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All Tree Amplitudes of 6D (2, 0) Supergravity: Interacting Tensor Multiplets and the K3 Moduli Space

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We present a twistor-like formula for the complete tree-level S matrix of 6D (2, 0) supergravity coupled to 21 abelian tensor multiplets. This is the low-energy effective theory that corresponds to Type IIB superstring theory compactified on a K3 surface. The formula is expressed as an integral over the moduli space of certain rational maps of the punctured Riemann sphere. By studying soft limits of the formula, we are able to explore the local moduli space of this theory, $\frac{SO(5,21)}{SO(5)\times SO(21)}$. Finally, by dimensional reduction, we also obtain a new formula for the tree-level S matrix of 4D $\mathcal{N}=4$ Einstein-Maxwell theory.

INTRODUCTION

To describe scattering amplitudes of supersymmetric theories in higher dimensions, [1, 2] introduced a six-dimensional rational map formalism in the spirit of [3–5]. Using this formalism, extremely compact formulas were found for n-point scattering amplitudes of a wide range of interesting theories, including maximally supersymmetric gauge theories and supergravity in diverse dimensions, as well as Dirac-Born-Infeld type world-volume theories of probe D branes and the M5 brane in flat space. In the case of the M5 brane [1], which contains a self-dual (chiral) 2-form tensor field, the formalism circumvents a common difficulty in formulating a covariant action principle due to the self-duality constraint.

In this article, we continue to explore the utility of the 6D rational maps and spinor-helicity formalism and present the tree-level S matrix for the theory of 6D (2, 0) supergravity. This chiral theory arises as the low-energy limit of Type IIB string theory compactified on a K3 surface [6] and is particularly interesting because it describes the interaction of self-dual 2-form gauge fields and gravitons.

To describe massless scattering in 6D, it is convenient to introduce spinor-helicity variables [7] that solve the on-shell constraint and carry chiral little group data,

$$p_i^{AB} = \lambda^A_{i,a}\lambda^B_{i,b}\epsilon^{ab} := \langle \lambda_i^A\, \lambda_i^B\rangle. \tag{1}$$

Here, and throughout, $i = 1, \dots, n$ labels the n particles, $A = 1, 2, 3, 4$ is a spinor index of the $Spin(5, 1)$ Lorentz group, and $a = 1, 2$ is a left-handed index of the $SU(2)_L \times SU(2)_R$ massless little group. This is the only non-trivial little-group information that enters for (2, 0) supersymmetry, and the supermultiplets are chiral–the (2, 0) supergravity multiplet and a number of (2, 0) tensor multiplets which contain a self dual 2-form $H = dB$, $H = *H$. The tensor multiplets transform as singlets of $SU(2)_R$, whereas the gravity multiplet is as a triplet; later we will introduce the doublet index $\hat{a}$ for $SU(2)_R$.

We also introduce a flavor index $f_i = 1, \dots, 21$ to label the 21 tensor multiplets; this is the number that arises in 6D from compactification of the various NS and R fields of Type IIB superstring theory on a K3 surface. It is also the unique number for which the gravitational anomalies cancel [8]. Therefore for 21 tensor multiplets, the theory is anomaly free and has a UV completion as a string theory. We assume that we are at generic points of the moduli space, where perturbative amplitudes are well-defined [9]. In this abelian theory, the tensor multiplets only interact via graviton exchange. In particular, there are no interactions that change the flavor.

In the rational-map formulation, amplitudes for n particles are expressed as integrals over the moduli space of rational maps from the n-punctured Riemann sphere to the space of spinor-helicity variables. In general, the amplitudes take the following form [1, 2, 10],

$$A_n^{6D} = \int d\mu_n^{\rm 6D}\, \mathcal{I}_L\, \mathcal{I}_R\,, \tag{2}$$

where $d\mu_n^{\rm 6D}$ is the measure encoding the 6D kinematics and the product $\mathcal{I}_L\,\mathcal{I}_R$ is the integrand that contains the dynamical information of the theories, including supersymmetry. The measure is given by

$$d\mu_n^{\rm 6D} = \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^m d^8\rho_k}{{\rm vol}\,({\rm SL}(2,\mathbb{C})_\sigma \times {\rm SL}(2,\mathbb{C})_\rho)} \frac{1}{V_n^2} \prod_{i=1}^n E_i^{\rm 6D}, \tag{3}$$

and $n = 2m+2$ (we will discuss $n = 2m+1$ later). The coordinates σ_i label the n punctures. They are determined up to an overall ${\rm SL}(2,\mathbb{C})_\sigma$ Möbius group transformation, whose "volume" is divided out in a standard way. Also,

$$E_i^{\rm 6D} = \delta^6 \left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i)\, \rho^B(\sigma_i)\rangle}{\prod_{j\neq i}\sigma_{ij}} \right), \tag{4}$$

with $\sigma_{ij}=\sigma_i-\sigma_j$. These maps are given by degree-m polynomials

$$\rho_a^A(\sigma)=\sum_{k=0}^{m}\rho_{a,k}^A\,\sigma^k, \tag{5}$$

which are determined up to an overall $\mathrm{SL}(2,\mathbb{C})_\rho$ transformation, whose volume is divided out. This group is a complexification of $SU(2)_L$. This construction implies, in particular, that the integrals are completely localized on the $(n-3)!$ solutions of the scattering equations [10]. It is an important but non-trivial fact that the bosonic constraints $E_i^{6\mathrm{D}}$ impose the on-shell condition, conservation of momentum, and the scattering equations.

Now consider $n=2m+1$, for which we have [2],

$$d\mu_n^{6\mathrm{D}}=\frac{\left(\prod_{i=1}^n d\sigma_i\prod_{k=0}^{m-1}d^8\rho_k\right)\,d^4\omega\,\langle\xi d\xi\rangle}{\mathrm{vol}\;(\mathrm{SL}(2,\mathbb{C})_\sigma,\mathrm{SL}(2,\mathbb{C})_\rho,\mathrm{T})}\,\frac{1}{V_n^2}\prod_{i=1}^n E_i^{6\mathrm{D}}. \tag{6}$$

The polynomials $\rho_a^A(\sigma)$ now are given by

$$\rho_a^A(\sigma)=\sum_{k=0}^{m-1}\rho_{a,k}^A\,\sigma^k+\omega^A\xi_a\,\sigma^m, \tag{7}$$

and there is a shift symmetry $\mathrm{T}(\alpha)$ acting on ω^A: $\omega^A\to\omega^A+\alpha\langle\xi\rho_{m-1}^A\rangle$, which we also have to mod out.

We will now review the integrand factors for non-chiral 6D $(2,2)$ supergravity, since they are relevant for this article. For $(2,2)$ supergravity, we have,

$$\mathcal{I}_L=\det{}'S_n,\quad \mathcal{I}_R=\Omega_F^{(2,2)}, \tag{8}$$

where S_n is a $n\times n$ matrix, with entries given as following,

$$[S_n]_{ij}=\begin{cases}\frac{p_i\cdot p_j}{\sigma_{ij}} & \text{if}\quad i\neq j\\ 0 & \text{if}\quad i=j\end{cases} \tag{9}$$

The matrix has rank $(n-2)$, and the reduced Pfaffian and determinant are defined as

$$\mathrm{Pf}'S_n=\frac{(-1)^{i+j}}{\sigma_{ij}}\mathrm{Pf}S_{ij}^{ij},\quad \det{}'S_n=\left(\mathrm{Pf}'S_n\right)^2. \tag{10}$$

Here $S_{i,j}^{i,j}$ means that the i-th and j-th rows and columns of S_n are removed, and the reduced Pfaffian is independent the choice of i,j [11]. $\Omega_F^{(2,2)}$ is a fermionic function of Grassmann coordinates $\eta_i^{Ia},\tilde{\eta}_i^{\hat{I}\hat{a}}$, which we use to package the supermultiplet of on-shell states into a 'superfield',

$$\begin{aligned}\Phi^{(2,2)}(\eta,\tilde{\eta})\;&=\;\phi+\cdots+\eta_a^I\eta_{I,b}\tilde{\eta}_{\hat{a}}^{\hat{I}}\tilde{\eta}_{\hat{I},\hat{b}}G^{ab\hat{a}\hat{b}}\\&\quad+\cdots+(\eta)^4(\tilde{\eta})^4\bar{\phi},\end{aligned} \tag{11}$$

where $G^{ab\hat{a}\hat{b}}$ is the graviton. Here $I,\hat{I}=1,2$ are the R-symmetry indices corresponding to a $SU(2)\times SU(2)$ subgroup of the full $USp(4)\times USp(4)$ R-symmetry. The fermionic function $\Omega_F^{(2,2)}$ imposes the conservation of supercharge, which may be viewed as a double copy,

$$\Omega_F^{(2,2)}=\Omega_F^{(2,0)}\,\Omega_F^{(0,2)}, \tag{12}$$

with

$$\Omega_F^{(2,0)}=V_n\prod_{k=0}^{m}\delta^4\left(\sum_{i=1}^n C_{a,k;i,b}\,\eta_i^{Ib}\right), \tag{13}$$

where $V_n=\prod_{i<j}\sigma_{ij}$ is the Vandermonde determinant. The $n\times 2n$ matrices $C_{a,k;i,b}=(W_i)_a^b\sigma_i^k$ and $(W_i)_a^b$ can be expressed in terms of $\rho_a^A(\sigma_i)$ via

$$p_i^{AB}W_{i,b}^a=\frac{\rho^{[A,a}(\sigma_i)\,\lambda_{i,b}^{B]}}{\prod_{j\neq i}\sigma_{ij}}, \tag{14}$$

which is independent of the choice of A,B, and satisfies $\det W_i=\prod_{j\neq i}\sigma_{ij}^{-1}$. The matrix $C_{a,k;i,b}$ is a symplectic Grassmannian which was used in [2] as an alternative way to impose the 6D scattering equations. $\Omega_F^{(0,2)}$ is the conjugate of $\Omega_F^{(2,0)}$, and the definition is identical to $\Omega_F^{(2,0)}$ with the understanding that we use the right-handed variables, such as $\tilde{\eta}_{\hat{a}}^{\hat{I}},\tilde{\lambda}_{\hat{A}\hat{a}},\tilde{\rho}_{\hat{A}\hat{a}},(\tilde{W}_i)_{\hat{a}}^{\hat{b}}$, etc.

For $n=2m+1$, the integrands take a slightly different form. For the fermionic part, we have

$$\begin{aligned}\Omega_F^{(2,0)}\;&=V_n\prod_{k=0}^{m-1}\delta^4\left(\sum_{i=1}^n C_{a,k;i,b}\,\eta_i^{Ib}\right)\\&\quad\times\delta^2\left(\sum_{i=1}^n\xi^aC_{a,m;i,b}\,\eta_i^{Ib}\right).\end{aligned} \tag{15}$$

whereas the $n\times n$ matrix S_n is modified to an $(n+1)\times(n+1)$ matrix, which we denote $\widehat{S}_n$. It is given by

$$[\widehat{S}_n]_{ij}=\begin{cases}\dfrac{p_i\cdot p_j}{\sigma_{ij}} & \text{if}\quad i\neq j,\\[2ex] \;\;0 & \text{if}\quad i=j,\end{cases} \tag{16}$$

where $i,j=1,2,\ldots,n,\star$, $\sigma_\star$ is a reference puncture, and the reference momentum $p_\star$ is given by

$$p_\star^{AB}=\frac{2\,q^{[A}p^{B]C}(\sigma_\star)\tilde{q}_C}{q^D[\tilde{\rho}_D(\sigma_\star)\,\tilde{\xi}]\langle\rho^E(\sigma_\star)\,\xi\rangle\tilde{q}_E}, \tag{17}$$

and q and $\tilde{q}$ are arbitrary spinors.

6D $(2,0)$ SUPERGRAVITY

Let us now consider the 6D $(2,0)$ supergravity theory that contains 21 tensor multiplets and the graviton multiplet. The superfield of the $(2,0)$ tensor multiplet is a singlet of the $SU(2)_L\times SU(2)_R$ little group, i.e., the superfield is a scalar,

$$\begin{aligned}\Phi(\eta)\;&=\;\phi+\eta_a^I\psi_I^a+\eta_a^1\eta_b^2B^{ab}\\&\quad+\eta^{1a}\eta_a^2\phi_{12}+\ldots+(\eta)^4\bar{\phi},\end{aligned} \tag{18}$$

where $I, J = 1, 2$ for $(2,0)$ supersymmetry, and $a, b = 1, 2$ are the $SU(2)_L$ little-group indices as before. The graviton multiplet transforms as a $(\mathbf{1}, \mathbf{3})$ of the little group, so the superfield carries explicit $SU(2)_R$ indices,

$$\Phi_{\hat{a}\hat{b}}(\eta) = B_{\hat{a}\hat{b}} + \ldots + \eta^I_a \eta_{I,b} G^{ab}_{\hat{a}\hat{b}} + \ldots + (\eta)^4 \bar{B}_{\hat{a}\hat{b}}\,, \quad (19)$$

and $\Phi_{\hat{a}\hat{b}}(\eta) = \Phi_{\hat{b}\hat{a}}(\eta)$. We see that both the tensor multiplet $\Phi(\eta)$ and graviton multiplet $\Phi_{\hat{a}\hat{b}}(\eta)$ can be obtained from 6D $(2,2)$ superfield in (11) via SUSY reductions [12][13],

$$\begin{aligned} \Phi(\eta) &= \int d\tilde{\eta}^{\hat{1}}_{\hat{a}} d\tilde{\eta}^{\hat{2}\,\hat{a}}\, \Phi^{(2,2)}(\eta, \tilde{\eta})\big|_{\tilde{\eta}\to 0}\,, \\ \Phi_{\hat{a}\hat{b}}(\eta) &= \int d\tilde{\eta}^{\hat{I}}_{\hat{a}} d\tilde{\eta}_{\hat{I}\hat{b}}\, \Phi^{(2,2)}(\eta, \tilde{\eta})\big|_{\tilde{\eta}\to 0}\,. \end{aligned} \quad (20)$$

These integrals have the effect of projecting onto the right-handed $USp(4)$ R-symmetry singlet sector, which reduces $(2,2) \to (2,0)$. Using the reduction, the amplitudes of $(2,0)$ supergravity with n_1 supergravity multiplets and n_2 tensor multiplets of the same flavor ($n_1 + n_2 = n$) can be obtained from the $(2,2)$ supergravity amplitude directly via

$$A^{(2,0)}_{n_1,n_2}(\eta) = \int \prod_{i\in n_1} d\tilde{\eta}^{\hat{I}}_{i,\hat{a}_i} d\tilde{\eta}_{i,\hat{I}\hat{b}_i} \prod_{j\in n_2} d\tilde{\eta}^{\hat{1}}_{j,\hat{a}_j} d\tilde{\eta}^{\hat{2}\,\hat{a}_j}_j\, A^{(2,2)}_n(\eta, \tilde{\eta}).$$

Note $A^{(2,2)}_n(\eta, \tilde{\eta}) \sim \eta^{2n}\tilde{\eta}^{2n}$, so the integration removes all $\tilde{\eta}$'s. The fermionic integral can be performed straightforwardly using (8), and (13) (or (15) for odd n), and we obtain

$$A^{(2,0)}_{n_1,n_2}(\eta) = \int d\mu^{\rm 6D}_n\, \tilde{M}^{n_1 n_2}_{\hat{a}\hat{b}}\, V_n\, {\det}' S_n\, \Omega^{(2,0)}_F, \quad (21)$$

where $\tilde{M}^{n_1 n_2}_{\hat{a}\hat{b}}$, which we will define shortly, is obtained by integrating out $\Omega^{(0,2)}_F$.

We begin with n even, as the odd-n case works in a similar fashion. Introducing the $n \times n$ matrix

$$\tilde{M}_{\hat{a}_1\cdots\hat{a}_n} = \begin{pmatrix} \tilde{C}_{\hat{1},0;1,\hat{a}_1} & \tilde{C}_{\hat{1},0;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{1},0;n,\hat{a}_n} \\ \tilde{C}_{\hat{1},1;1,\hat{a}_1} & \tilde{C}_{\hat{1},1;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{1},1;n,\hat{a}_n} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{C}_{\hat{1},m;1,\hat{a}_1} & \tilde{C}_{\hat{1},m;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{1},m;n,\hat{a}_n} \\ \tilde{C}_{\hat{2},0;1,\hat{a}_1} & \tilde{C}_{\hat{2},0;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{2},0;n,\hat{a}_n} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{C}_{\hat{2},m;1,\hat{a}_1} & \tilde{C}_{\hat{2},m;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{2},m;n,\hat{a}_n} \end{pmatrix}, \quad (22)$$

$\tilde{M}^{n_1 n_2}_{\hat{a}\hat{b}}$ is given by

$$\tilde{M}^{n_1 n_2}_{\hat{a}\hat{b}} = \det \tilde{M}_{\hat{a}_1\cdots\hat{a}_n} \det \tilde{M}_{\hat{b}_1\cdots\hat{b}_n}. \quad (23)$$

The indices $\hat{a}_i, \hat{b}_i$ are contracted if $i \in n_2$, whereas for $j \in n_1$ we symmetrize $\hat{a}_j, \hat{b}_j$. This corresponds to constructing little-group singlets for tensors and triplets for gravitons. After the contraction and symmetrization, the result of (23) simplifies drastically [14]

$$\tilde{M}^{n_1 n_2}_{\hat{a}\hat{b}} \to \frac{{\rm Pf} X_{n_2}}{V_{n_2}}\, \tilde{M}^{n_1 0}_{\hat{a}\hat{b}}\,, \quad (24)$$

where X_{n_2} is a $n_2 \times n_2$ anti-symmetric matrix given by

$$[X_{n_2}]_{ij} = \begin{cases} \frac{1}{\sigma_{ij}} & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j\,, \end{cases} \quad (25)$$

and $\tilde{M}^{n_1 0}_{\hat{a}\hat{b}}$ only contains the graviton multiplets. Let us remark that the simplification (24) (especially the appearance of ${\rm Pf} X_{n_2}$) will be crucial for the generalization to amplitudes with multiple tensor flavors which is relevant for IIB superstring theory on K3.

At this point in the analysis, we obtain the tree-level S matrix of 6D $(2,0)$ supergravity with a single flavor of tensor multiplets:

$$A^{(2,0)}_{n_1,n_2}(\eta) = \int d\mu^{\rm 6D}_n \frac{{\rm Pf} X_{n_2}}{V_{n_2}}\, \tilde{M}^{n_1 0}_{\hat{a}\hat{b}}\, V_n\, {\det}' S_n\, \Omega^{(2,0)}_F. \quad (26)$$

For odd n, the matrix $\tilde{M}_{\hat{a}_1\cdots\hat{a}_n}$ is given by

$$\tilde{M}_{\hat{a}_1\cdots\hat{a}_n} = \begin{pmatrix} \tilde{\xi}^{\hat{b}}\tilde{C}_{\hat{b},m;1,\hat{a}_1} & \tilde{\xi}^{\hat{b}}\tilde{C}_{\hat{b},m;2,\hat{a}_2} & \cdots & \tilde{\xi}^{\hat{b}}\tilde{C}_{\hat{b},m;n,\hat{a}_n} \\ \tilde{C}_{\hat{1},0;1,\hat{a}_1} & \tilde{C}_{\hat{1},0;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{1},0;n,\hat{a}_n} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{C}_{\hat{1},m-1;1,\hat{a}_1} & \tilde{C}_{\hat{1},m-1;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{1},m-1;n,\hat{a}_n} \\ \tilde{C}_{\hat{2},0;1,\hat{a}_1} & \tilde{C}_{\hat{2},0;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{2},0;n,\hat{a}_n} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{C}_{\hat{2},m-1;1,\hat{a}_1} & \tilde{C}_{\hat{2},m-1;2,\hat{a}_2} & \cdots & \tilde{C}_{\hat{2},m-1;n,\hat{a}_n} \end{pmatrix}$$

and the amplitudes take the same form

$$A^{(2,0)}_{n_1,n_2}(\eta) = \int d\mu^{\rm 6D}_n \frac{{\rm Pf} X_{n_2}}{V_{n_2}}\, \tilde{M}^{n_1 0}_{\hat{a}\hat{b}}\, V_n\, {\det}' \hat{S}_n\, \Omega^{(2,0)}_F. \quad (27)$$

The factor ${\rm Pf} X_{n_2}$ requires the non-vanishing amplitudes to contain an even number of tensors, as expected since they come in pairs. We have verified that this formula produces the correct low-point amplitudes. For instance, at 4 point one can check the tensor and graviton-tensor amplitudes [8]:

$$\begin{aligned} A^{(2,0)}_{0,4}(\eta) &= \delta^8(Q)\left(\frac{1}{s_{12}} + \frac{1}{s_{23}} + \frac{1}{s_{31}}\right), \quad (28) \\ A^{(2,0)}_{2,2}(\eta) &= \frac{\delta^8(Q)[1_{\hat{a}_1} 2_{\hat{a}_2} 3_{\hat{a}_3} 4_{\hat{a}_4}][1^{\hat{a}_1} 2^{\hat{a}_2} 3_{\hat{b}_3} 4_{\hat{b}_4}]}{s_{12}\, s_{23}\, s_{31}} + {\rm sym}\,, \end{aligned}$$

where $[1_{\hat{a}_1} 2_{\hat{a}_2} 3_{\hat{a}_3} 4_{\hat{a}_4}] = \epsilon_{ABCD} \lambda^A_{1\hat{a}_1} \lambda^B_{2\hat{a}_2} \lambda^C_{3\hat{a}_3} \lambda^D_{4\hat{a}_4}$, and $\delta^8(Q) = \delta^8(\sum_{i=1}^4 \lambda^A_{i,a} \eta^a_i)$ is a fermionic delta function that imposes the conservation of the total supercharge.

Multiple flavors

As we have emphasized, the simplification (24) is crucial for the generalization to multiple tensor flavors, which is required for the 6D $(2,0)$ supergravity. Indeed, the formula takes a form very similar to that of a (non-supersymmetric) Einstein-Maxwell theory worked out by Cachazo, He and Yuan [11]. In that case, in passing from single-$U(1)$ photons to multiple-$U(1)$ ones, one simply replaced the matrix X_n by $\mathcal{X}_n$ [11],

$$[\mathcal{X}_n]_{ij} = \begin{cases} \frac{\delta_{f_i f_j}}{\sigma_{ij}} & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j \end{cases} \tag{29}$$

which allows the introduction of multiple distinct flavors: namely, f_i, f_j are flavor indices, and $\delta_{f_i f_j} = 1$ if particles i, j are of the same flavor, otherwise $\delta_{f_i f_j} = 0$. Inspired by this result, we are led to a proposal for the complete tree-level S matrix of 6D $(2,0)$ supergravity with multiple flavors of tensor multiplets:

$$\boxed{A^{(2,0)}_{n_1,n_2}(\eta) = \int d\mu^{\rm 6D}_n \, \frac{{\rm Pf}\mathcal{X}_{n_2}}{V_{n_2}} \, \tilde{M}^{n_1 0}_{\hat{a}\hat{b}} \, V_n \, \det{}' S_n \, \Omega^{(2,0)}_F .} \tag{30}$$

Again, the 6D scattering equations and integrands take different forms depending on whether n is even or odd. Since n_2 is necessarily even, this is equivalent to distinguishing whether n_1 is even or odd. Also, as we will show later, when reduced to 4D the proposed formula reproduces the amplitudes of (supersymmetric) Einstein-Maxwell theory, which is one of the consistency checks of our result.

Equation (30) is our main result, which is a localized integral formula that describes all tree-level superamplitudes of abelian tensor multiplets (with multiple flavors) coupled to gravity multiplets. We can verify that it has all the correct properties. For instance, due to the fact that all the building blocks of the formulas come from either 6D $(2,2)$ supergravity or Einstein-Maxwell theory, they all behave properly in the factorization limits, and transform correctly under the Möbius symmetry of the n-punctured Riemann sphere. It also reproduces explicit amplitudes: for instance, the 4-point superamplitude of tensors [15]

$$A^{(2,0)}_{0,4}(\eta) = \delta^8(Q) \left(\frac{\delta^{f_1 f_2} \delta^{f_3 f_4}}{s_{12}} + \frac{\delta^{f_1 f_3} \delta^{f_2 f_4}}{s_{13}} + \frac{\delta^{f_2 f_3} \delta^{f_1 f_4}}{s_{23}} \right),$$

or higher-point amplitudes such as the amplitude for six scalars belonging to tensor multiplets,

$$\begin{aligned} A(\phi^{f_1}, \phi^{f_1}_{12}, \bar\phi^{f_2}, \bar\phi^{f_2}, \phi^{f_3}, \phi^{f_3}_{12}) &= \left[\frac{s_{23} s_{46}}{s_{123}} + \frac{s_{24} s_{36}}{s_{124}} \right] \\ &\quad - \left[\frac{s_{34} s_{61}}{s_{612}} + \frac{s_{34} s_{25}}{s_{134}} \right] + s_{34} , \end{aligned}$$

where we assume that the flavors f_1, f_2, f_3 are all different.

THE K3 MODULI SPACE FROM SOFT LIMITS

Type IIB string theory compactified on K3 has a well studied moduli space described by the coset [16],

$$\mathcal{M}_{(2,0)} = SO(5,21;\mathbb{Z}) \backslash SO(5,21)/(SO(5) \times SO(21)) .$$

The discrete group is invisible at the level of tree level field theory, so we concern ourselves with the local form of the moduli space of 6D $(2,0)$ supergravity, which is $\frac{SO(5,21)}{SO(5)\times SO(21)}$. The dimension of this coset is 105, which corresponds precisely to the 105 scalars contained in the 21 tensor multiplets. These scalars are Goldstone bosons which describe the breaking of the non-compact group to the compact $SO(5)\times SO(21)$, which are the R-symmetry and flavor symmetry, respectively. Therefore, the scalars obey soft theorems [17], which are the tools to explore the structure of the moduli space of the theory directly from the S matrix.

As expected, we find that the amplitudes behave like pion amplitudes with "Adler's zero" [18] in the single soft limit. For $p_1 \to 0$, we have,

$$A_n(\phi^{f_1}_1, 2, \ldots, n) \to \mathcal{O}(p_1) , \tag{31}$$

and the same for other scalars in the tensor multiplet. The commutator algebra of the coset space may be explored by considering double soft limits for scalars. Beginning with the flavor symmetry, we find for $p_1, p_2 \to 0$

$$A_n(\phi^{f_1}_1, \bar\phi^{f_2}_2, \ldots) \to \frac{1}{2} \sum_{i=3}^{n} \frac{p_i \cdot (p_1 - p_2)}{p_i \cdot (p_1 + p_2)} R^{f_1 f_2}_i A_{n-2} , \tag{32}$$

where f_i's are flavor indices, and $R^{f_1 f_2}_i$ is a generator of the unbroken $SO(21)$, which may be viewed as the result of the commutator of two broken generators. $R^{f_1 f_2}_i$ acts on superfields as

$$\begin{aligned} R^{f_1 f_2}_i \Phi^{f_2}_i &= \Phi^{f_1}_i , \quad R^{f_1 f_2}_i \Phi^{f_1}_i = -\Phi^{f_2}_i , \\ R^{f_1 f_2}_i \Phi^{f_3}_i &= 0, \qquad R^{f_1 f_2}_i \Phi_{i,\hat{a}\hat{b}} = 0, \end{aligned} \tag{33}$$

where $f_3 \neq f_1, f_2$. Therefore, this flavor symmetry generator exchanges tensor multiplets of flavor f_1 with ones of f_2, and sends all other tensor multiplets and the graviton multiplet to 0. We can also consider the $SO(5)$ R-symmetry generators by taking soft limits of two scalars which do not form a R-symmetry singlet. For instance

$$A_n(\phi_1, \phi^{IJ}_2, \ldots) \to \frac{1}{2} \sum_{i=3}^{n} \frac{p_i \cdot (p_1 - p_2)}{p_i \cdot (p_1 + p_2)} R_{i,IJ} A_{n-2} , \tag{34}$$

with

$$R_{i,IJ} = \frac{\partial}{\partial \eta^I_{i,a}} \frac{\partial}{\partial \eta^{J,a}_i} . \tag{35}$$

Similarly, other choices of soft scalars lead to the remaining R-symmetry generators:

$$R^{IJ}_i = \eta^I_{i,a} \, \eta^{J,a}_i , \quad R^I_{i,J} = \eta^I_{i,a} \frac{\partial}{\partial \eta^J_{i,a}} . \tag{36}$$

Finally, we consider the cases where two soft scalars carry different flavors and do not form an R-symmetry singlet. This actually leads to new soft theorems:

$$A_n(\phi_1^{f_1}, \phi_2^{f_2,IJ}, \ldots) \to \sum_{i=3}^{n} \frac{p_1 \cdot p_2}{p_i \cdot (p_1 + p_2)} R_i^{f_1 f_2} R_{i,IJ} A_{n-2},$$

and similarly for other R-symmetry generators. The results of the soft limits now contain both flavor and R-symmetry generators, reflecting the direct product structure in $\frac{SO(5,21)}{SO(5)\times SO(21)}$. This is a new phenomenon that is not present in pure supergravity [17, 19] or pion amplitudes.

4D $\mathcal{N}=4$ EINSTEIN-MAXWELL THEORY

One may compactify 6D $(2,0)$ supergravity on T^2 to obtain a 4D $\mathcal{N}=4$ supergravity theory. The tree amplitudes of this theory capture the leading low-energy behavior of Type IIB (or Type IIA) superstring theory on K3 $\times T^2$. At a generic point in the moduli space, this is an $\mathcal{N}=4$ supergravity multiplet coupled to a certain number of Maxwell multiplets.

In the language of spinor-helicity variables, the reduction to 4D can be obtained by decomposing the 6D spinor as $A \to \alpha = 1, 2,\ \dot{\alpha} = 3, 4$. Then, the compact momenta are $P_i^{\alpha\beta} = P_i^{\dot{\alpha}\dot{\beta}} = 0$, which is implemented by:

$$\lambda_a^A \to \lambda_+^{\alpha} = 0, \quad \lambda_-^{\dot{\alpha}} = 0. \tag{37}$$

The 6D tensor superfield becomes an $\mathcal{N}=4$ vector multiplet in 4D, in a non-chiral form [1, 20],

$$\begin{aligned}\Phi(\eta_a) \to V_{\mathcal{N}=4}(\eta_+, \eta_-) = \phi + \eta_-^{\hat{I}} \psi_{\hat{I}}^- + \ldots \\ + (\eta_+)^2 A^+ + (\eta_-)^2 A^- + \cdots + (\eta_+)^2(\eta_-)^2 \bar{\phi}. \end{aligned}\tag{38}$$

This manifests an $SU(2)\times SU(2)$ subgroup of the $SU(4)$ R-symmetry, and is neutral under the $U(1)$ little group.

Dimensional reduction of the $\Phi^{\hat{a}\hat{b}}(\eta)$ superfield is analogous, except we must consider separately the 3 cases of $\Phi^{\hat{+}\hat{-}}$, $\Phi^{\hat{+}\hat{+}}$, $\Phi^{\hat{-}\hat{-}}$. In 4D these become another vector multiplet and a pair of positive and negative helicity graviton superfields, respectively:

$$\Phi^{\hat{+}\hat{-}}(\eta_a) \to V_{\mathcal{N}=4}(\eta_+, \eta_-)\,, \tag{39}$$

$$\begin{aligned}\Phi^{\hat{+}\hat{+}}(\eta_a) \to \mathcal{G}^+_{\mathcal{N}=4}(\eta_+, \eta_-) = A^+ + \eta_-^{\hat{I}} \psi_{\hat{I}}^{-+} + \ldots \\ + (\eta_+)^2 G^{++} + (\eta_-)^2 \phi + \cdots + (\eta_+)^2(\eta_-)^2 \bar{A}^+\,,\end{aligned} \tag{40}$$

$$\begin{aligned}\Phi^{\hat{-}\hat{-}}(\eta_a) \to \mathcal{G}^-_{\mathcal{N}=4}(\eta_+, \eta_-) = \bar{A}^- + \eta_-^{\hat{I}} \Psi_{\hat{I}}^{--} + \ldots \\ + (\eta_-)^2 G^{--} + (\eta_+)^2 \bar{\phi} + \cdots + (\eta_+)^2(\eta_-)^2 A^-\,.\end{aligned} \tag{41}$$

We see the on-shell spectrum of the 4D supergravity theory consists of the $\mathcal{G}^+$ and $\mathcal{G}^-$ superfields coupled to 22 abelian $\mathcal{N}=4$ Maxwell multiplets.

With this set up, we are ready to perform the dimensional reduction of the 6D formula (30) [21]. First, the 6D measure reduces to

$$d\mu^{4\mathrm{D}} = \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^d d^2\rho_k \prod_{k=0}^{\tilde{d}} d^2\tilde{\rho}_k}{\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{GL}(1,\mathbb{C})} \frac{1}{R(\rho)R(\tilde{\rho})} \prod_{i=1}^n E_i^{4D}$$

where $R(\rho)$, $R(\tilde{\rho})$ are the resultants of the two components of $\rho^{\alpha}(\sigma)$, $\tilde{\rho}^{\dot{\alpha}}(\sigma)$, respectively. The 4D scattering equations are given by

$$E_i^{4D} = \delta^4\left(p_i^{\alpha\dot{\alpha}} - \frac{\rho^{\alpha}(\sigma_i)\tilde{\rho}^{\dot{\alpha}}(\sigma_i)}{\prod_{j\neq i}\sigma_{ij}}\right), \tag{42}$$

and the polynomials may have different degrees,

$$\rho^{\alpha}(\sigma) = \sum_{k=0}^{d} \rho_k^{\alpha}\, \sigma^k, \quad \tilde{\rho}^{\dot{\alpha}}(\sigma) = \sum_{k=0}^{\tilde{d}} \tilde{\rho}_k^{\dot{\alpha}}\, \sigma^k, \tag{43}$$

with $d + \tilde{d} = n-2$. The 2×2 matrix $(\tilde{W}_i)_{ab}$ reduces to

$$(\tilde{W}_i)_{\hat{+}\hat{+}} = (\tilde{W}_i)_{\hat{-}\hat{-}} = 0,\ (\tilde{W}_i)_{\hat{+}\hat{-}} = t_i,\ (\tilde{W}_i)_{\hat{-}\hat{+}} = \tilde{t}_i. \tag{44}$$

with $t_i = \frac{\lambda_i^{\alpha}}{\rho^{\alpha}(\sigma_i)}$, $\tilde{t}_i = \frac{\tilde{\lambda}_i^{\dot{\alpha}}}{\tilde{\rho}^{\dot{\alpha}}(\sigma_i)}$, which is independent of $\alpha, \dot{\alpha}$, and $t_i\tilde{t}_i = \prod_{j\neq i}\frac{1}{\sigma_{ij}}$. As for the integrand, the parts that reduce to 4D non-trivially are

$$\tilde{M}^{n_1}_{\hat{a}\hat{b}} \to \tilde{T}^{n_1}_{\hat{a}\hat{b}}, \qquad \det{}'S_n \to R^2(\rho)R^2(\tilde{\rho})V_n^{-2}. \tag{45}$$

Assume we have m_1 $\mathcal{G}^+$ superparticles and m_2 $\mathcal{G}^-$, with $m_1 + m_2 = n_1$ [22], and we find $\tilde{T}^{n_1}$ is given by

$$\tilde{T}^{n_1} = T_+^{m_1} T_-^{m_2} = \left(V_{m_1}^2 \prod_{i\in m_1} t_i^2\right)\left(V_{m_2}^2 \prod_{j\in m_2} \tilde{t}_j^2\right), \tag{46}$$

where $V_{m_1} = \prod_{i<j}\sigma_{ij}$ for $i, j \in m_1$, and similarly for V_{m_2}. We therefore obtain the general formula for the amplitudes of 4D $\mathcal{N}=4$ Einstein-Maxwell theory:

$$A_n^{\mathcal{N}=4} = \int d\mu^{4\mathrm{D}} \frac{\mathrm{Pf}\mathcal{X}_{n_2}}{V_{n_2}V_n} T_+^{m_1} T_-^{m_2} R^2(\rho) R^2(\tilde{\rho})\, \Omega_F^{\mathcal{N}=4}, \tag{47}$$

where $\Omega_F^{\mathcal{N}=4}$ implements the 4D $\mathcal{N}=4$ supersymmetry, arising as the reduction of $\Omega_F^{(2,0)}$,

$$\Omega_F^{\mathcal{N}=4} = \prod_{k=0}^{d} \delta^2\Big(\sum_{i=1}^n t_i \sigma_i^k \eta_{i+}^I\Big) \prod_{k=0}^{\tilde{d}} \delta^2\Big(\sum_{i=1}^n \tilde{t}_i \sigma_i^k \eta_{i-}^{\hat{I}}\Big). \tag{48}$$

The formula should be understood as summing over all non-zero $d, \tilde{d}$ obeying $d + \tilde{d} = n-2$. However, it is clear from the superfields that we should require

$$d = \frac{n_2}{2} + m_1 - 1, \qquad \tilde{d} = \frac{n_2}{2} + m_2 - 1, \tag{49}$$

recall n_2 can only be even. This shows that only the middle sector contributes to the amplitudes of pure vector multiplets. This is expected due to the fact that the graviton only couples to two opposite-helicity photons. We have checked (47) against many explicit amplitudes, and also verified that the integrand is identical to that of [11] for certain component amplitudes.

DISCUSSION AND CONCLUSION

In this article we have presented a formula for all tree-level scattering amplitudes of the low-energy effective theory of type IIB superstring theory compactified on a K3 surface. The formula for a single flavor of tensor multiplet is constructed via a SUSY reduction of the one for 6D $(2,2)$ supergravity amplitudes. We observe an important simplification in deriving the single-flavor formula, particularly the appearance of the object $\mathrm{Pf}X_n$. This is crucial for generalizing the formula to multiple flavors (21 to be precise), which is required for type IIB superstring theory on K3 at a point where tensors only interact via gravitons. By studying soft limits of the formula, we were able to explore the moduli space of the theory, which led to new soft theorems that describe the direct product structure of $SO(5)\times SO(21)$. By dimensional reduction of the 6D formula, we further deduced a new formula for amplitudes of 4D $\mathcal{N}=4$ Einstein-Maxwell theory. As we discussed, 6D $(2,0)$ supergravity with 21 tensor multiplets is free of anomalies and has a UV completion; so it would be of interest to extend our formula to include α' corrections, perhaps along the lines of [23]. Also, a recent paper [24] appeared which introduces a different form of the 6D scattering equations which treats amplitudes with even and odd points on equal footing, but uses a different formalism for supersymmetry. It will be interesting to study our formula in this form of the scattering equations.

Our results provide an S matrix confirmation of various properties of $(2,0)$ supergravity and the dimensionally reduced theory as predicted by string dualities. If one considers Type IIB superstring theory compactified on $\mathrm{K3}\times T^2$, standard U-dualities will imply equivalence to the Type IIA superstring theory on the same geometry or the heterotic string theory compactified to 4D on a torus. The formulas discussed in this paper should apply to all these cases, at least at generic points of the moduli space where there are no CFTs in 6D or nonabelian gauge symmetries in 4D. At these points, we see the coset structure of the local moduli space through double soft scalar theorems. While IIB in 10D has a dilaton which sets the coupling, in 6D this scalar field is one of the 105 moduli fields belonging to the 21 tensor multiplets, and appears on an equal footing with the other 104 scalar fields. To make sense of all this, in 4D there should be several alternative perturbation expansions of the same theory based on the various dual point of view. However, at tree level, their low-energy effective descriptions should coincide, which is clear from our formula.

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