim \frac{1}{R}\, v^*(\zeta;\epsilon) \qquad \text{(II. 5. 1)}$$
$$\tilde{p} \sim p^*(\zeta;\epsilon)$$

then

$$u^* + \frac{df^*}{d\zeta} = \frac{\alpha}{\varkappa} F(1 - \frac{\alpha}{R\epsilon}\zeta + \dots)$$

$$\frac{1}{\varkappa R\epsilon}(F u^* + F' f^*) + \frac{1}{R\epsilon^3} p^* = \frac{1}{i\alpha} \frac{d}{d\zeta}(\bar{\mu} \frac{du^*}{d\zeta})$$

$$\frac{dp^*}{d\zeta} - \frac{1}{R^2} \frac{\alpha^2}{\varkappa} F f^* = \frac{\epsilon}{R} \frac{\alpha^2}{\varkappa^2}(1 - \frac{\alpha}{R\epsilon}\zeta + \dots)$$

Therefore as $R \to \infty$, $\epsilon \to 0$

$$u^* + \frac{df^*}{d\zeta} = \frac{\alpha}{\varkappa} F + (\text{T.S.T})^*$$

$$\frac{d}{d\zeta}(\bar{\mu} \frac{du^*}{d\zeta}) = 0 + (\text{T.S.T})$$

$$\frac{dp^*}{d\zeta} = 0 + (\text{T.S.T.})$$

Where (T.S.T.) denotes transcendentally small terms.

---

* Since $R = \frac{1}{\epsilon} e^{\varkappa/\epsilon - A}$, $A = B + 2\tilde{\pi}$,
$(R\epsilon^n)^{-1} \to \epsilon^{-n+1} e^{A - \frac{\varkappa}{\epsilon}} \to$ (T.S.T.) for any finite n .

Hence if we assume expansions of the form

$$u^*(\zeta;\epsilon) = u_o^*(\zeta) + g_1^*(\epsilon) u_1^*(\zeta) + \dots$$

$$g_1^*(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0$$

we obtain

$$p_o^* = \text{const.}$$
$$u_o^* = \frac{1}{\varkappa} \tau_o^* F(\zeta)$$
$$f_o^* = \frac{1}{\varkappa}(\alpha - \tau_o^*) \int_0^\zeta F(\zeta_1) d\zeta_1$$

and

$$p_1^* = \text{const.}$$

$$u_1^* = \frac{1}{\varkappa}\, \tau_1^*\, F(\zeta)$$

$$f_1^* = -\frac{1}{\varkappa}\, \tau_1^* \int_0^{\zeta} F(\zeta_1) d\zeta_1$$

provided $O(R\epsilon^3 g_1^*) > 1$

Then two-term inner solutions are

$$\tilde{u}_{inner} \sim \frac{\epsilon}{\varkappa}[\tau_o^* F(\zeta) + g_1^*(\epsilon)\tau_1^* F(\zeta) + \ldots]$$

$$\tilde{f}_{inner} \sim \frac{1}{\varkappa R}\left[(\alpha-\tau_o^*)\int_0^{\zeta} F\, \alpha\zeta_1 - g_1^*\, \tau_1^* \int_0^{\zeta} F\, d\zeta_1 + \ldots\right]$$

$$\tilde{p}_{inner} \sim p_o^* + g_1^*\, p_1^* + \ldots$$

Matching with Intermediate Solution:

Since for large values of ζ

$$F \sim \log\zeta + B$$

we obtain

$$\tilde{u}_{inner} \sim \frac{\epsilon}{\varkappa}[\tau_o^* + g_1^*\, \tau_1^* + \ldots](\log\zeta + B)$$

$$\tilde{f}_{inner} \sim \frac{1}{\varkappa R}[(\alpha-\tau_o^*) - g_1^*\, \tau_1^* + \ldots][\zeta\log\zeta + (B-1)\zeta]$$

$$\tilde{p}_{inner} \sim p_o^* + g_1^*\, p_1^* + \ldots$$

Expressed in terms of the intermediate variable, these become

$$\tilde{u}_{inner} \sim [\tau_o^* + g_1^*\, \tau_1^* + \ldots][1 + \frac{\epsilon}{\varkappa}\log\epsilon + \frac{\epsilon}{\varkappa}(\log\hat{\eta} - 2\tilde{\pi})]$$

$$\tilde{f}_{inner} \sim \epsilon[(\alpha-\tau_o^*) - g_1\, \tau_1^* + \ldots]\hat{\eta}[1 + \frac{\epsilon}{\varkappa}\log\epsilon + \frac{\epsilon}{\varkappa}(\log\hat{\eta} - 2\tilde{\pi} - 1)]$$

$$\tilde{p}_{inner} \sim p_o^* + g_1^*\, p_1^* + \ldots$$

On the other hand, the two-term intermediate expansions for small values of $\hat{\eta}$ are

$$\tilde{u}_{inter} \sim \alpha - a_o(\gamma + \tfrac{1}{2}\log\frac{i\alpha}{\varkappa} + \tfrac{1}{2}\log\hat{\eta}) + \ldots$$

$$- \frac{\epsilon}{\varkappa}\log\epsilon\,[\alpha + \tfrac{1}{2}a_o + a_1(\gamma + \tfrac{1}{2}\log\frac{i\alpha}{\varkappa} + \tfrac{1}{2}\log\hat{\eta}) + \ldots]$$

$$\tilde{f}_{inter} \sim \epsilon\left\{\hat{f}_o(0) + \frac{\varkappa}{2i\alpha}a_o + \ldots + \frac{\epsilon}{\varkappa}\log\epsilon\,[\hat{f}_1(0) + 2\alpha\hat{\eta} + \frac{i\varkappa}{2\alpha}(a_o - a_1)] + \ldots\right\}$$

Therefore

$$a_o = a_1 = \hat{f}_o(0) = \hat{f}_1(0) = 0$$

$$\tau_o^* = \alpha, \quad \tau_1^* = -2\alpha, \quad g_1^* = \frac{\epsilon}{\varkappa}\log\epsilon$$

With these values for the constants, the three-term intermediate solution for small $\hat{\eta}$ is

$$\tilde{u}_{inter} \sim \alpha - \alpha\frac{\epsilon}{\varkappa}\log\epsilon - \frac{\epsilon}{\varkappa}[\hat{p}_2(0) + i\varkappa^2\alpha + a_2(\gamma + \tilde{\pi} + \tfrac{1}{2}\log\frac{i\alpha}{\varkappa}$$

$$+ \varkappa\alpha^2\hat{\eta}(\log\hat{\eta} - 2\tilde{\pi})]$$

On the other hand the three-term inner solution in terms of the intermediate variable is

$$\tilde{u}_{inner} \sim \alpha - 2\frac{\epsilon}{\varkappa}\log\epsilon + \frac{\epsilon}{\varkappa}[\tau_2^* + \alpha(\log\hat{\eta} - 2\tilde{\pi})]$$

Hence

$$a_2 = -4\alpha$$

$$\tau_2^* = 4\alpha(\gamma + \tilde{\pi} + \tfrac{1}{2}\log\frac{i\alpha}{\varkappa}) - \hat{p}_2(0) - i\varkappa^2\alpha$$

Summarizing the results to $O(\epsilon)$, we obtained

$$\tilde{u}_{outer} \sim \alpha\, e^{-\alpha\eta}\Big[1 + \epsilon\Big\{A_1 + W(\eta) - 2\alpha e^{2\alpha\eta}\int_\eta^\infty W e^{-2\alpha\eta_1}d\eta_1$$

$$- iK\Delta^*[C' - \alpha C + \tfrac{1}{2}\alpha^2(e^{2\alpha\eta}\int_\eta^\infty C e^{-2\alpha\eta_1}d\eta_1 - \int_o^\eta C\,d\eta_1)]\Big\} + \ldots\Big]$$

$$\tilde{f}_{outer} \sim \epsilon\, e^{-\alpha\eta}\Big\{A_1 + 2\alpha e^{2\alpha\eta}\int_\eta^\infty W e^{-2\alpha\eta_1}d\eta_1 + iK\Delta^*\alpha[-C$$

$$+ \tfrac{1}{2}(e^{2\alpha\eta}\int_\eta^\infty C e^{-2\alpha\eta_1}d\eta_1 - \int_o^\eta C(\eta_1)d\eta_1)]\Big\} + \ldots$$

$$\tilde{p}_{outer} \sim -\alpha\, 2^{-\alpha\eta}\Big\{1 + \epsilon\,[A_1 - 2\alpha\, e^{2\alpha\eta} \int_\eta^\infty W\, e^{-2\alpha\eta_1}\, d\eta_1$$

$$+ \tfrac{1}{2} i K \Delta^* \alpha^2 (e^{2\alpha\eta} \int_\eta^\infty C\, e^{-2\alpha\eta_1} - \int_0^\eta C\, d\eta_1)] + \dots\Big\}$$

$$\tilde{u}_{inter} \sim \alpha - \alpha \frac{\epsilon}{\kappa} \log\epsilon - \frac{\epsilon}{\kappa}\Big\{\hat{p}_2(0) + i\,\kappa\,\alpha^2 + \kappa\,\alpha^2\,\hat{\eta} + \alpha(\log\hat{\eta} - 2\tilde{\pi})$$

$$+ 4\, 2\, K_o[2(\tfrac{i\alpha}{\kappa}\hat{\eta})^{\frac{1}{2}}]\Big\}$$

$$\tilde{\tau}_{inter} \sim \epsilon\,\Big\{(\frac{\epsilon}{\kappa}\log\epsilon)2\alpha\hat{\eta} + \dots\Big\}$$

$$\tilde{p}_{inter} \sim -\alpha + \epsilon\,\alpha^2\,(4\beta_1 + i\,K\Delta^*\,\beta_2\,\alpha)$$

$$\tilde{u}_{inner} \sim \alpha\frac{\epsilon}{\kappa}\Big\{1 - 2\frac{\epsilon}{\kappa}\log\epsilon + \frac{\tau_2^*}{\alpha}\,\frac{\epsilon}{\kappa}\Big\}\, F(\zeta)$$

$$\tilde{\tau}_{inner} \sim \frac{\alpha}{\kappa R}\Big\{2\frac{\epsilon}{\kappa}\log\epsilon - \tau_2^*\,\frac{\epsilon}{\kappa}\Big\}\int_0^\zeta F(\zeta_1)d\zeta_1$$

$$\tilde{p}_{inner} \sim \tilde{p}_{inter}$$

and

$$A_1 = -\alpha(2\beta_1 + \tfrac{1}{2} i\kappa K\,\Delta^*\,\beta_2\,\alpha)$$

$$\hat{p}_2(0) = -2\,\alpha\, A_1$$

$$\frac{\tau_2^*}{\alpha} = 2\,A_1 + 4(\gamma + \tilde{\pi} + \tfrac{1}{2}\log\frac{i\alpha}{\kappa}) - i\,\kappa^2$$

By adding these expansions in three regions and subtracting the common parts we obtain composite expansions which are valid uniformly in $0 \le \eta < \infty$;

$$\tilde{u} \sim \alpha\, e^{-\alpha\eta}\Big[\!\Big[2 - U(\eta) + \epsilon\Big\{A_1 - 2\alpha e^{2\alpha\eta}\int_\eta^\infty W\, e^{-2\alpha\eta_1}\, d\eta_1$$

$$- i\,K\,\Delta^*\Big[C' - \alpha\, C + \tfrac{1}{2}\alpha^2(e^{2\alpha\eta}\int_\eta^\infty C\, e^{-2\alpha\eta_1}\, d\eta_1 - \int_0^\eta C\, d\eta_1)$$

$$- 4\,\kappa\,\alpha\, K_o\,[2(\tfrac{i\alpha}{\kappa}\hat{\eta})^{\frac{1}{2}}]\Big]\Big\}$$

$$+ \frac{\epsilon}{\kappa}(1 - 2\frac{\epsilon}{\kappa}\log\epsilon + \frac{\tau_2^*}{\alpha}\frac{\epsilon}{\kappa})\,[F(\zeta) - (\log\zeta + B)] + \dots\Big]\!\Big]$$

For $\tilde{\tau}$ and $\tilde{p}$, $\tilde{\tau}_{outer}$ and $\tilde{p}_{outer}$ are uniformly valid in $0 \le \eta < \infty$.

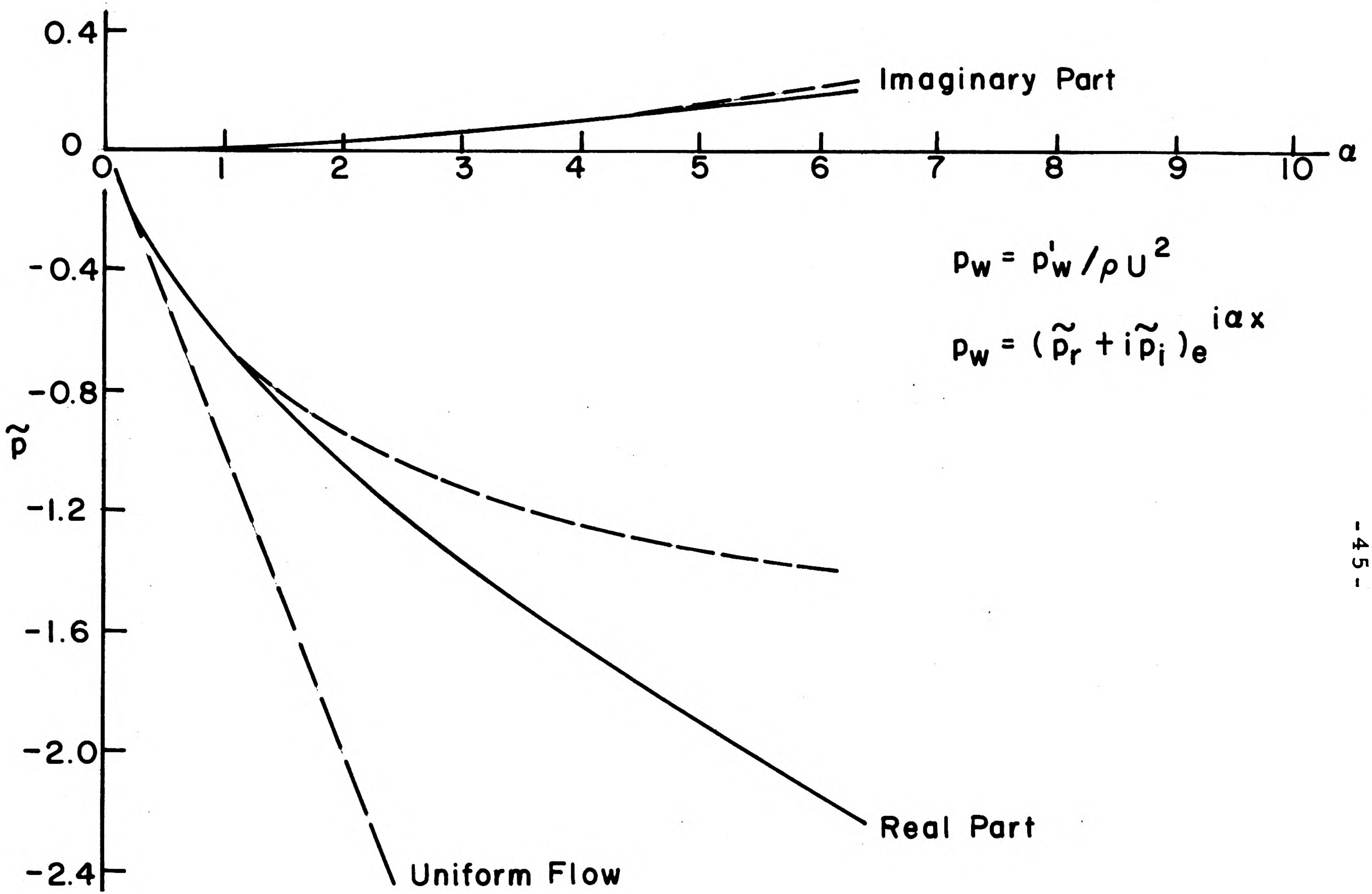

FIG II-1 SURFACE PRESSURE

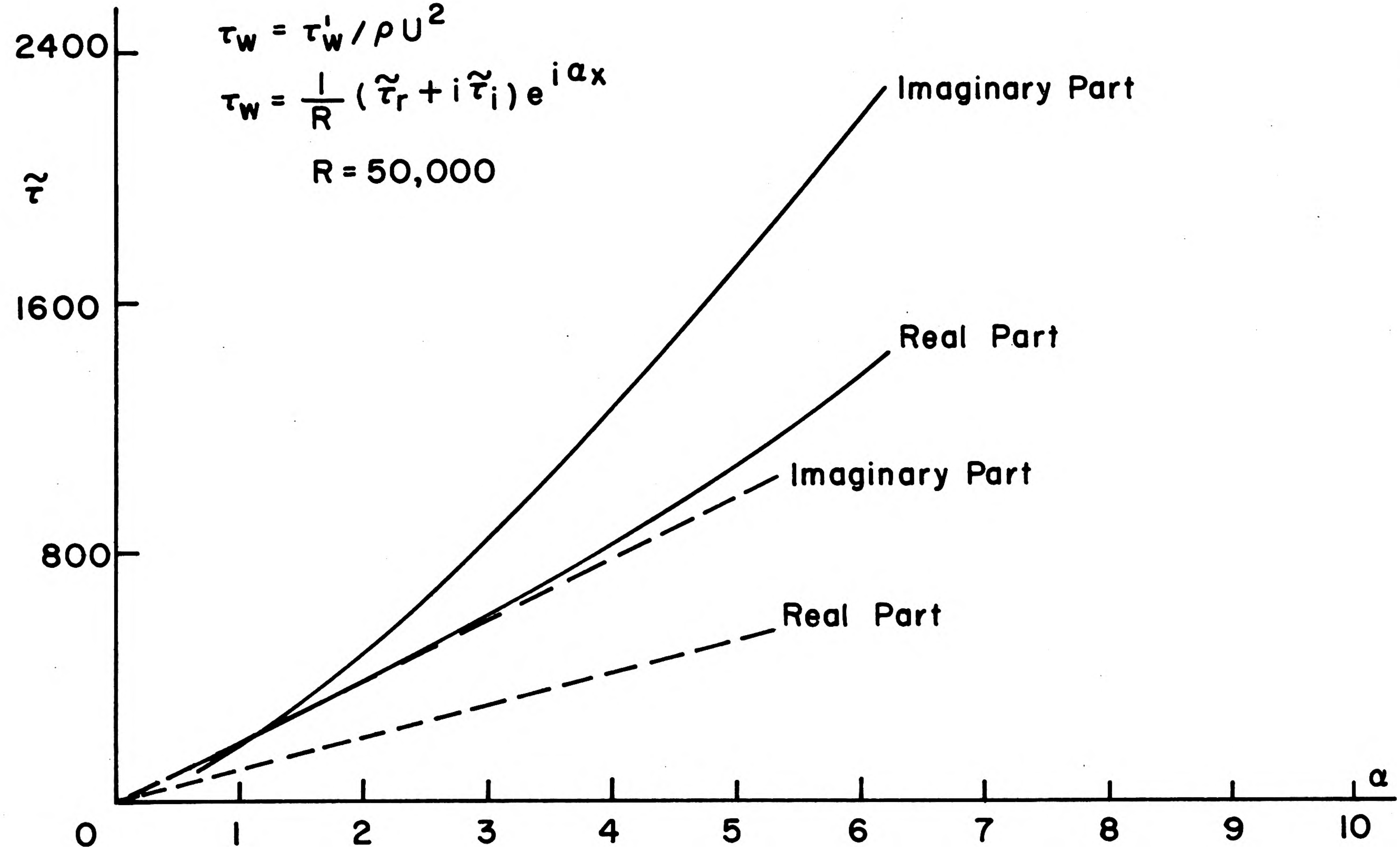

FIG II-2 SURFACE SHEAR STRESS

APPENDIX III

COMPRESSIBLE FLAT-PLATE BOUNDARY LAYER

The undisturbed velocity profile in compressible turbulent boundary layer to be used in the linear perturbation analysis described in the main body of this report was computed numerically by assuming a turbulent eddy viscosity.

Compressible turbulent boundary layer along a flat plate is governed by the following set of equations:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial \tau}{\partial y}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial q}{\partial y} + \frac{\partial}{\partial y}(\tau u)$$

$$H = C_p T + \tfrac{1}{2} u^2$$

$$p = R \rho T .$$

For the shear stress and heat flux appearing in the momentum and energy equations, we assume

$$\tau = (\mu + \mu_R)\frac{\partial u}{\partial y}$$

$$q = (k + k_R)\frac{\partial T}{\partial y}$$

Here μ_R is the eddy-viscosity coefficient and is assumed to have the following form:*

$$\mu_R = \begin{cases} K \rho u_e \delta_k^* \Gamma(y/\delta) & \text{for } y_1 \le y < \delta \\ \rho (\kappa y E)^2 \left|\frac{\partial u}{\partial y}\right| & \text{for } 0 \le y \le y_1 \end{cases}$$

$$E = 1 - \exp(-y/A)$$

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* Cebeci, Smith, and Mosinskis. AIAA Journal, vol. 8, no. 11, 1970.

where y_1 is the value of y at which these two expressions give the same value. Γ is the intermittency factor and is taken to be

$$\Gamma\left(\frac{y}{\delta}\right) = \frac{1}{2}\left\{1 - \operatorname{erf}\left[5\left(\frac{y}{\delta} - 0.78\right)\right]\right\}$$

as in low speed turbulent boundary layer,* and

$$\delta_k^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy$$

Eddy thermal conductivity k_R is related to μ_R through eddy Prandtl number

$$Pr_R = \mu_R C_p / k_R$$

Introduce the following transformation

$$\xi = K \rho_e u_e^2 \int_0^x \delta_k^* dx$$

$$\eta = \frac{u_e}{(2\xi)^{\frac{1}{2}}} \int_0^y \zeta \, dy$$

and for stream function and total enthalpy

$$\psi = (2\xi)^{\frac{1}{2}} f(\xi, \eta)$$

$$H = H_e \, g(\xi, \eta) \quad .$$

Then we obtain

$$u = u_e \frac{\partial f}{\partial \eta}$$

and the momentum and energy equations become

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* Klebanoff. NACA TN 3178, 1954

$$\frac{\partial}{\partial\eta}\left(C_R \frac{\rho}{\rho_e} \frac{\partial^2 f}{\partial\eta^2}\right) + f \frac{\partial^2 f}{\partial\eta^2} = 2\xi\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\xi\partial\eta} - \frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2}\right)$$

$$\frac{\partial}{\partial\eta}\left(\frac{C_R}{Pr_T}\frac{\rho}{\rho_e}\frac{\partial g}{\partial\eta}\right) + f\frac{\partial y}{\partial\eta} - \frac{u_e^2}{2H_e}\frac{\partial}{\partial\eta}\left[\left(\frac{1}{Pr_T}-1\right)C_R\frac{\rho}{\rho_e}\frac{\partial}{\partial\eta}\left(\frac{\partial f}{\partial\eta}\right)^2\right]$$

$$= 2\xi\left(\frac{\partial f}{\partial\eta}\frac{\partial g}{\partial\xi} - \frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}\right)$$

where

$$C_R = (\mu+\mu_R)/(K\,\rho_e u_e\,\delta_k^*)$$

$$Pr_T = (\mu+\mu_R)\,C_p/(k+k_R) \quad .$$

Let

$$f(\xi,\eta) = \eta - u_\tau(\xi)F(\xi,\eta)$$

$$u_\tau \equiv (\tau_w/\rho_w)^{\frac{1}{2}}/u_e$$

Then the momentum equation becomes

$$\frac{\partial}{\partial\eta}\left(C_R\frac{\rho}{\rho_e}\frac{\partial^2 F}{\partial\eta^2}\right) + (\eta - u_\tau F)\frac{\partial^2 F}{\partial\eta^2}$$

$$= 2\xi\frac{\partial^2 F}{\partial\xi\partial\eta}$$

$$- 2\xi\left[u_\tau\frac{\partial F}{\partial\eta}\frac{\partial^2 F}{\partial\xi\partial\eta} - \frac{u_\tau'}{u_\tau}\left(\frac{\partial F}{\partial\eta}\right)^2 - u_2\left(\frac{\partial F}{\partial\xi} + \frac{u_\tau'}{u_\tau}F\right)\frac{\partial^2 F}{\partial\eta^2}\right]$$

If u_τ and u_τ'/u_τ are both small compared to unity, then we obtain

$$\frac{d}{d\eta}\left(C_R\frac{\rho}{\rho_e}\frac{d^2 F}{\partial\eta^2}\right) + \eta\frac{d^2 F}{d\eta^2} = 0$$

namely the similarity of velocity defect -- the velocity-defect law, provided that C_R and ρ/ρ_e are functions of η only.

The last condition is violated in a very thin viscous sublayer near the wall, but we expect the following system

$$\frac{d}{d\eta}\left(C_R \frac{\rho}{\rho_e} \frac{d^2f}{d\eta^2}\right) + f\frac{d^2f}{d\eta^2} = 0$$

$$\frac{d}{d\eta}\left(C_R \frac{\rho}{\rho_e} \frac{dg}{d\eta}\right) + f\frac{dg}{d\eta} = \frac{u_e^2}{2H_e}\frac{d}{d\eta}\left[\left(\frac{1}{PR_T} - 1\right)\frac{\rho}{\rho_e}\frac{d}{d\eta}\left(\frac{df}{d\eta}\right)^2\right]$$

$$f'(0) = 0,\ f(0) = \text{const.},\ g(0) = \text{const.}$$

$$f'(\infty) = g(\infty) = 1$$

to give a good first approximation to the flat-plate turbulent boundary-layer solution.

A comparison of this approximation and experimental data is shown in Fig. III-1.

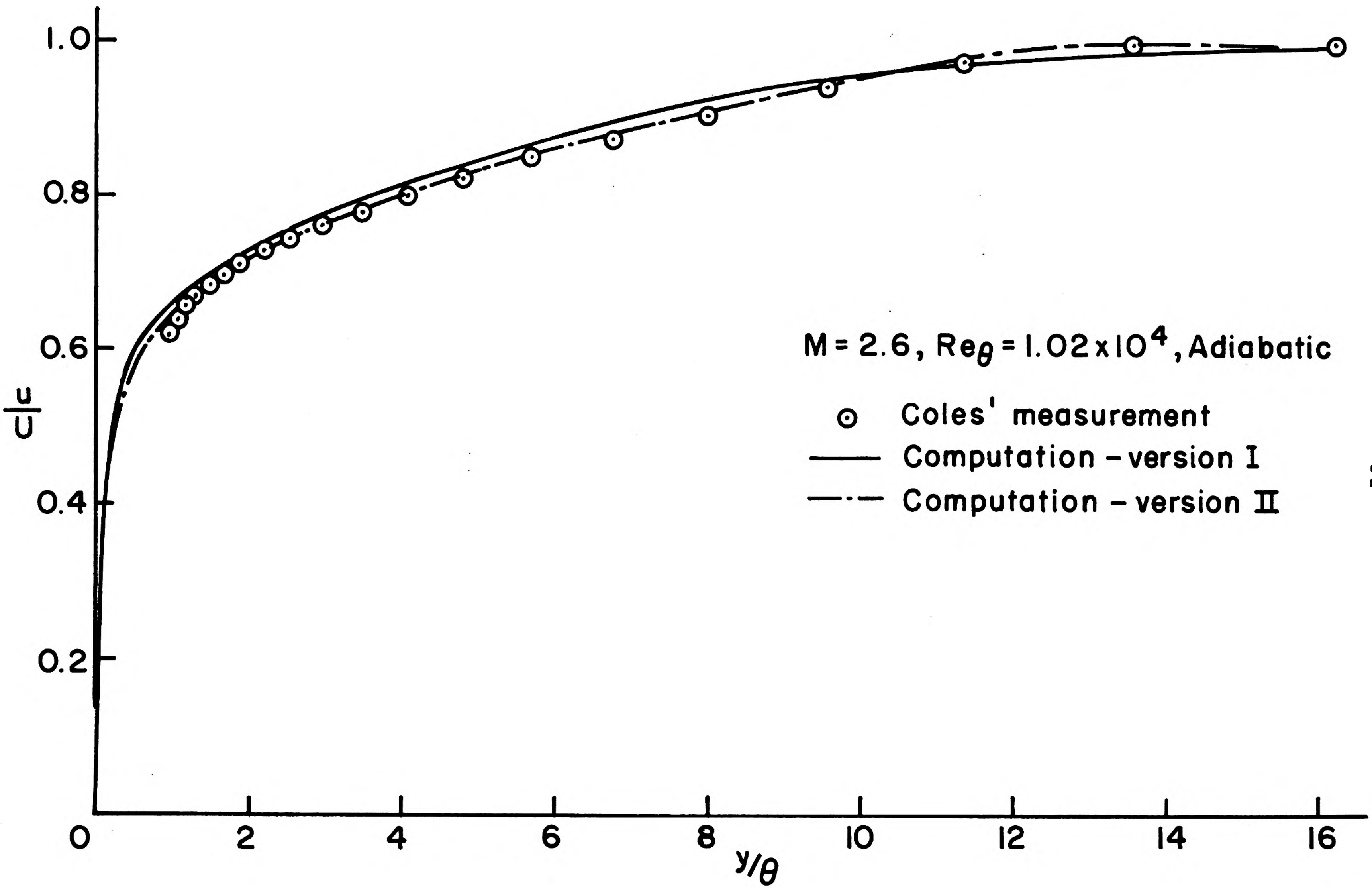

FIG. III-1 FLAT-PLATE TURBULENT BOUNDARY LAYER

APPENDIX IV

CROSS HATCHING STUDIES - LIQUID LAYER MODEL WITH INVISCID EXTERNAL FLOW

1. Introduction

The appearance of a remarkably regular cross-hatched surface pattern on recovered reentry vehicles has generated considerable interest on the part of fluid mechanicians. The regularity and "cleanness" of the pattern tend to call for some relatively simple explanations. However, no theoretical explanation offered so far has been completely convincing. On the other hand, since the first systematic investigation by Larsen, the experimental studies have provided much more information which tends to screen out most, if not all. of the explanations. Much understanding on the phenomena in terms of the necessary conditions for the occurrence of cross-hatched pattern has been achieved through the experiments, but little progress on the physical mechanism is made from the theoretical end. The present work is intended to offer a plausible model which will be capable of explaining the phenomena. The basic philosophy is to use the simplest possible model first and increase its complexity only when the simplified model fails to explain certain observed phenomena. The approach is based mainly on the refracting and reflecting properties of the shear flow in the supersonic-turbulent boundary layer for small disturbances generated by a wavy surface. As these disturbances propagate outward along curved Mach lines, a certain fraction of the energy is reflected back toward the surface because of the sharp gradient in the mean Mach number. Thus, a certain waviness pattern produces a distribution of surface heat transfer rate that is shifted in phase with respect to the surface waviness by an amount that depends on the ratio of wave-length to boundary layer thickness, the free stream Mach number, and the Mach number distribution across the boundary layer, etc. In the first part of this study, an inviscid, linear analysis is used for th external flow. A liquid layer is then incorporated with this external inviscid calculation in order to search for the causes of the occurrence of certain wave angle and wave-length.

2. External Flow

2. 1. Formulation

The external undisturbed flow is assumed to be a parallel shear flow and a thin sublayer (less than 1 or 2% of the boundary layer thickness) near the wall is neglected in this phase of study. The edge of the sublayer is taken to be the location where $u/u_e = 0.5$. The undisturbed velocity distribution was assumed to be a power-law profile, and the exponent was obtained by a curve-fit to D. Coles' measurement of supersonic adiabatic turbulent boundary layer. The perturbed flow field due to an obliqued wavy surface in such a prescribed mean flow is then considered in the following.

The coordinates are made dimensionless by the boundary layer thickness δ_B. The three velocity components are

$$u_e [U(y) + u], \ u_e v, \ u_e w \ (u, v, w << U)$$

and

$$p_e + p_e u_e^2 p, \ \rho_e[\bar{\rho}(y) + \sigma] \ (p << \frac{1}{\gamma M_e^2} \ \sigma << \bar{\rho})$$

gives the pressure and the density respectively.

It may be shown that viscous effects are negligible when the scale length of the disturbances is of the same order of magnitude as the boundary layer thickness. Then, the linearized equations for the perturbations are

$$\frac{1}{\bar{\rho}}\left(U\frac{\partial\sigma}{\partial x}+\frac{d\bar{\rho}}{dy}v\right)+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \qquad \text{(IV. 2. 1)}$$

$$U\frac{\partial u}{\partial x}+\frac{dU}{dy}v+\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x} = 0 \qquad \text{(IV. 2. 2)}$$

$$U\frac{\partial v}{\partial x}+\frac{1}{\bar{\rho}}\frac{\partial p}{\partial y} = 0 \qquad \text{(IV. 2. 3)}$$

$$U\frac{\partial w}{\partial x}+\frac{1}{\bar{\rho}}\frac{\partial p}{\partial z} = 0 \qquad \text{(IV. 2. 4)}$$

$$U\frac{\partial p}{\partial x}-a^{-2}\left(U\frac{\partial\sigma}{\partial x}+\frac{d\rho}{dy}v\right) = 0 \qquad \text{(IV. 2. 5)}$$

where $a^{-2} = \dfrac{\gamma p_e}{\rho_e u_e^2 \bar{\rho}}$

The phase speed of the wavy surface is assumed to be small compared with the air speed, and hence, the wall is given by

$$\bar{y}_w = \epsilon\, e^{ik(\lambda x + \nu z)} \qquad \text{(IV. 2. 6)}$$

where k is the nondimensional wave number in the direction normal to the wave front; λ and ν are the directional cosine of the normal, related by

$$\lambda^2 + \nu^2 = 1 \qquad \text{(IV. 2. 7)}$$

Then the linearized boundary condition at the wall is

$$v(x, 0, z) = i\,k\,\lambda\,\epsilon\,U(0)\,e^{ik(\lambda x + \nu z)} \qquad \text{(IV. 2. 8)}$$

The boundary condition far away from the wall is that only simple outgoing waves exist outside the boundary layer; namely, if $Qe^{ik(\lambda x + \nu z)}$ stands for the perturbation quantities, we have

$$Qy + ik\beta_e Q = 0 \qquad \text{for} \qquad y \geq 1 \qquad \text{(IV. 2. 9)}$$

where $\beta_e^2 = \lambda^2 M_e^2 - 1$, and M_e is the Mach number of the uniform external flow.

Eliminating all perturbations but p from equations (IV. 2. 1-5) and putting

$$p = \epsilon P(y) e^{ik(\lambda x + \nu z)} \tag{IV. 2. 10}$$

we obtain the equation for P

$$\frac{d^2P}{dy^2} - \frac{2}{M} \frac{dM}{dy} \frac{dP}{dy} + k^2(\lambda^2 M^2 - 1)P = 0 \tag{IV. 2. 11}$$

where $M = U(y)/\bar{a}(y) = \frac{\bar{u}(y)}{u_\infty}$

The boundary condition for P at the wall is

$$Py(0) = k^2 \lambda^2 M_o^2/M_e^2 \tag{IV. 2. 12}$$

where $M_o \equiv M(0)$

By changing λM to M_n, (IV. 2. 11) and (IV. 2. 12) become identical to those for a non-oblique wave with the reduced Mach number. We will then consider the system

$$\frac{d^2P}{dy^2} - \frac{2}{M_n} \frac{dM_n}{dy} \frac{dP}{dy} + k^2(M_n^2 - 1)P = 0$$

$$Py(0) = k^2 M_n^2(0)/M_e^2 \tag{IV. 2. 13}$$

$$Py + i k \beta_e P = 0 \qquad \text{for} \qquad y \geq 0$$

$$\beta_e = \sqrt{M_{ne}^2 - 1}$$

The problem stated by (IV. 2. 13) has been studied by Lighthill.[1] For arbitrary $M_n(y)$ and k, no closed-form solution is possible. Lighthill gave

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[1] M. J. Lighthill. "Reflection at a Laminar Boundary Layer of a Weak Steady Disturbance to a Supersonic Stream, Neglecting Viscosity and Heat Conduction," Quart. Jour. Mech. and Appl. Math., Vol. 3, part 3, 1950.
M. J. Lighthill, "On Boundary Layers and Upstream Influence. II. Supersonic Flows without Separation," Proc. Roy. Soc. A, Vol. 217, p. 478, 1953.

the asymptotic solution for large values of k. However, as the wave length decreases (k increases) one will expect the inner viscous sublayer, which has been neglected in the present formulation, to become important since we are interested in the whole range of wave number, the system (IV. 2. 13) is solved numerically for a given Mach number distribution and wave number k. The details are given in Section 4. It should be noted that for a given free stream Mach number M_e, the solution also depends on the wave sweep angle θ ($\lambda = \sin\theta$).

Now, tentatively, we will assume that the perturbation of the heat transfer rate, hence the ablation rate, is proportional to the pressure perturbation on the wall. If we let the perturbed surface to be given by

$$\bar{y}_w = \epsilon\, e^{ik(\lambda x + \nu z - ct)} \tag{IV. 2. 14}$$

where $c = c_r + i\, C_i$ the complex wave velocity. In the above analysis, a steady wave pattern was considered, i. e., $c_r = 0$. Notice that the wave velocity is also normalized by the freestream velocity u_e. Therefore, the assumption of a standing wave pattern should be a fairly good approximation. Then, we may write

$$\frac{d\bar{y}_w}{dt} \sim -p(0) \tag{IV. 2. 15}$$

or, using (IV. 2. 14) we have

$$(k\, c_i) \sim P_r(0) \tag{IV. 2. 16}$$

Thus, the amplification rate of the perturbed surface is directly proportional to the magnitude of the real part of the pressure disturbance on the wall. Numerical results will be given in the following section.

2.2. Results and Discussions

The Mach number distribution is assumed to be given by

$$\frac{M_n(y)}{M_{ne}} = \left(\frac{y + \delta_s}{1 + \delta_s}\right)^n \qquad n = 0.274 \qquad \text{(IV.2.17)}$$

where δ_s represents the sublayer thickness defined as the distance from wall to the location where $u/u_e = 0.5$. The subscript e refers to the edge condition. The wall temperature is taken to be equal to the free-stream static temperature. Computations were carried out for $M_e = 2.6$ and 6.0, and for a range of sweep angles of the wavy surface. Fig. IV-1 shows the calculated magnitude and phase shift of the pressure perturbation at wall as functions of the sweep angle θ and the wave number k for $M_e = 2.6$. For small wave numbers, the phase shift of the pressure is close to 90^o for larger wave angles and increases as the wave angle decreases, reaches 180^o at the Mach angle where $(M_n)_e = 1$. Further decrease in the wave angle leads to a complete subsonic layer and a constant phase shift of 180^o. On the other hand, for large wave numbers, the phase shift is larger than 90^o even at large wave angles and reaches 180^o as the wave angle decreases before $(M_n)_e = 1$. The magnitude of the pressure perturbation in general increases as the wave angle increases except for small wave numbers which peaks at the Mach angle.

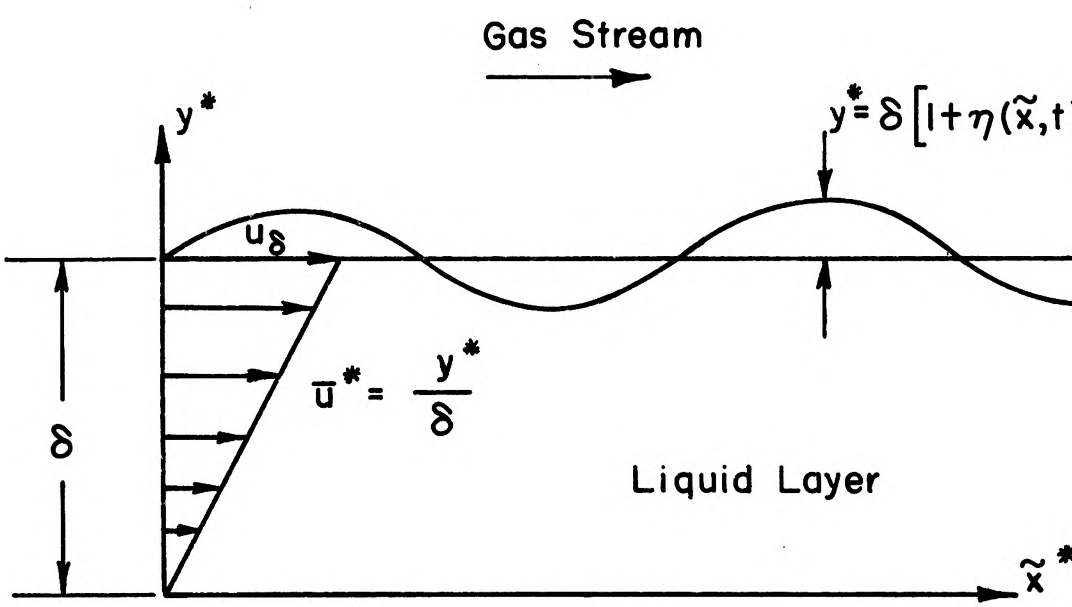
Since we assume that the perturbation of the ablation rate is directly proportional to the pressure perturbation at wall, Fig. IV-2 gives essentially the rate of amplification of the surface perturbation. For small wave numbers, this quantity exhibits a maximum at a value of θ near the Mach wave angle. This peak moves to a larger angle as the wave number increases. The magnitude at the peak also increases with the wave number. Fig. IV-3 shows a similar result for $M_e = 6.0$.

In conclusion, the present simple model predicts the most unstable wave angle for a given wave number and edge Mach number. However, there is no way of predicting the most unstable wave number. It is the purpose of the second part of this study to provide a model for selection of the most unstable wave number.

3. Liquid Layer Model

A thin liquid layer is assumed to exist between the external gas flow and the ablating material. The stability of this layer is then investigated in a similar manner to the analysis of Nachtsheim.* However, the results from the previous inviscid pressure calculation will be used instead of a uniform supersonic external stream as considered by Nachtsheim. Only a brief summary will be given here and one may refer to the paper of Nachtsheim for more details.

3.1. Formulation



Schematic of the Flow

* Nachtsheim, P. R., "Analysis of the stability of a thin liquid film adjacent to a high-speed gas stream," NASA TN-D-4976, Jan. 1969.

The following assumptions will be applied in the present analysis.

(1) The amplitude of the wall perturbation is small as compared to the thickness of the liquid layer δ, i.e. $\overline{\eta} << 1$. So, the boundary condition may be evaluated at $\overline{y} = y/\delta = 1$.

(2) The mean velocity profile is taken to be linear inside the liquid layer.

(3) Density and Viscosity are taken to be constant inside the liquid layer and their perturbation ignored.

(4) The perturbation of shearing stresses of the external gas stream is assumed to be negligible.

(5) The gravitational field, included in the analysis of Nachtsheim, is ignored.

In this liquid layer analysis, the coordinates are made dimensionless by the thickness of the layer δ, and the velocity components and pressure are given by

$$u_\delta(\overline{u} + u'), \quad u_\delta v', \quad u_\delta w', \quad p_e + p_\delta u_\delta^2 p'$$

respectively. Here u_δ is the velocity of the liquid layer at the interface in the direction normal to the wave front. The time is normalized by δ/u_δ. The governing linearized equations for the perturbations are then given by

$$\begin{aligned}
&\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \\
&\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \frac{\partial \overline{u}}{\partial y} v' + \frac{\partial p'}{\partial x} = \frac{1}{R}\nabla^2 u' \\
&\frac{\partial v'}{\partial t} + \overline{u}\frac{\partial v'}{\partial x} + \frac{\partial p'}{\partial y} = \frac{1}{R}\nabla^2 v' \\
&\frac{\partial w'}{\partial t} + \overline{u}\frac{\partial w'}{\partial x} + \frac{\partial p'}{\partial z} = \frac{1}{R}\nabla^2 w'
\end{aligned} \tag{IV. 3. 1}$$

where

$$\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$R = \frac{u_\delta \delta}{\nu_\delta}$$

Now, the perturbations are assumed to be proportional to $\exp[i\alpha(\lambda x+\nu z-ct)]$, where c is the wave speed normalized by u_δ and α is equal to $k(\delta/\delta_B)$. Then equations (IV.3.1) become

$$\begin{aligned}
& i\alpha\lambda u' + v'_y + i\alpha\nu w' = 0 \\
& -i\alpha c u' + \bar{u} i\alpha\lambda u' + \bar{u}_y v' + i\alpha\lambda p' = \frac{1}{R}\left(\frac{d^2}{dy^2} - \alpha^2\right)u' \\
& -i\alpha c v' + \bar{u} i\alpha\lambda v' \qquad + p'_y = \frac{1}{R}\left(\frac{d^2}{dy^2} - \alpha^2\right)v' \\
& -i\alpha c w' + \bar{u} i\alpha\lambda w' \qquad + i\alpha\nu p' = \frac{1}{R}\left(\frac{d^2}{dy^2} - \alpha^2\right)w'
\end{aligned} \tag{IV.3.2}$$

Let $\lambda u' + \nu w' = \tilde{u}, \quad v' = \tilde{v}, \quad \tilde{x} = \lambda x + \nu z$

Then, equations (IV.3.2) can be reduced to

$$\begin{aligned}
& i\alpha\tilde{u} + \tilde{v}_y = 0 \\
& -i\alpha c\tilde{u} + i\alpha\bar{u}^*\tilde{u} + \bar{u}^*_y\tilde{v} + i\alpha p' = \frac{1}{R}\left(\frac{d^2}{dy^2} - \alpha^2\right)\tilde{u} \\
& -i2c\tilde{v} + i\alpha\bar{u}^*\tilde{v} \qquad + p'_y = \frac{1}{R}\left(\frac{d^2}{dy^2} - \alpha^2\right)\tilde{v}
\end{aligned} \tag{IV.3.3}$$

where

$$\bar{u}^* = \lambda\bar{u} = y \tag{IV.3.4}$$

Because of the linear velocity profile assumed, the mean shear force is constant throughout the liquid layer and is equal to the external shear at the interface, i.e.

$$\tau_w = \mu_\delta \frac{u_\delta/\lambda}{\delta} \equiv \tau_o \rho_e u_e^2 \tag{IV.3.5}$$

The appropriate boundary conditions are

$$\tilde{u}(0) = \tilde{v}(0) = 0 \tag{IV.3.6a}$$

and at interface y = 1 (linearized)

$$\tilde{v} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial \tilde{x}} \qquad \text{(IV. 3. 6b)}$$

$$\tau' = \mu_\delta \left(\frac{\partial \tilde{u}}{\partial y} + i\alpha \tilde{v} \right) = 0 \qquad \text{(IV. 3. 6c)}$$

$$\rho_\delta u_\delta^2 p' - \frac{2 u_\delta \mu_\delta}{\delta} \left(\frac{\partial \tilde{v}}{\partial y} \right) = \rho_e u_e^2 p_o - T \frac{\partial^2 \eta}{\partial \tilde{x}^2} \qquad \text{(IV. 3. 6d)}$$

where T is the surface tension per unit length and p_o is the external pressure perturbation given by (IV. 2. 10) evaluated at y = 0.

The interface can be expressed as

$$\eta = \epsilon \left(\frac{\delta_B}{\delta} \right) e^{i\alpha(\tilde{x} - ct)} \qquad \text{(IV. 3. 7)}$$

Then, by substituting

$$\begin{aligned} \tilde{u} &= f'(y) e^{i\alpha(\tilde{x} - ct)} \\ \tilde{v} &= -i\alpha f(y) e^{i\alpha(\tilde{x} - ct)} \\ p' &= \pi(y) e^{i\alpha(\tilde{x} - ct)} \end{aligned} \qquad \text{(IV. 3. 8)}$$

into eq. (IV. 3. 3), we obtain, after elimination of the pressure, the governing equation for f(y)

$$(y-e)(f'' - \alpha^2 f) = \frac{1}{i\alpha R} (f^{iv} - 2\alpha^2 f'' + \alpha^4 f) \qquad \text{(IV. 3. 9)}$$

This is the well-known Orr-Sommerfeld equation with a linear mean velocity profile. The boundary conditions (IV. 3. 6) then become

$$f(0) = f'(0) = 0 \qquad \text{(IV. 3. 10a)}$$

$$f(1) = \epsilon \left(\frac{\delta_B}{\delta} \right) (c-1) \qquad \text{(IV. 3. 10b)}$$

$$f''(1) + \alpha^2 f(1) = 0 \qquad \text{(IV. 3. 10c)}$$

$$f(1) - (1-c) f'(1) + \frac{1}{i\alpha R} [f'''(1) - 3\alpha^2 f'(1)] \qquad \text{(IV. 3. 10d)}$$

$$= \frac{\rho_e u_e^2}{\rho_\delta u_\delta^2} \epsilon P(0;k) + \frac{T}{\rho_\delta u_\delta^2} \epsilon \left(\frac{\delta_B}{\delta} \right) \alpha^2$$

Now, we may use the skin friction coefficient defined by Eq. (IV.3.5) to further simplify (IV.3.10d), since

$$\frac{\rho_e u_e^2}{\rho_\delta u_\delta^2} = \frac{\tau_w}{\tau_o}\frac{1}{\rho_\delta u_\delta^2} = \frac{\mu_\delta u_\delta/\lambda}{\tau_o \delta \rho_\delta u_\delta^2} = \frac{1}{\tau_o \lambda R}$$

hence, (IV.3.10d), combining with (IV.3.10b), becomes

$$f(1) - (1-c)f'(1) + \frac{1}{i\alpha R}\left[f'''(1) - 3\alpha^2 f'(1)\right] = \frac{f(1)}{\tau_o \lambda R(c-1)}\left[\left(\frac{\delta}{\delta_B}\right)P(0;k) + \frac{\alpha^2}{W^2}\right] \qquad \text{(IV.3.11)}$$

Where W is the Weber's number defined by

$$W^2 = \frac{\rho_e u_e^2 \delta}{T} \qquad \text{(IV.3.12)}$$

Eq. (IV.3.11) can be regrouped into the form

$$f'''(1) - B_1 f'(1) - B_2 f(1) = 0 \qquad \text{(IV.3.11a)}$$

with

$$B_1 = 3\alpha^2 + i\alpha R(1-c)$$

$$B_2 = -i\alpha R + \frac{i\alpha}{\tau_o \lambda (c-1)}\left[\left(\frac{\delta}{\delta_B}\right)P(0;k) + \frac{\alpha^2}{W^2}\right]$$

The homogeneous equation, together with the homogeneous boundary conditions, forms an eigenvalue problem. The eigenvalue satisfies a complex relation of the form

$$F(\alpha, R, c_r, c_i) = 0 \qquad \text{(IV.3.13)}$$

The eigenvalues are obtained numerically by using a linear search procedure. The details of the method of solution is given in Section 5. We should just point out the fact that the following parameters appear in the problem and affect the eigenvalues.

$$W^2,\ (\frac{\delta}{\delta_B}),\ \tau_o,\ \lambda,\ M_\infty \qquad \text{(IV.3.14)}$$

The effect of M_∞ appears through its influence on the external pressure perturbation at wall P(0). When these parameters are fixed, Eq. (IV.3.13) gives the rate of amplification (αc_i) as functions of the wave number α, for a given Reynolds number R. The results of the numerical integration will be given in the next section.

3.2. Results and Discussion

Fig. IV-4 shows a plot of the amplification rate αc_i as function of the wave number at $M_\infty = 6.0$ for the conditions indicated on the figure. For this case, αc_i has a peak at $\alpha_m = 0.65$, corresponding to the wave length approximately equal to the boundary layer thickness. The Weber's number used corresponds to a liquid with surface tension of the order of 500 dynes/cm (as compared to that of water of 72 dynes/cm). Disturbances with short wave length are stabilized by the effect of surface tension. Now, by plotting the value of α_m versus the wave angle θ, a peak can be located which uniquely determined the preferred wave angle and wave length for a given flow.

Fig. IV-5 shows the corresponding wave propagation velocity as function of wave number α. For small α, it is always greater than one (fast wave). However, it decreases for large α and, in fact, becomes less than one for sufficiently large α. We may also note that the maximum value of c_r is about 3.4. Since this is normalized by the surface speed of the liquid film, which is quite small in absolute term, the assumption of a standing wave pattern to the external flow is justified.

Figs. IV-6 and IV-7 shows the variations of (αc_i) and c_r with α for $M_\infty = 2.6$. The reason for the occurrence of the peak amplification rate at such a large wave number is due mainly to the use of a small surface tension (T = 50 dynes/cm) we should recall that the present analysis becomes increasingly inaccurate as the wave number increases because of the external inviscid approximation. Thus, a refined model which includes the sublayer structure should be constructed in order to provide a better estimate of the preferred wave angle and wave length. Moreover, an additional phase shift between the pressure perturbation and the heat transfer perturbation may exist which again calls for the inclusion of the viscous sublayer in the external flow calculation.

4. Numerical Solution of the External Perturbed Pressure Field

4.1. Analysis

The system to be solved is a second order ordinary differential equation of the form

$$\frac{d^2P}{dy^2} - \frac{2}{M_n}\frac{dM_n}{dy}\frac{dP}{dy} + k^2(M_n^2 - 1)P = 0 \qquad \text{(IV.4.1.)}$$

with boundary conditions

$$Py(0) = k^2 \; M_n^2(0)/M_e^2 \qquad \text{(IV.4.2a)}$$

$$Py(\infty) + i k\beta_e P(\infty) = 0; \;\; \beta_e = \sqrt{M_{n_e}^2 - 1} \quad \text{for } M_{n_e} > 1 \qquad \text{(IV.4.2b)}$$

$$Py(\infty) - k\,\beta_e' P(\infty) = 0; \;\; \beta_e' = \sqrt{1 - M_{n_e}^2} \quad \text{for } M_{n_e} < 1 \qquad \text{(IV.4.2c)}$$

Eq. (IV.4.1) can be rewritten as

$$\frac{d}{dy}\left(\frac{1}{M_n^2}\frac{dP}{dy}\right) + k^2\left(1 - \frac{1}{M_n^2}\right)P = 0 \qquad \text{(IV.4.3)}$$

Now let

$$P(y) = [Z_1(y) + i\, Z_2(y)]C\, e^{i\varphi_\infty} \tag{IV.4.4a}$$

$$\frac{1}{M_n^2}\frac{dP}{dy} = [Z_3(y) + i\, Z_4(y)]C\, e^{i\varphi_\infty} \tag{IV.4.4b}$$

then

$$\frac{d}{dy}\begin{Bmatrix} Z_1 \\ Z_2 \end{Bmatrix} = M_n^2 \begin{Bmatrix} Z_3 \\ Z_4 \end{Bmatrix} \tag{IV.4.5a}$$

$$\frac{d}{dy}\begin{Bmatrix} Z_3 \\ Z_4 \end{Bmatrix} = -\,k^2\left(1 - \frac{1}{M_n^2}\right)\begin{Bmatrix} Z_1 \\ Z_2 \end{Bmatrix} \tag{IV.4.5b}$$

where C and φ_∞ account for the proper normalization to be determined by the boundary condition at wall.

Eqs. (IV.4.5) forms a set of four coupled first order ordinary differential equations. The equations are then integrated using the following conditions at $y = 1$

$\left(\frac{dM_n}{dy} \approx 0 \text{ for } y \geq 1\right)$

$$\begin{aligned} Z_1(1) &= 1.0 \\ Z_2(1) &= 0 \\ Z_3(1) &= -a_1 \\ Z_4(1) &= -a_2 \end{aligned} \tag{IV.4.6}$$

where

$$\left.\begin{aligned} a_1 &= 0 \\ a_2 &= k\,\beta_e/(M_e^2 \sin^2 \end{aligned}\right\} \text{ for } M_{n_e} \geq 1$$

and

$$\left.\begin{aligned} a_1 &= k\,\beta_e'/(M_e^2 \sin^2\theta \\ a_2 &= 0 \end{aligned}\right\} \text{ for } M_{n_e} < 1$$

The normalization constants are determined by satisfying the boundary condition at y = 0 (IV. 4. 2a) which gives

$$C = \frac{k^2}{M_e^2 \sqrt{Z_3^2(0)+Z_4^2(0)}}$$

$$\varphi_\infty = -\tan^{-1} \frac{Z_4(0)}{Z_3(0)} \qquad \text{(IV. 4. 7)}$$

The solution is then uniquely determined from (IV. 4. 4)

5. Numerical Calculation for the Stability of a Thin Liquid Layer

The equation to be solved is

$$(y-c)\left(\frac{d^2}{dy^2} - \alpha^2\right)f = \frac{1}{i\alpha R}\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 f \qquad \text{(IV. 5. 1)}$$

with the boundary conditions

$$f(0) = f'(0) = 0$$

$$f''(1) + \alpha^2 f(1) = 0 \qquad \text{(IV. 5. 1a)}$$

$$f'''(1) - B_1 f'(1) - B_2 f(1) = 0$$

where

$$B_1 = 3\alpha^2 + i\alpha R(1-c)$$

$$B_2 = -i\alpha R + \frac{i\alpha}{\tau_o \lambda} \frac{1}{c-1}\left[\left(\frac{\delta}{\delta_B}\right)P(0) + \frac{\alpha^2}{W^2}\right]$$

This system constitutes an eigenvalue problem for the four parameters α, R, c_v and c_i. When any two of these parameters are given, the other two are uniquely determined. In the present analysis, we will demonstrate the method of finding the eigenvalues c_r and c_i when α and R are given. Let

$$f'' - \alpha^2 f = \zeta \qquad \text{(IV. 5. 2)}$$

then Eq. (IV. 5. 1) becomes

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\zeta - i\alpha R(y-c)\zeta = 0 \qquad \text{(IV. 5. 3)}$$

Now let

$$\begin{cases} f_n = g_n + i\,h_n \\ \zeta_n = \xi_n + i\,\eta_n \end{cases} \qquad n = 1, 2 \qquad \text{(IV. 5. 4)}$$

We have,

$$\begin{cases} g_n'' - \alpha^2 g_n = \xi_n \\ h_n'' - \alpha^2 h_n = \eta_n \end{cases} \qquad \text{(IV. 5. 5)}$$

$$\begin{cases} \xi_n'' - (\alpha^2 + \alpha R c_i)\xi_n + \alpha R(y - c_r)\eta_n = 0 \\ \eta_n'' - (\alpha^2 + \alpha R c_i)\eta_n - \alpha R(y - c_r)\xi_n = 0 \end{cases} \qquad \text{(IV. 5. 6)}$$

The corresponding boundary conditions at $y = 1$ are

$$\begin{cases} \zeta_n(1) \equiv f_n''(1) - \alpha^2 f_n(1) = -2\,\alpha^2\, f_n(1) \\ \zeta_n'(1) \equiv f_n'''(1) - \alpha^2 f_n'(1) = (B_1 - \alpha^2) f_n'(1) + B_2\, f_n(1) \end{cases} \qquad \text{(IV. 5. 7)}$$

The index n refers to two linearly independent solutions of the equation which can be obtained numerically by using two linearly independent initial conditions, e.g.,

$$f_1(1) = 1, \quad f_1'(1) = 0\,; \qquad f_2(1) = 0, \quad f_2'(1) = 1 \qquad \text{(IV. 5. 8)}$$

then from (IV. 5. 7)

$$\zeta_1(1) = -2\alpha^2, \quad \zeta_1'(1) = B_2\,; \qquad \zeta_2(1) = 0, \quad \zeta_2'(1) = B_1 - \alpha^2 \qquad \text{(IV. 5. 9)}$$

Two sets of solutions ζ_1 and ζ_2 are then obtained by integrating (IV. 5. 6) with the initial conditions (IV. 5. 9). Solutions to (IV. 5. 5) satisfies (IV. 5. 8) and are given by

$$\begin{aligned}
g_1 &= \cosh\alpha(y-1) - \frac{1}{2\alpha}\left[e^{\alpha y}\int_y^1 \xi_1\, e^{-\alpha y_1}\, dy_1 - e^{-\alpha y}\int_y^1 \xi_1 e^{\alpha y_1}\, dy_1\right] \\
h_1 &= - \frac{1}{2\alpha}\left[e^{\alpha y}\int_y^1 \eta_1\, e^{-\alpha y_1}\, dy_1 - e^{-\alpha y}\int_y^1 \eta_1\, e^{\alpha y_1}\, dy_1\right] \\
g_2 &= \frac{1}{\alpha}\sinh\alpha(y-1) - \frac{1}{2\alpha}\left[e^{\alpha y}\int_y^1 \xi_2\, e^{-\alpha y_1}\, dy_1 - e^{-\alpha y}\int_y^1 \xi_2\, e^{\alpha y_1}\, dy_1\right] \\
h_2 &= - \frac{1}{2\alpha}\left[e^{\alpha y}\int_y^1 \eta_2\, e^{-\alpha y_1}\, dy_1 - e^{-\alpha y}\int_y^1 \eta_2\, e^{\alpha y_1}\, dy_1\right]
\end{aligned} \qquad \text{(IV. 5. 10)}$$

Solution to Eq. (IV. 5. 1) can be constructed from these two linearly independent solutions

$$f = C_1 f_1 + C_2 f_2 \tag{IV. 5. 11}$$

where C_1 and C_2 are complex constants. The boundary conditions at $y = 0$ then require

$$C_1 f_1(0) + C_2 f_2(0) = 0$$

$$C_1 f_1'(0) + C_2 f_2'(0) = 0$$

In order to have a non-trivial solution, we must have

$$D \equiv f_1(0)\, f_2'(0) - f_1'(0) f_2(0) = 0 \tag{IV. 5. 12}$$

For arbitrary set of parameters α, R, c_r, c_i, condition (IV. 5. 12) will not be satisfied. Thus, an iteration procedure is needed to search for the correct eigenvalues.

We consider the case when α and R are given. In order to search for the eigenvalue $c = c_r + i c_i$, we need to find $(\partial D/\partial c)$, then a new estimate of the eigenvalue can be obtained by

$$\Delta c = - D/(\frac{\partial D}{\partial c}) \tag{IV. 5. 13}$$

and

$$c_{new} = c_{old} + \Delta c$$

To evaluate $\frac{\partial D}{\partial c}$ we also need to know $\zeta c = \frac{\partial \zeta}{\partial c}$. The governing equation for ζc can be easily obtained from (IV. 5. 3)

$$(\frac{d^2}{dy^2} - \alpha^2)\zeta_{nc} - i\alpha R(y- \quad)\zeta_{nc} + i\alpha R\zeta_n = 0 \tag{IV. 5. 14}$$

These equations are to be integrated simultaneously with (IV.5.6) using the initial conditions

$$
\begin{aligned}
\zeta_{1c}(1) &= 0 \\
\zeta'_{1c}(1) &= \frac{\partial B_2}{\partial c} = -\frac{i\alpha}{T_o \lambda (c-1)^2}\left[\left(\frac{\delta}{\delta_B}\right)P(0) + \frac{\alpha^2}{W^2}\right] \\
\zeta_{2c}(1) &= 0 \\
\zeta'_{2c}(1) &= \frac{\partial B_1}{\partial c} = -\, i\alpha R
\end{aligned}
\tag{IV.5.15}
$$

A computer program has been written to search for the eigenvalues.

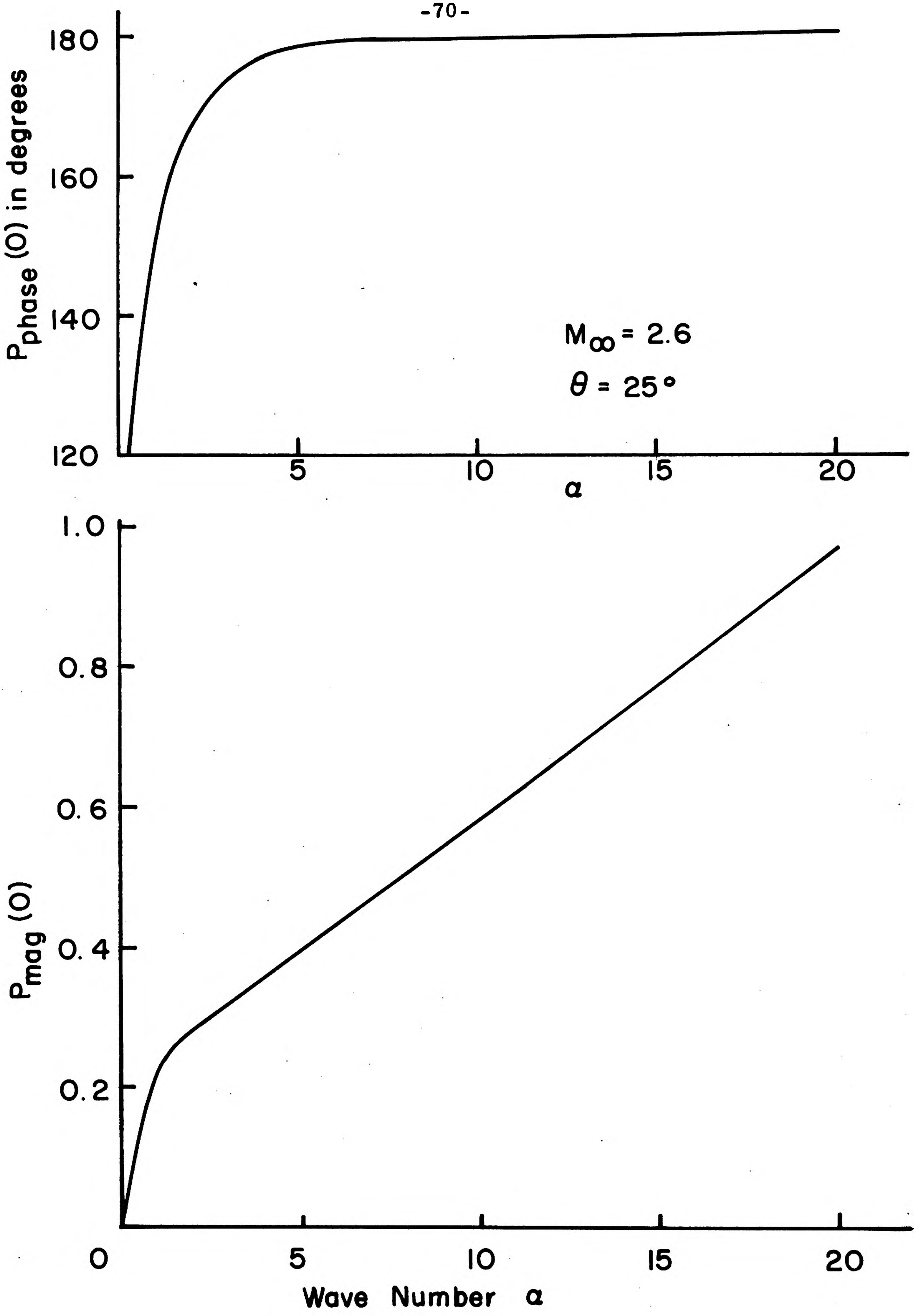


FIG. IV-1a SURFACE PRESSURE AT M_∞ = 2.6

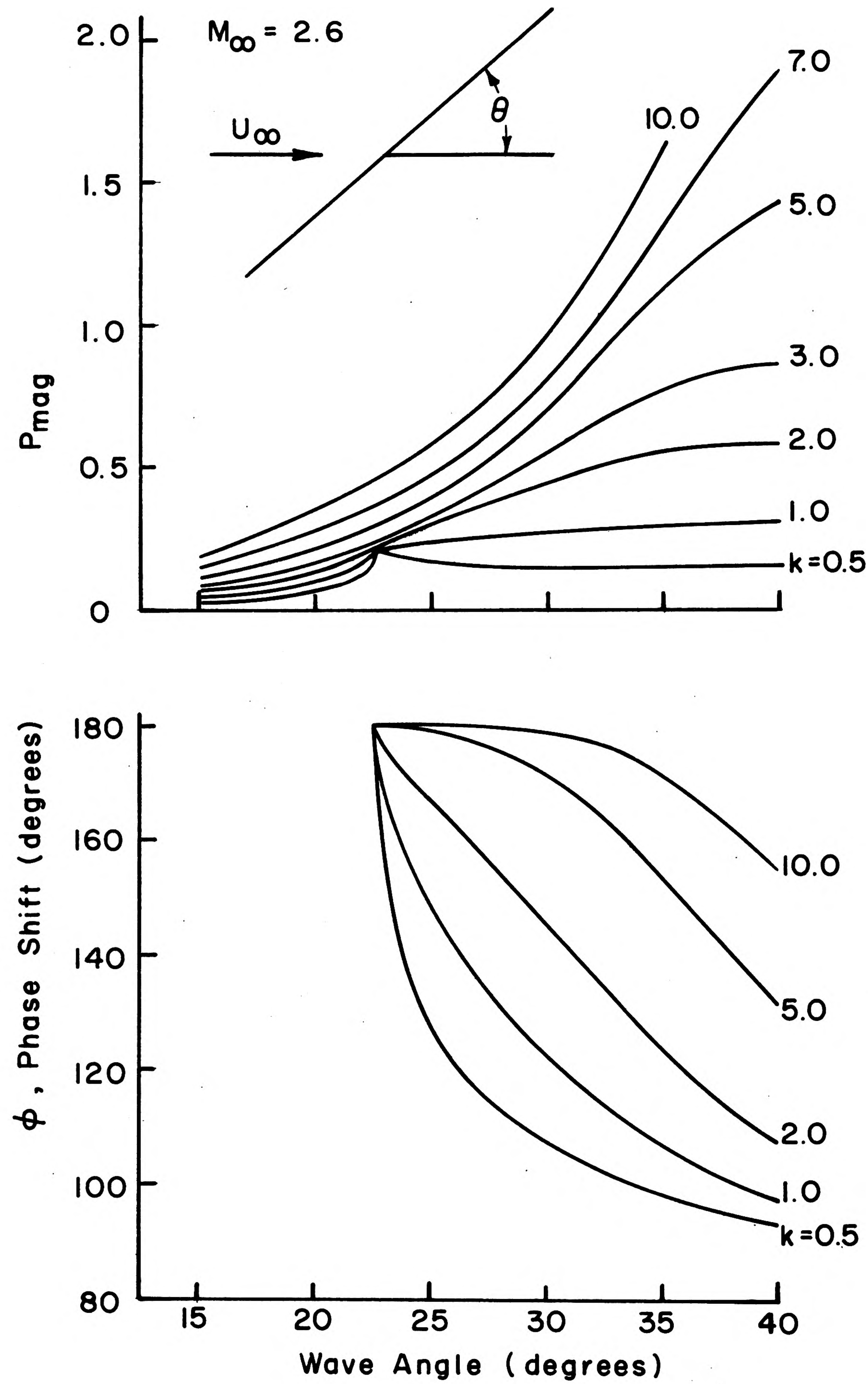


FIG. IV-1b SURFACE PRESSURE AT $M_\infty = 2.6$

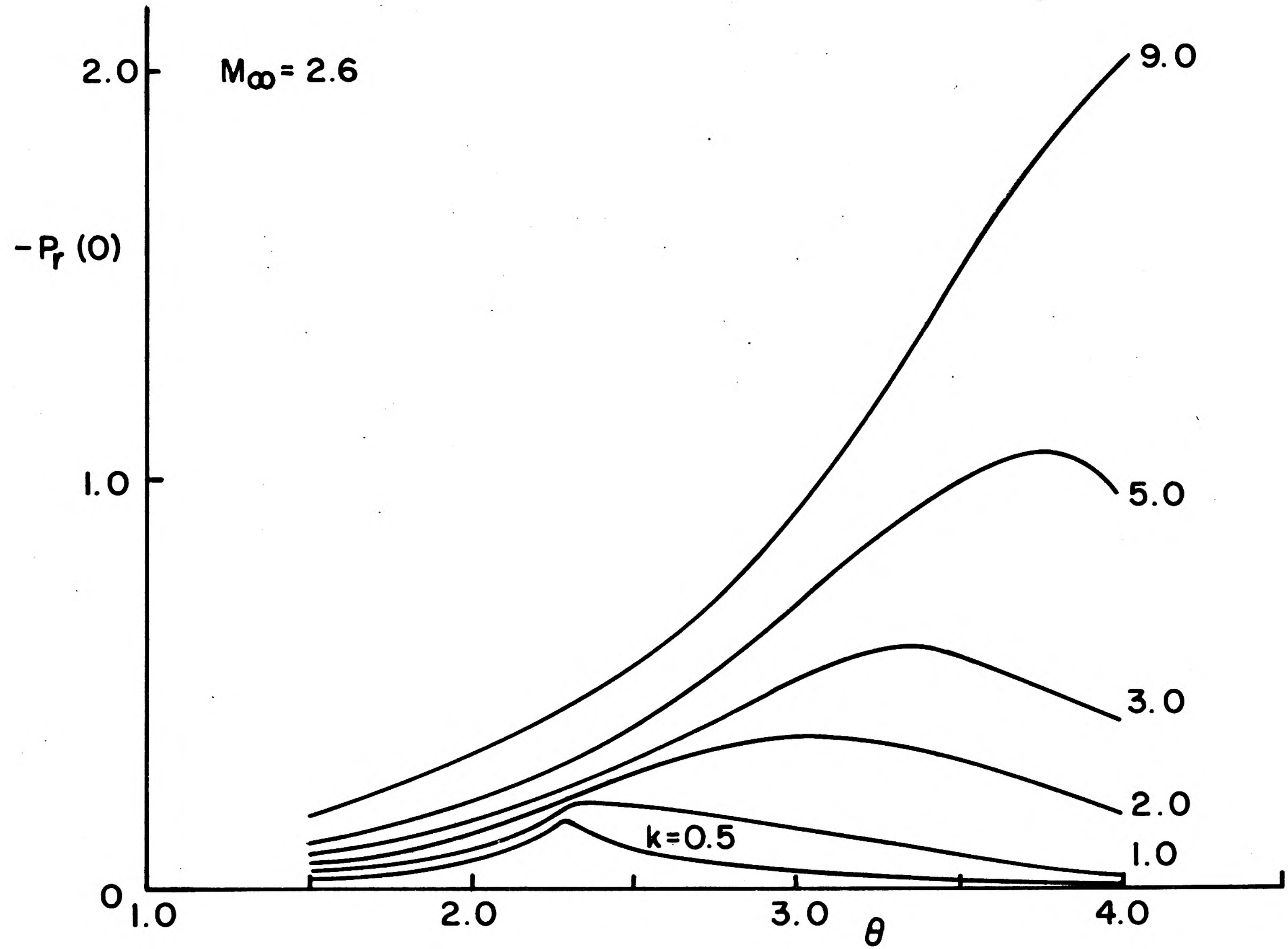


FIG. IV-2 SURFACE PRESSURE AT $M_{\infty} = 2.6$ - REAL PART

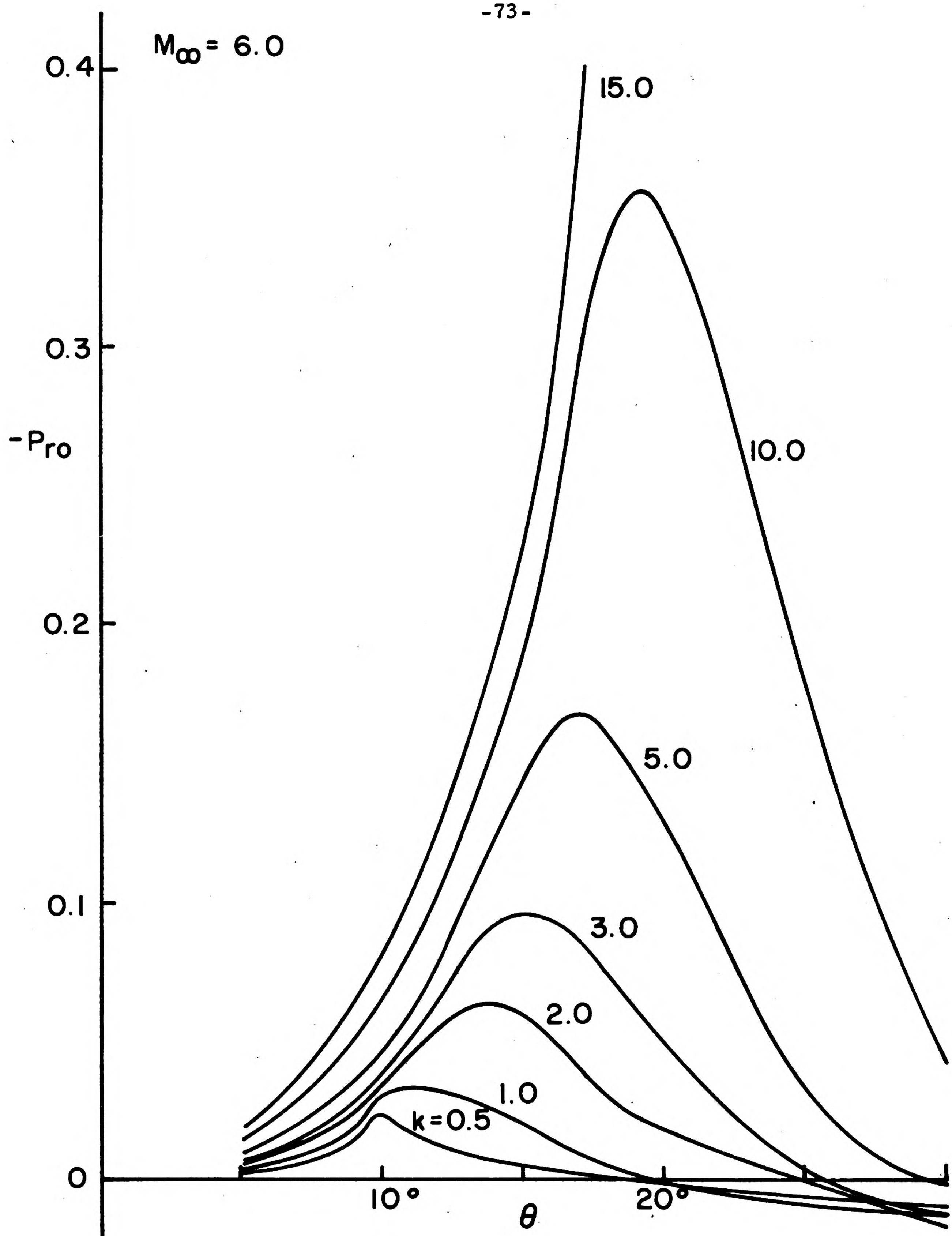


FIG. IV-3 SURFACE PRESSURE AT $M_\infty = 6$ - REAL PART

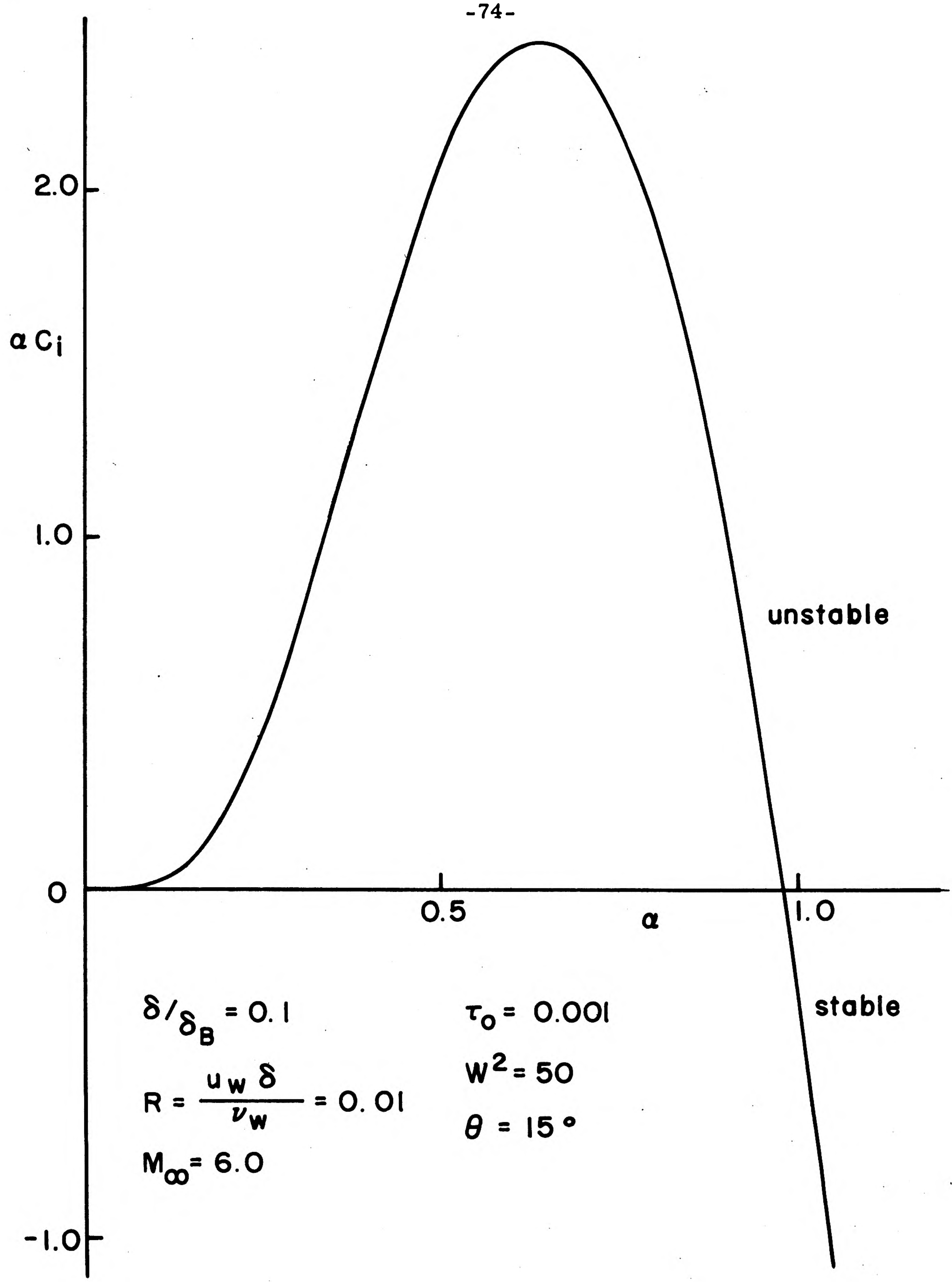


FIG. IV-4 AMPLIFICATION RATE VS. WAVE NUMBER α

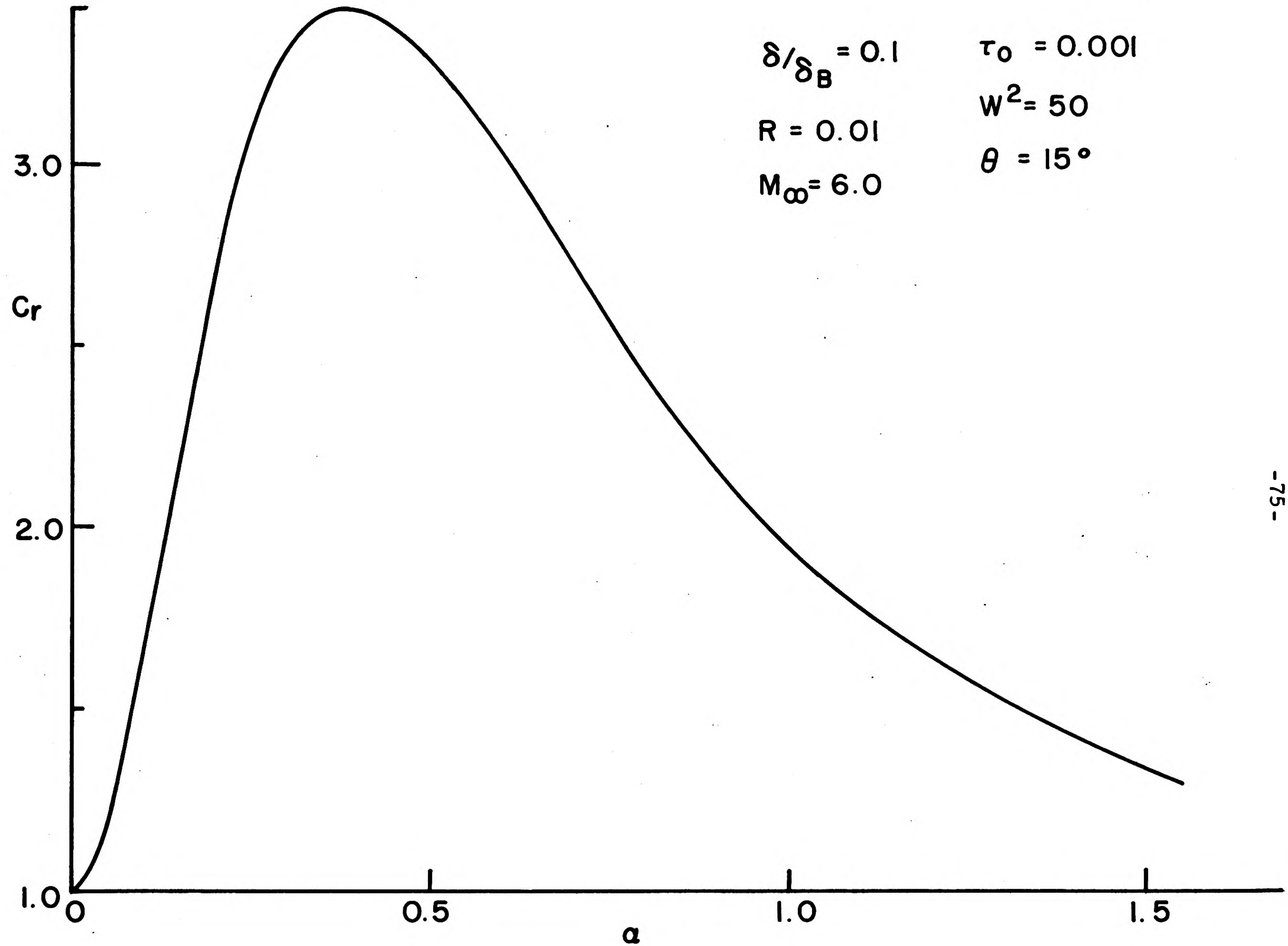


FIG. IV-5 WAVE PROPAGATION VELOCITY, C_r VS. WAVE NUMBER α

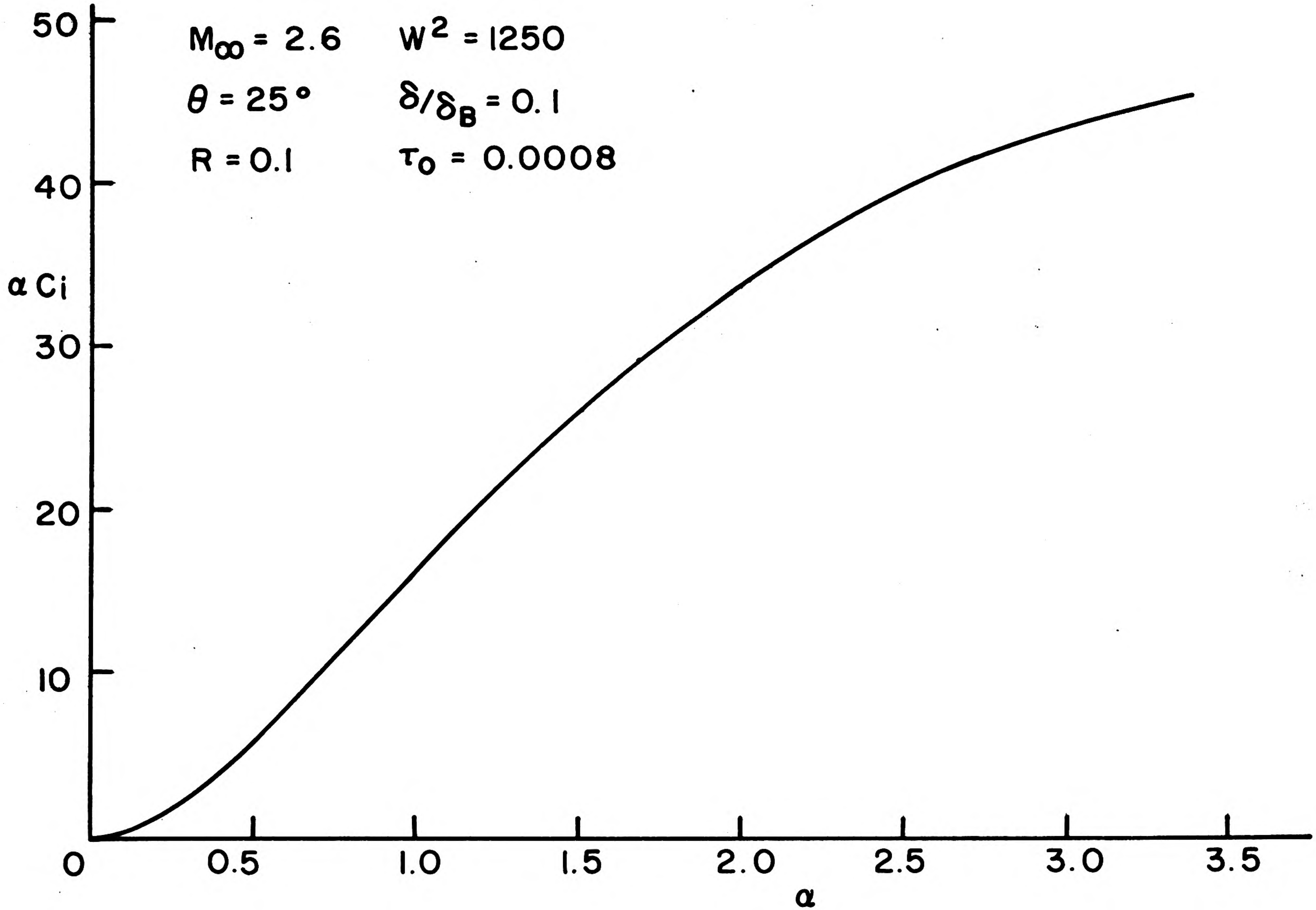


FIG. IV-6 AMPLIFICATION RATE VS. WAVE NUMBER

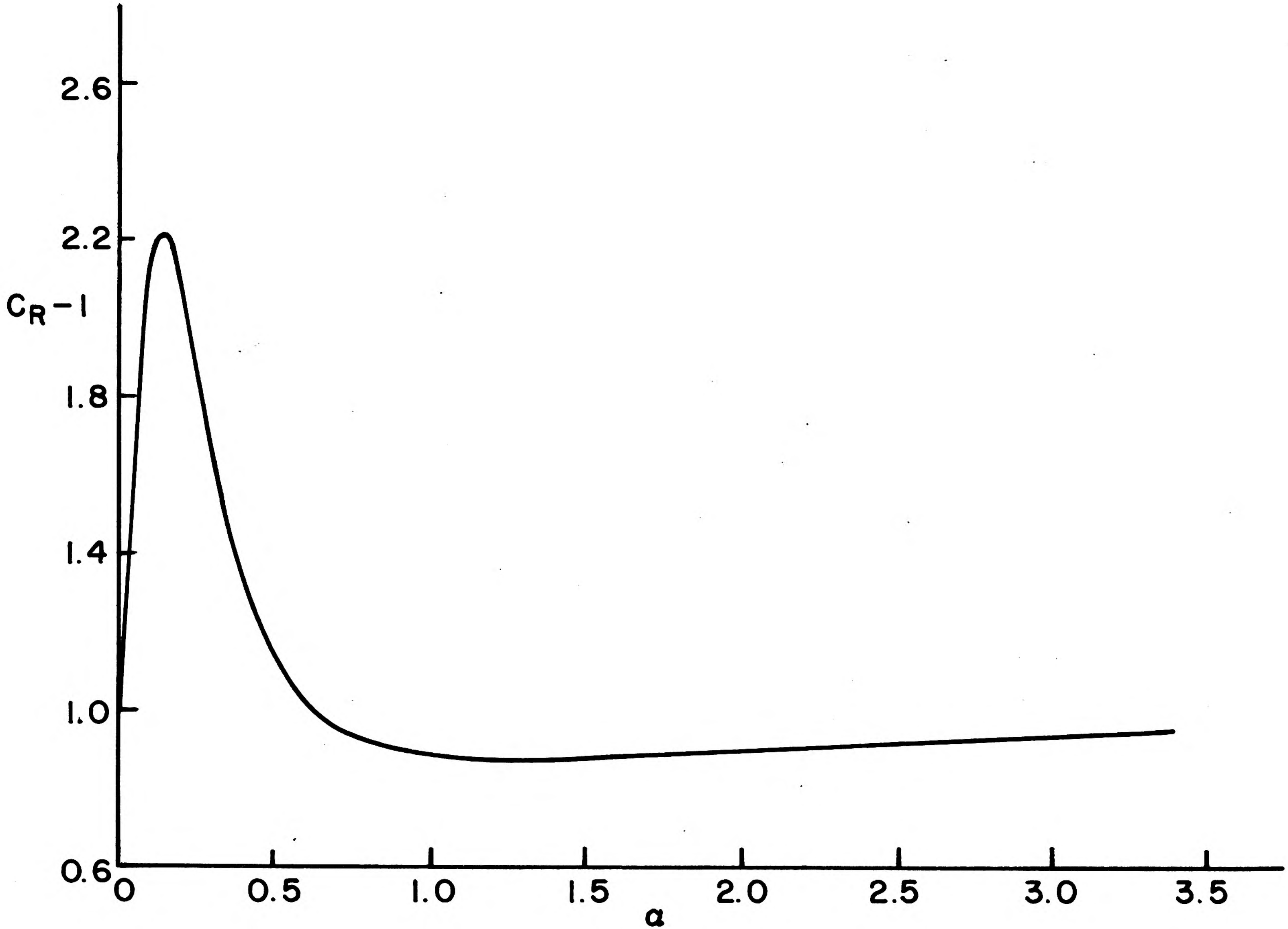


FIG. IV-7 WAVE PROPAGATION VELOCITY

APPENDIX V

STABILITY OF SUBLIMING SURFACE

From the mass and energy balances at the interface between the air and the solid which is subliming under the aerodynamic heating, we obtain

$$\begin{cases} \rho_s \dfrac{\partial \bar{y}}{\partial t} = -\dot{m}' & \text{(V. 1)} \\ q_a' = q_s' + \dot{m}' L_v & \text{(V. 2)} \end{cases}$$

where ρ_s = density of solid material

$\bar{y}$ = surface perturbation

$\dot{m}'$ = perturbation in ablation rate

q_a' = perturbation in aerodynamic heating

q_s' = perturbation in internal heat conduction

L_v = heat of sublimation

For undisturbed, steady, one-dimensional ablation, we have

$$\begin{cases} \bar{q}_s = \rho_s v_s L_T & \text{(V. 3)} \\ \bar{q}_a = \rho_s v_s (L_T + L_v) & \text{(V. 4)} \end{cases}$$

L_T = $c_s (T_w - T_{in})$

= heat required to raise the ablator from the interior temperature to the surface temperature

v_s = steady ablation speed.

By dividing the perturbations by the corresponding steady rates, (1) and (2) become

$$\frac{1}{v_s}\frac{\partial \bar{y}}{\partial t} = -\left(1+\frac{L_T}{L_v}\right)\frac{q_a'}{\bar{q}_a} + \frac{L_T}{L_v}\frac{q_s'}{\bar{q}_s} \tag{V.5}$$

If we assume traveling-wave type perturbations, i.e.

$$\bar{y} = \epsilon\, e^{i\alpha(x-ct)} \quad , \qquad \epsilon \ll 1$$

$$\frac{q_a'}{\bar{q}_a} \equiv \epsilon\, Q_a(\alpha, c; M_e, R) e^{i\alpha(x-ct)}$$

$$\frac{q_s'}{\bar{q}_s} \equiv \epsilon\, Q_s(\alpha, c; M_e, R) e^{i\alpha(x-ct)}$$

$$R \equiv \frac{\rho_\infty^* u_\infty^* \delta^*}{\mu_\infty^*} \quad , \quad \text{Reynolds number} \quad ,$$

(δ^* is the boundary layer edge, not the displacement thickness)

then

$$\frac{i\alpha c}{v_s} = \left(1+\frac{L_T}{L_v}\right)Q_a(\alpha, c) - \frac{L_T}{L_v}Q_s(\alpha, c) \tag{V.6}$$

This equation yields the wave speed and the rate of amplification of small perturbations, when Q_a and Q_s are obtained from small-perturbation analyses of aerodynamic heating and internal heat conduction, respectively, for the surface perturbation $\bar{y} = \epsilon \exp(i\alpha x - i\alpha ct)$.

If it is assumed that the specific heat and thermal conductivity of the solid are constant, we obtain

$$Q_s = \frac{v_s}{2\kappa_s}\left\{1 - \left[1 + 4\frac{\alpha\kappa_s}{v_s}\left(\frac{\alpha\kappa_s}{v_s} - \frac{ic}{v_s}\right)\right]^{\frac{1}{2}}\right\} \tag{V.7}$$

$$\kappa_s = k_s/\rho_s c_s \qquad \text{thermal diffusivity}$$

For the purpose of obtaining Q_a, the following assumptions are made:

(1) the boundary layer is turbulent,

(2) the perturbations are inviscid except in a thin sublayer (not necessarily identical to the viscous sublayer) adjacent to the wall,

(3) molecular and turbulent transport is important only in the sublayer.

The inviscid small perturbations are computed by truncating the mean velocity near the wall below $\bar{u}/u_\infty = 0.5$ (no rigorous justification for this choice).

The solution in the sublayer is obtained by an integral method with the inviscid perturbations at the wall at the boundary condition at the sublayer edge. Then

$$Q_a = a_p \hat{p}_o + a_v \frac{i\alpha c}{v_s} \tag{V.8}$$

where

$$\hat{p}_o = \frac{p'}{p_\infty} \Big/ \epsilon \exp(i\alpha x - i\alpha ct), \quad \text{from inviscid solution}$$

$$\begin{cases} a_p \equiv \dfrac{\beta}{\lambda} \cdot \dfrac{1}{R} \left(\dfrac{\mu_w k_w}{Pr} \right)^{\frac{1}{2}} \dfrac{1}{\gamma M_o^2} \left(\dfrac{dT_t}{dy} \right)_o & \text{(V.9a)} \\ a_v \equiv \beta \left[(T_{t_o} - T_w) - i^{\frac{3}{2}} \left(\dfrac{k_w}{\alpha\, Pr\, R\, \rho_o U_o} \right)^{\frac{1}{2}} \dfrac{\rho_o}{\rho_w} \left(\dfrac{dT_t}{dy} \right)_o \right] & \text{(V.9b)} \end{cases}$$

Here

$R = \rho_\infty^* u_\infty^* \delta^*/u_\infty^*$, Reynolds number based on boundary layer thickness

$Pr = \mu^* C_p^*/k^*$ for air

$\gamma = C_p/C_v$ for air

$y = y^*/\delta^*$ (δ^* boundary-layer thickness)

T_t = total temperature

All unstarred variables are made dimensionless by dividing by the corresponding values at ∞. $(\ )_o$ refers to the values at the velocity cut-off point in the inviscid calculation, i.e. $U_o = \bar{u}/u_\infty = 0.5$ in the present analysis. $(\ )_w$ denotes the values at the wall.

$$\lambda = \bar{\dot{m}} / \rho_\infty^* u_\infty^* = \rho_s v_s / \rho_\infty^* u_\infty^*$$

$$\beta \equiv C_p^* T_\infty^* / (L_v + L_T) \quad \text{with } C_p^* T_\infty^* \text{ for air.}$$

Sample computations shown in the accompanying figures were carried out for

$M_\infty = 2.5$, $T_o = 4700^oR$, $R = 10^5$

$T_w = 1210^oR$, $L_v = 750$ Btu/lb., $L_T = 175$ Btu/lb.

$\lambda = 0.001$

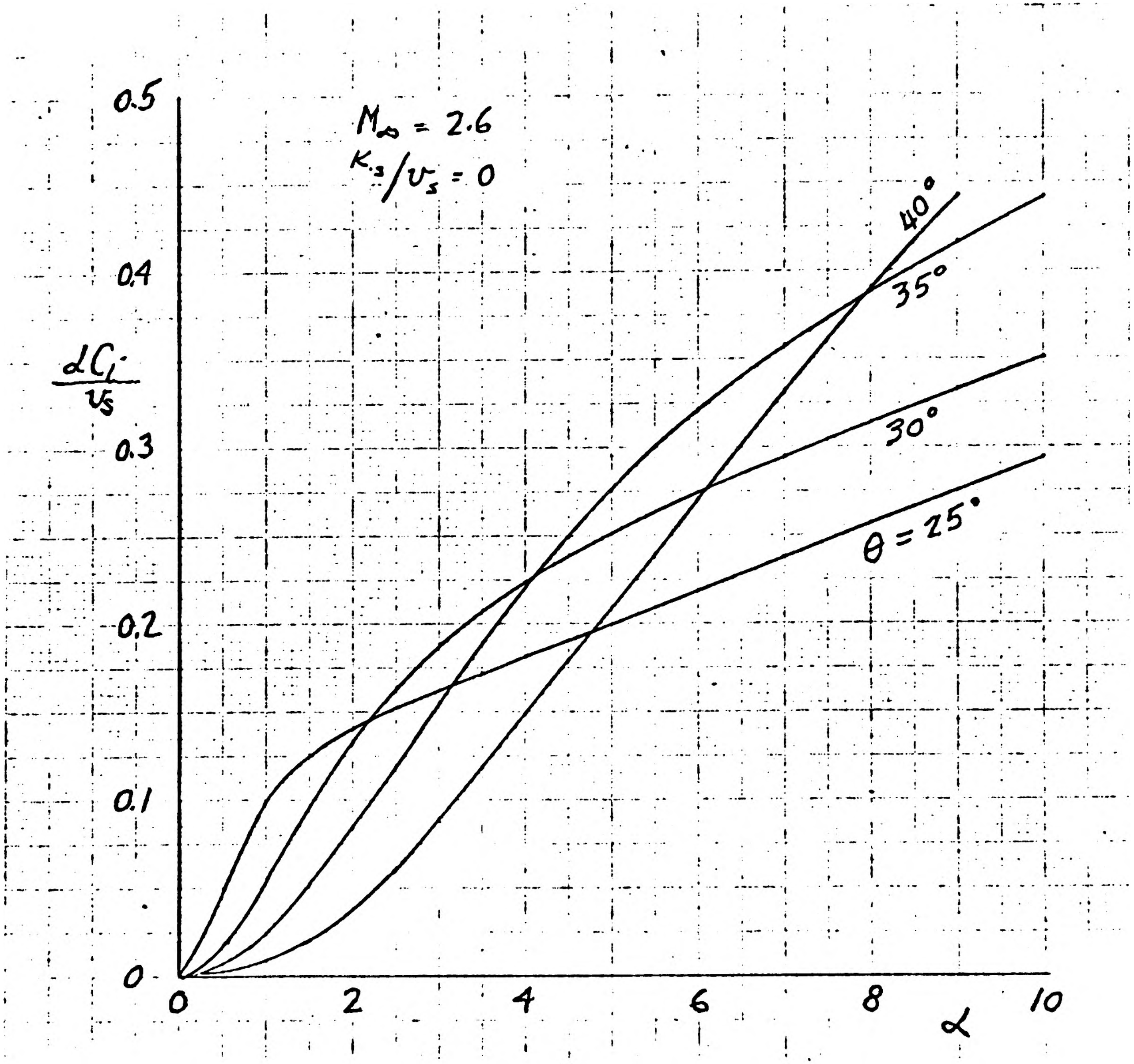


Fig. V-1 AMPLIFICATION RATES AGAINST WAVE NUMBER

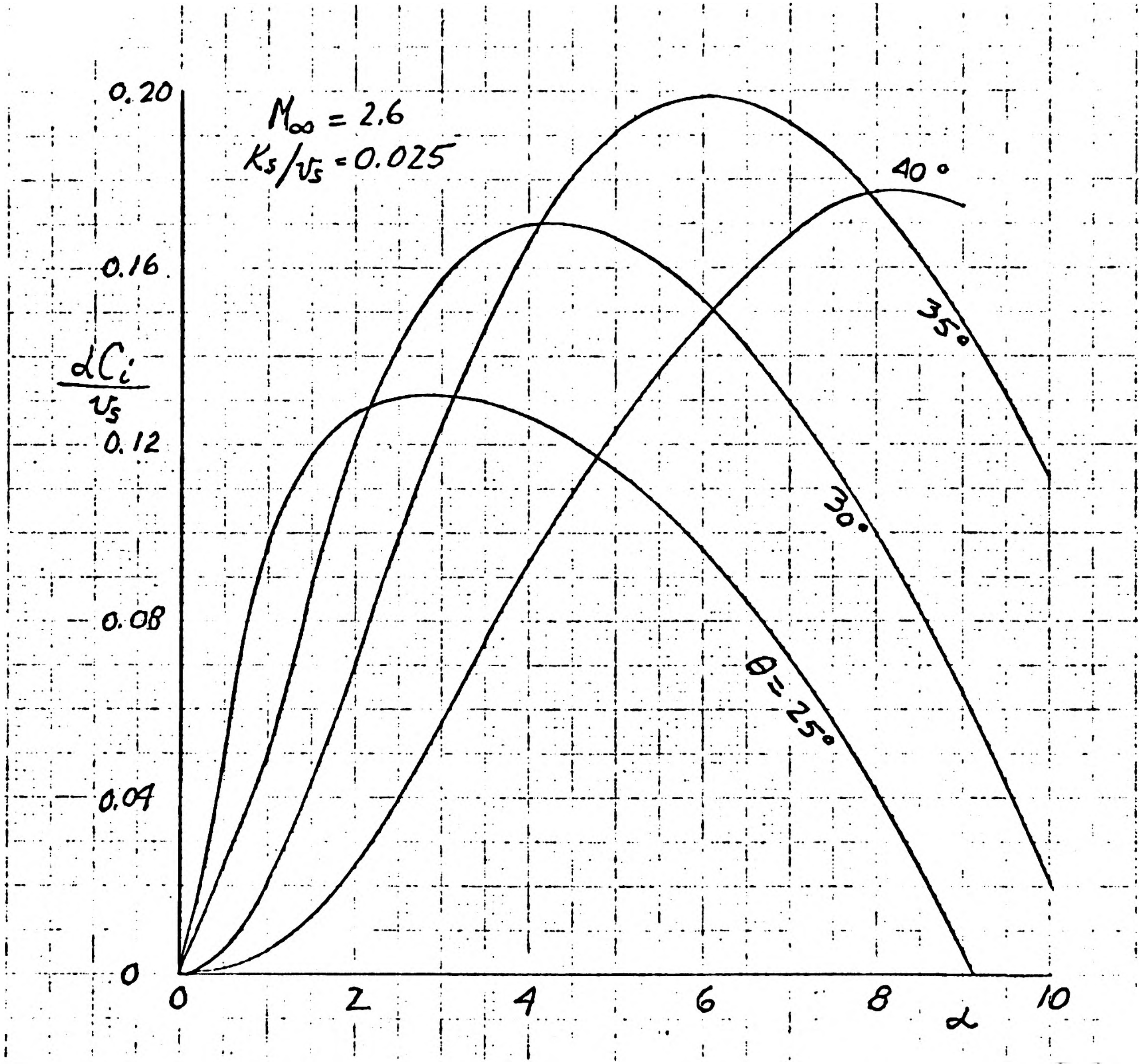


Fig. V-1 AMPLIFICATION RATES AGAINST WAVE NUMBER

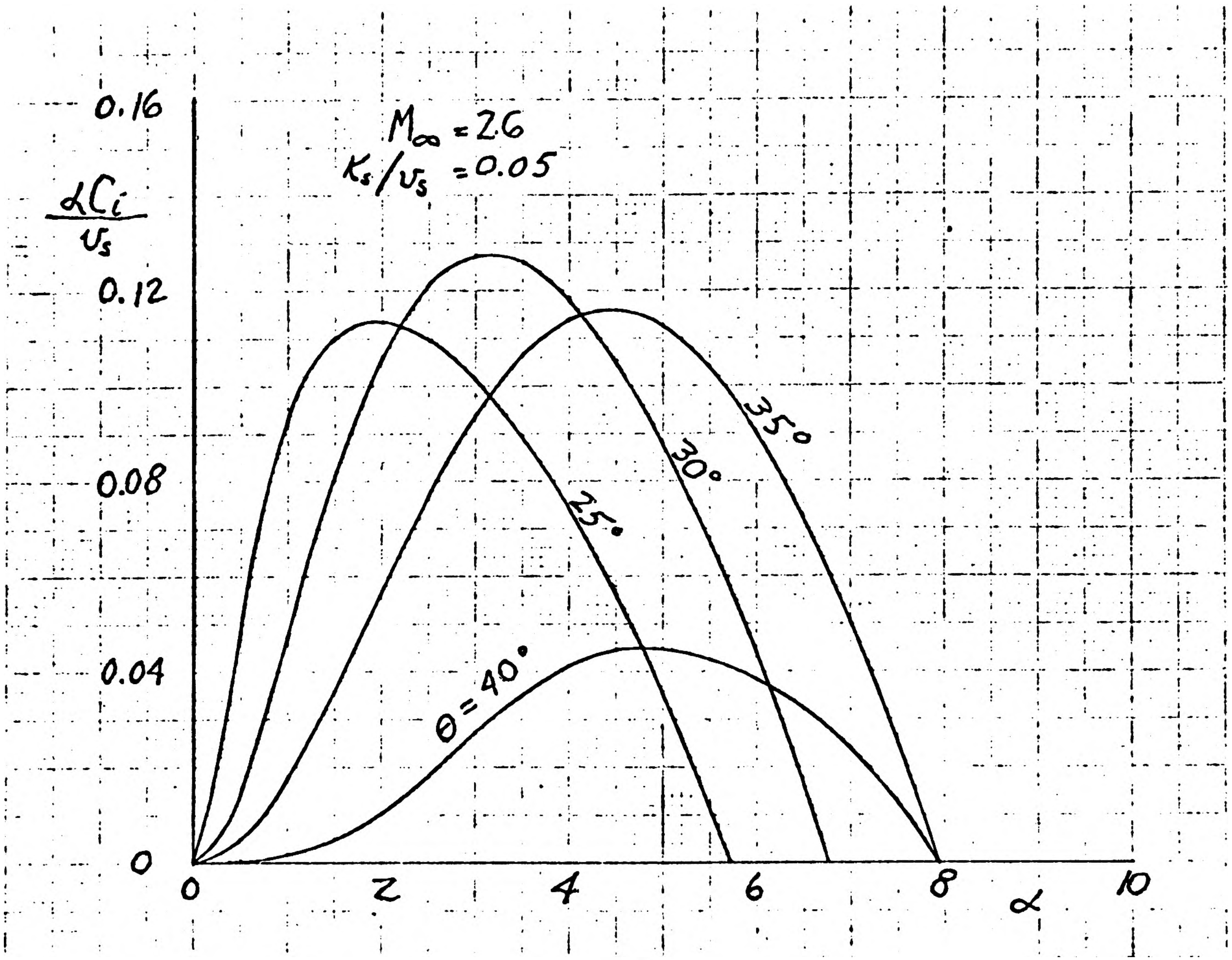
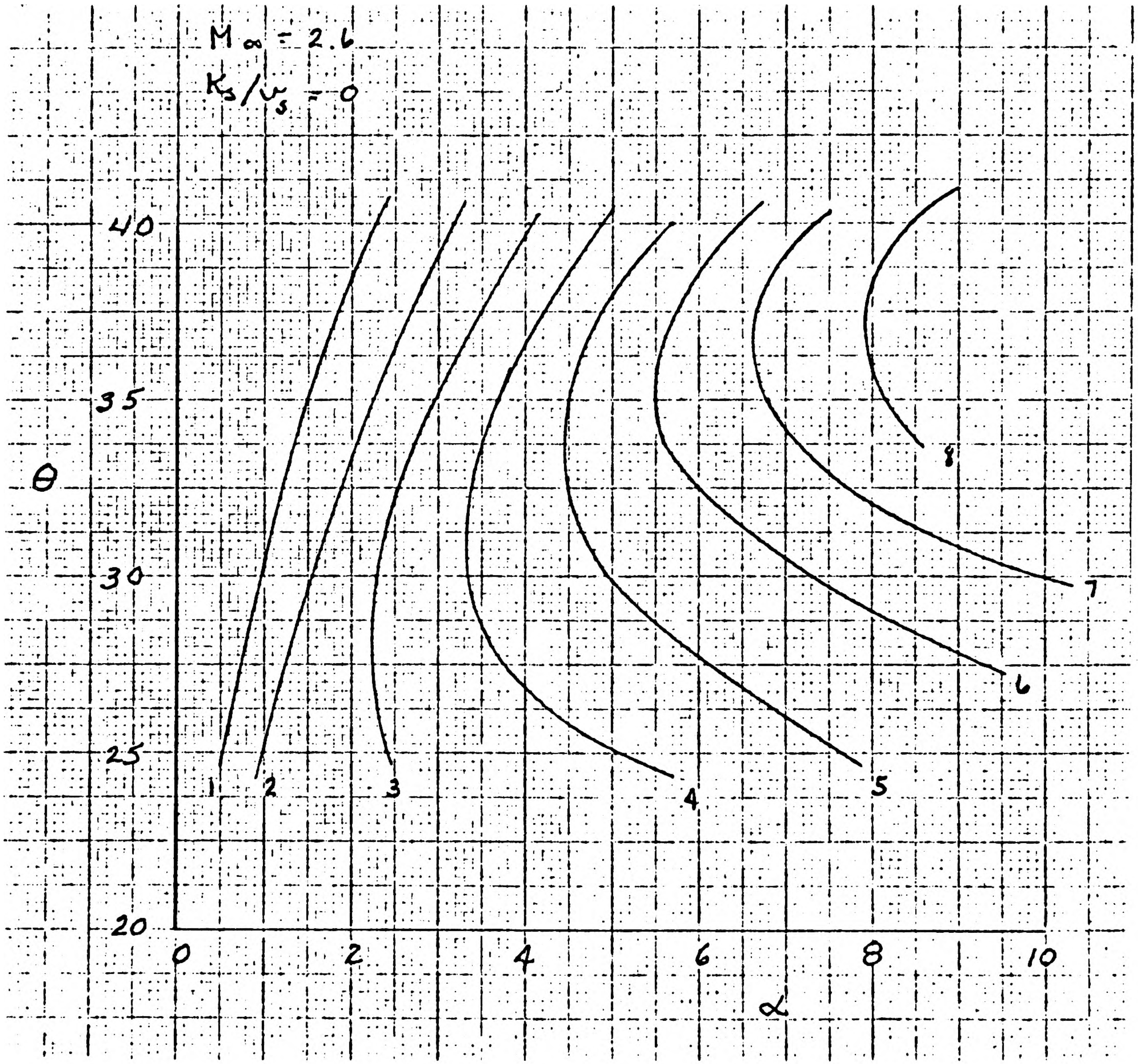


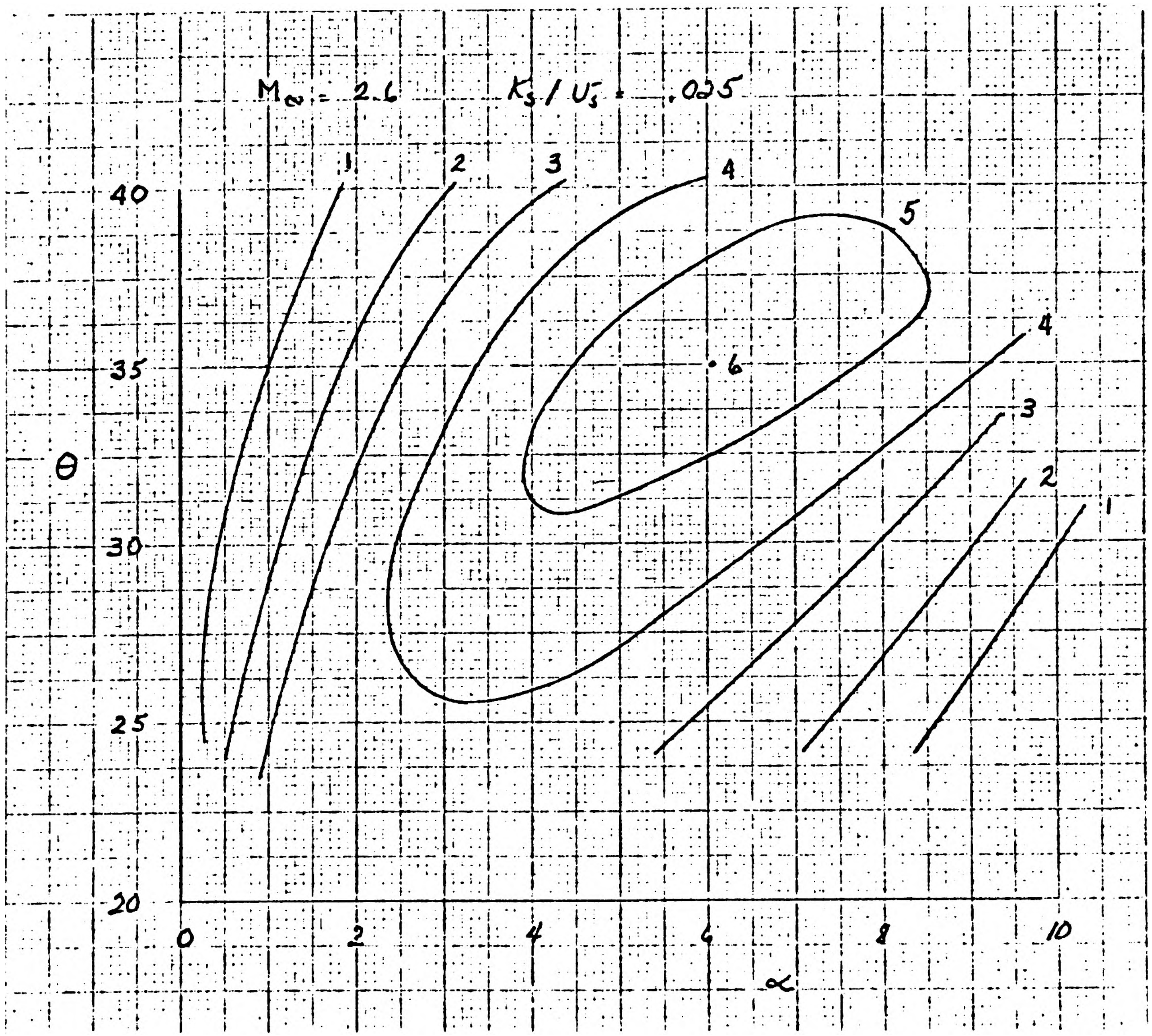
Fig. V-1 AMPLIFICATION RATES AGAINST WAVE NUMBER



$\alpha c_i/V_s$

1. .05
2. .10
3. .15
4. .20
5. .25
6. .30
7. .35
8. .40

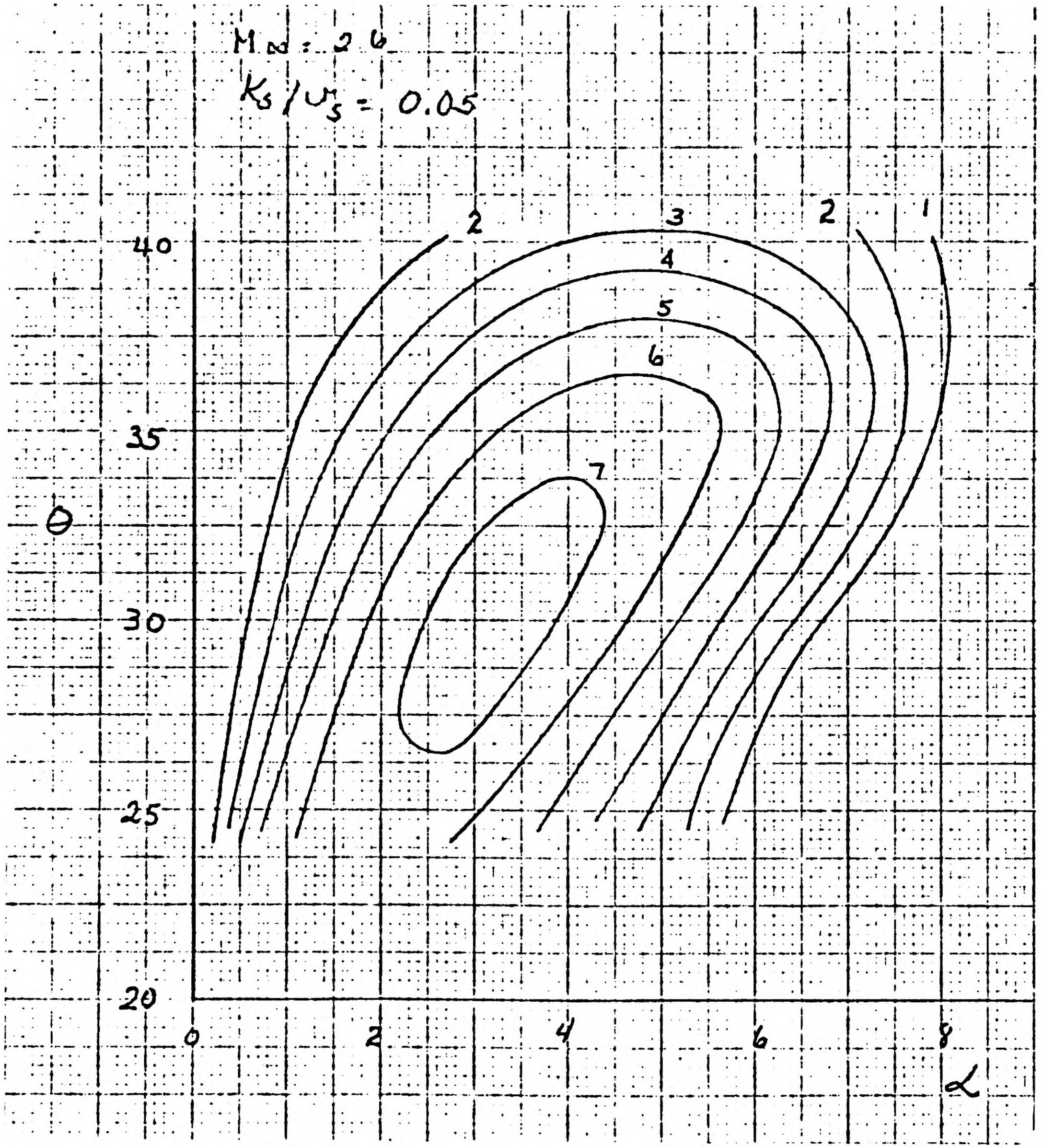
Fig. V-2 CONTOURS OF CONSTANT AMPLIFICATION



$\alpha c_i / V_s$

1. .02
2. .06
3. .10
4. .14
5. .18
6. .20

Fig. V-2 CONTOURS OF CONSTANT AMPLIFICATION



$\alpha c_i/V_s$

1. 0
2. .02
3. .04
4. .06
5. .08
6. .10
7. .12

Fig. V-2 CONTOURS OF CONSTANT AMPLIFICATION

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1. ORIGINATING ACTIVITY (Corporate author) California Institute of Technology Pasadena, California 91109	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Stability Theory for Cross Hatching,Part I.Linear stability Theory

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Final

5. AUTHOR(S) (Last name, first name, initial)

Lester Lees, Toshi Kubota, Denny R-S Ko

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
April 1972	94	8

8a. CONTRACT OR GRANT NO. F 04701-68-C-0151 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

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A linear instability theory was developed which couples small perturbations in surface shape of a subliming surface and resulting perturbations in aerodynamic heat-transfer rates. Equations governing compressible turbulent boundary layers were linearized for small perturbations whose streamwise scale lengths are comparable with the undisturbed boundary layer thickness. For turbulent shear stress, the mixing-length approximation was employed, and the turbulent Prandtl number was assumed to be unity for turbulent heat flux. Heat conduction within the ablating solid was analyzed by linearizing the boundary condition small amplitude. Sample computations were made for a teflon surface ablating under turbulent boundary layer at Mach 2.6, and the results indicate that the small surface perturbations are unstable within certain regions in the wave number - wave angle space.

DD FORM 1 JAN 64 1473

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14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
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