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Coulomb Branch Quantization and Abelianized Monopole Bubbling

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Abstract

We develop an approach to study Coulomb branch operators in 3D $\mathcal{N} = 4$ gauge theories and the associated quantization structure of their Coulomb branches. This structure is encoded in a one-dimensional TQFT subsector of the full 3D theory, which we describe by combining several techniques and ideas. The answer takes the form of an associative non-commutative star-product algebra on the Coulomb branch. For "good" and "ugly" theories (according to Gaiotto-Witten classification), we also have a trace map on this algebra, which allows to compute correlation functions and, in particular, guarantees that the star-product satisfies a truncation condition. This work extends previous work on Abelian theories to the non-Abelian case by quantifying the monopole bubbling that describes screening of GNO boundary conditions. In our approach, the monopole bubbling is determined from the algebraic consistency of the OPE. This also yields a physical proof of the Bullimore-Dimofte-Gaiotto abelianization description of the Coulomb branch.

Contents

1 Introduction

Gauge theories in three dimensions contain a special type of local defect operators called monopole operators defined by requiring certain singular behavior of the gauge field close to the insertion point [1]. These operators play important roles in the dynamics of these theories, and in particular in establishing various interesting properties such as infrared dualities between theories with different ultraviolet (UV) descriptions (see for instance [2–15] for recent examples). Because these operators are not polynomial in the Lagrangian fields, they are notoriously difficult to study, and so far most studies have focused on determining only their quantum numbers [1, 16–29]. The goal of this paper is to present the first example where we can directly compute operator product expansion (OPE) coefficients and correlation functions of monopole operators in 3D non-abelian gauge theories.

We focus on a class of 3D gauge theories with $\mathcal{N} = 4$ supersymmetry (eight Poincaré supercharges) constructed by coupling a vector multiplet with gauge group G to a matter hypermultiplet that transforms in some representation of G.[1] For a matter representation of sufficiently large dimension, these theories flow in the infrared (IR) to interacting superconformal field theories (SCFTs), whose correlation functions are generally intractable. However, as shown in [35, 36], these theories also contain one-dimensional protected subsectors whose correlation functions are topological, and one may hope that computations in these protected subsectors become tractable. This is indeed the case, as was shown in [37,38] and as will be shown here. While $\mathcal{N} = 4$ SCFTs have in general two inequivalent protected topological sectors, one associated with the Higgs branch and one with the Coulomb branch, it is the Coulomb branch sector that contains monopole operators, and hence this is the one that we focus on here. (The Higgs branch sector was studied in [37].) From the 3D SCFT point of view, the information contained in either of the two protected sectors is equivalent to that contained in the ($n \leq 3$)-point functions of certain 1/2-BPS operators in the SCFT [35–38].

The Coulomb branch protected sector consists of operators that belong to the cohomology of a certain supercharge $\mathcal{Q}^C$ that is a linear combination of a Poincaré and a superconformal charge.[2] As such, one may think that the protected sector mentioned above is emergent at the IR fixed point, and hence it may be inaccessible in the UV description. This is indeed true for SCFTs defined on $\mathbb{R}^3$. However, as was shown in [37,38], if one defines the QFT on a

[1] These theories do not allow the presence of Chern-Simons terms. It is possible to construct $\mathcal{N} = 4$ Chern-Simons-matter theories [30–34], but we do not study them here.

[2] Similar statements hold about the Higgs branch protected sector if one replaces $\mathcal{Q}^C$ with another supercharge $\mathcal{Q}^H$.

round S^3 instead of on $\mathbb{R}^3$, then the protected sector can be accessible in the UV because, on S^3, the square of $\mathcal{Q}^C$ does not contain special conformal generators. (Poincaré and special conformal generators get mixed together when mapping a CFT from $\mathbb{R}^3$ to S^3.) As we will explain, the square of $\mathcal{Q}^C$ includes an isometry of S^3 that fixes a great circle, and this is the circle where the 1D topological quantum field theory (TQFT) lives.

Previous work [37] used the idea of defining the QFT on S^3 together with supersymmetric localization to solve the 1D Higgs branch theory by describing a method for computing its structure constants. The Coulomb branch case is much more complicated because it involves monopole operators. A complete solution of the 1D Coulomb branch theory was obtained for abelian gauge theories in [38]. Building on the machinery developed in [38], we describe how to compute all observables within the 1D Coulomb branch topological sector of an arbitrary non-abelian 3D $\mathcal{N} = 4$ gauge theory by constructing "shift operators" whose algebra is a representation of the OPE of the 1D TQFT operators.

The mathematical physics motivation for studying the 1D TQFT is that it provides a "quantization" of the ring of holomorphic functions defined on the Coulomb branch $\mathcal{M}_C$. This can be explained as follows. The 3D theories we study have two distinguished branches of the moduli space of vacua: the Higgs branch and the Coulomb branch, which are each parametrized, redundantly, by VEVs of gauge-invariant chiral operators whose chiral ring relations determine the branches as generically singular complex algebraic varieties. While the Higgs branch chiral ring relations follow from the classical Lagrangian, those for the Coulomb branch receive quantum corrections. The Coulomb branch is constrained by SUSY to be a generically singular hyperkähler manifold of quaternionic dimension equal to the rank of G, which, with respect to a choice of complex structure, can be viewed as a complex symplectic manifold. The 1/2-BPS operators that acquire VEVs on the Coulomb branch, to be referred as Coulomb branch operators (CBOs), are monopole operators as well as operators built from the vector multiplet scalars. All the holomorphic functions on $\mathcal{M}_C$ are given by VEVs of the subset of CBOs that are chiral with respect to an $\mathcal{N} = 2$ subalgebra. Under the OPE these operators form a ring, which, as is well known, is isomorphic to the ring $\mathbb{C}[\mathcal{M}_C]$ of holomorphic functions on $\mathcal{M}_C$. It was argued in [36] that because the operators in the 1D TQFT are in one-to-one correspondence with chiral ring CBOs, the 1D TQFT is a deformation quantization of $\mathbb{C}[\mathcal{M}_C]$. Indeed, the 1D OPE induces a non-commutative associative product on $\mathbb{C}[\mathcal{M}_C]$ referred to as a star product, that in the limit $r \to \infty$ (r being the radius of S^3) reduces to the ordinary product of the corresponding holomorphic functions, and that at order $1/r$ gives the Poisson bracket of the corresponding holomorphic functions.

Note that both the quantization of [36] in the "$Q+S$" cohomology, and our quantization on a sphere are realizations of the older idea of obtaining a lower-dimensional theory by passing to the equivariant cohomology of a supercharge, which originally appeared in the context of Omega-deformation in 4D theories [39–41] and was also applied to 3D theories in [42–44].

Our procedure for solving the 1D Coulomb branch theory uses a combination of the cutting and gluing axioms, supersymmetric localization, as well as a consistency requirement that we refer to as polynomiality. We first cut S^3 into two hemispheres $HS^3_\pm$ along an equatorial $S^2 = \partial HS^3_\pm$ orthogonal to the circle along which operators live (see Figure 1). Correlators are then represented by an inner-product of wavefunctions generated by the path-integral on $HS^3_\pm$ with insertions of twisted CBOs. In [38, 45], it was shown that it is sufficient to consider such wavefunctions $\Psi_\pm(\mathcal{B}_{\rm BPS})$ with operator insertions only at the tip of $HS^3_\pm$, and evaluated on a certain class of half-BPS boundary conditions $\mathcal{B}_{\rm BPS}$. Insertions of twisted CBOs anywhere on the great semi-circles of $HS^3_\pm$ can then be realized, up to irrelevant $\mathcal{Q}^C$-exact terms, as simple shift operators acting on this restricted class of wavefunctions. It was shown in [38] that these shift operators can be fully reconstructed from general principles and knowledge of $\Psi_\pm(\mathcal{B}_{\rm BPS})$. Moreover, their algebra provides a faithful representation of the star product. Finally, one can determine expectation values of correlators (or, more abstractly, one can define an evaluation map on $\mathbb{C}[\mathcal{M}_C]$, known as the trace map in deformation quantization) by gluing $\Psi_+(\mathcal{B}_{\rm BPS})$ and $\Psi_-(\mathcal{B}_{\rm BPS})$ with an appropriate measure, as will be reviewed in Section 2.2.

The fact that the star product algebra can be determined independently of evaluating correlators is very useful. First, to calculate correlators using the above procedure involves solving matrix integrals, which can be complicated for gauge groups of rank > 1. In comparison, the star product can be inferred from the comparatively simple calculation of the wavefunctions $\Psi_\pm(\mathcal{B}_{\rm BPS})$. Second, the matrix models representing correlators diverge for "bad" theories in the sense of Gaiotto and Witten [30]. Nevertheless, as we will see, the HS^3 wavefunctions and the star product extracted from them are well-defined even in those cases. Therefore, we emphasize that our formalism works perfectly well even for bad theories, as far as the Coulomb branch and its deformation quantization are concerned. However, correlation functions cannot be computed for such theories, and the star products might not satisfy the truncation property introduced in [36].

On a more technical note, we provide a new way of analyzing "monopole bubbling" [46].

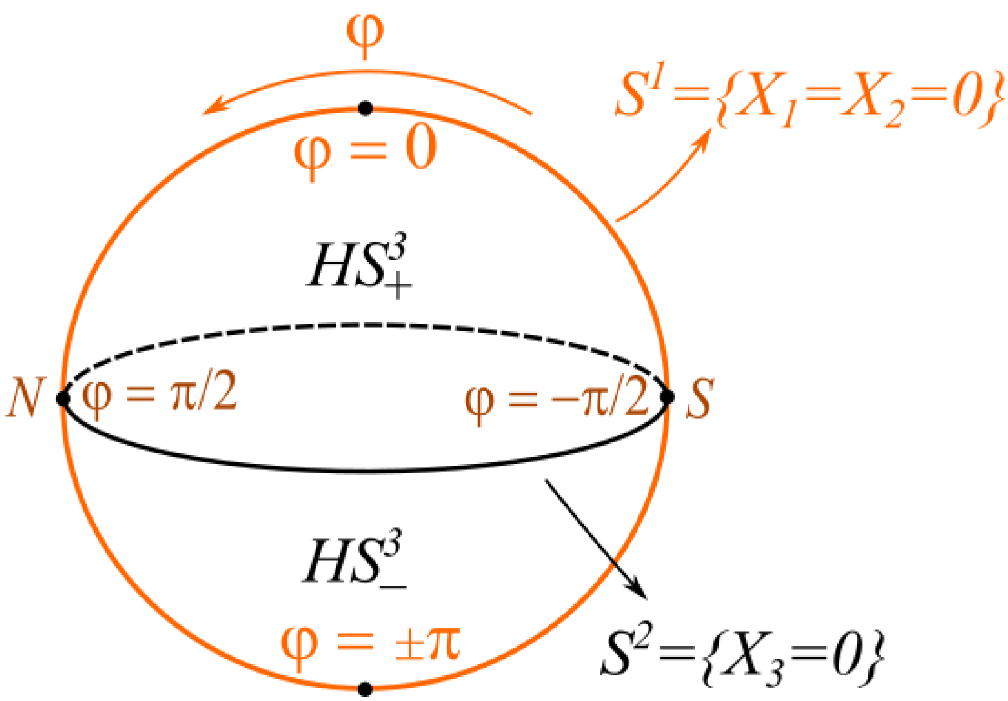

Figure 1: A schematic 2D representation of S^3 given by $X_1^2 + X_2^2 + X_3^2 + X_4^2 = r^2$. The 1D TQFT lives on the S^1 defined by $X_1 = X_2 = 0$ (red) and parametrized by the angle φ. The S^3 can be cut into two hemispheres $HS^3_\pm \cong B^3$ whose boundary forms an $S^2 = \partial HS^3_\pm$ (blue circle) defined by $X_4 = 0$. The 1D TQFT circle intersects this S^2 at two points identified with its North (N) and South (S) poles.

Monopole bubbling is a phenomenon by which the charge of a singular monopole is screened to a lower one by small 't Hooft-Polyakov monopoles. In our setup, this phenomenon is manifested through the fact that our shift operators for a monopole of given charge contain contributions proportional to those of monopoles of smaller charge with coefficients we refer to as bubbling coefficients. While we do not know a localization-based algorithm to obtain these coefficients in general, we propose that the requirement that the OPE of any two 1D TQFT operators should be a polynomial in the 1D operators uniquely determines the bubbling coefficients, up to operator mixing ambiguities. In Section 4 we provide many examples of gauge theories of small rank where we carry out explicitly our algorithm to determine the shift operators and bubbling coefficients. These results are also interesting for comparison with the literature on direct localization computations of bubbling in 4D, e.g., [47–49], which were subsequently refined by [50–52].

The main mathematical content of this work is a construction of deformation quantizations of Coulomb branches of 3D $\mathcal{N} = 4$ theories, which also satisfy the truncation condition of [36] in the case of good or ugly theories, as a consequence of the existence of the natural trace map (the one-point function).[3] By taking the commutative limit, we recover the ordinary Coulomb branch of our theory in the form of the "abelianization map" proposed by [53]. Therefore, our approach also provides a way to prove the abelianization map proposal of [53] starting from basic physical principles. In addition, the knowledge of bubbling coefficients vastly expands the domain of applicability to all Lagrangian 3D $\mathcal{N}=4$ theories of cotangent type. Finally, we expect that translating our approach into a language that uses the mathematical definition of Coulomb branches [54–57] might be of independent interest in the study of deformation quantization.

The rest of this paper is organized as follows. Section 2 contains a review of the setup of our problem as well as a derivation of the shift operators without taking into account bubbling. Section 3 discusses the dressing of monopole operators with vector multiplet scalars and sets up the computation of the bubbling coefficients. In Section 4 we provide explicit examples of shift operators and bubbling coefficients in theories with gauge groups of small rank. In Section 5 we discuss a few applications of our formalism: to determining chiral rings, to chiral ring quantization, and to computing correlation functions of monopole operators and performing checks of mirror symmetry. Many technical details and more examples are relegated to the Appendices.

2 Shift Operators

2.1 Setup

2.1.1 Theories

We study 3D $\mathcal{N}=4$ gauge theories of cotangent type, which is the same class of theories as those whose quantized Higgs branches were the subject of [37]. Coulomb branches of abelian gauge theories were scrutinized in [38] using different techniques, and here we extend those techniques to the case of general gauge groups $G \cong \prod_i G_i$, where each G_i is either simple or abelian. As the construction of such theories was detailed in [37,38], we only briefly mention it here.

The theories are built from the 3D $\mathcal{N}=4$ vector multiplet $\mathcal{V}$ taking values in the Lie algebra $\mathfrak{g}=\mathrm{Lie}(G)$, and from the 3D $\mathcal{N}=4$ hypermultiplet $\mathcal{H}$ valued in a (generally reducible) representation $\mathcal{R}$ of G. If we write this in terms of half-hypermultiplets, they take values in

[3] Such star-products are called "short" in an ongoing mathematical work on their classification, as we learned from P. Etingof.

$\mathcal{R} \oplus \bar{\mathcal{R}}$, which is what the name "cotangent type" stands for. More general representations of the half-hypermultiplets should also be possible to address using our techniques, however we do not do it in the current work.

Our focus is on such theories supersymmetrically placed on the round S^3 of radius r. There are several good reasons for choosing this background. One is that compactness makes the application of supersymmetric localization techniques more straightforward. But the most important reason, as should be clear to readers familiar with [37, 38], is that the sphere is a natural setting for deformation quantization of moduli spaces: the Coulomb and Higgs branches in such a background can be viewed as non-commutative, with $\frac{1}{r}$ playing the role of a quantization parameter. As in the 2D Ω-background in flat space [53], the result is an effective compactification of spacetime to a line.

Furthermore, quantized Coulomb and Higgs branch chiral rings are directly related to physical correlation functions, and in particular encode the OPE data of the BPS operators in the IR superconformal theory, whenever it exists. This relation equips the noncommutative star product algebra of observables with a natural choice of "trace" operation — the one-point function of the QFT — as well as natural choices of bases corresponding to operators that are orthogonal with respect to the two-point function and have well-defined conformal dimensions at the SCFT point. These extra structures are a significant advantage of quantization using the spherical background, and they are responsible for much of the progress we make in this paper.

The $\mathcal{N} = 4$ supersymmetric background on S^3 is based on the SUSY algebra $\mathfrak{s} = \mathfrak{su}(2|1)_\ell \oplus \mathfrak{su}(2|1)_r$, which also admits a central extension $\widetilde{\mathfrak{s}} = \widetilde{\mathfrak{su}(2|1)}_\ell \oplus \widetilde{\mathfrak{su}(2|1)}_r$, with central charges corresponding to supersymmetric mass and FI deformations of the theory. Note that in the flat space limit, $r \to \infty$, this algebra becomes the usual $\mathcal{N} = 4$ super-Poincaré, implying that all results of this paper should have a good $r \to \infty$ limit. All the necessary details on the SUSY algebra $\mathfrak{s}$, and how the vector and hypermultiplets transform under it, can be found either in Section 2 of [37] or in Section 2.1 and Appendix A.2 of [38]. Supersymmetric actions for $\mathcal{V}$, $\mathcal{H}$, and their deformations by mass and FI terms can also be found in those sections.

The SUSY algebra $\widetilde{\mathfrak{s}}$ contains two interesting choices of supercharges, $\mathcal{Q}^H$ and $\mathcal{Q}^C$. They satisfy the following relations:

$$(\mathcal{Q}^H)^2 = \frac{4i}{r}(P_\tau + R_C + ir\widehat{\zeta}), \qquad (\mathcal{Q}^C)^2 = \frac{4i}{r}(P_\tau + R_H + ir\widehat{m}), \tag{2.1}$$

where P_τ denotes a $U(1)$ isometry of S^3 whose fixed-point locus is a great circle parametrized by $\varphi \in (-\pi, \pi)$: call it $S^1_\varphi \subset S^3$.[4] Here, R_C and R_H are the Cartan generators of the usual $SU(2)_C \times SU(2)_H$ R-symmetry of $\mathcal{N}=4$ SUSY, which in terms of the inner $U(1)_\ell \times U(1)_r$ R-symmetry of $\mathfrak{s}$ are identified as:

$$R_H = \frac{1}{2}(R_\ell + R_r), \qquad R_C = \frac{1}{2}(R_\ell - R_r). \tag{2.2}$$

The notations $\widehat{\zeta}$ and $\widehat{m}$ stand for the FI and mass deformations, i.e., central charges of $\widetilde{\mathfrak{s}}$.

The most important features of $\mathcal{Q}^H$ and $\mathcal{Q}^C$ are that if we consider their action on the space of local operators and compute their equivariant cohomology, the answers have very interesting structures. The operators annihilated by $\mathcal{Q}^H$ are the so-called twisted-translated Higgs branch operators, whose OPE is encoded in quantization of the Higgs branch; such operators for the class of theories at hand were fully studied in [37]. Correspondingly, the cohomology of $\mathcal{Q}^C$ contains twisted-translated Coulomb branch operators, whose structure so far has been only explored for abelian theories in [38]. Such operators must be inserted along the great circle S^1_φ fixed by $(\mathcal{Q}^C)^2$, and their OPE encodes a quantization of the Coulomb branch. More details on twisted-translated operators are given in Appendix B.

2.1.2 Observables

The purpose of this work is to study the cohomology of $\mathcal{Q}^C$ and associated structures for general non-abelian gauge theories of cotangent type. The operators annihilated by $\mathcal{Q}^C$ are constructed from monopole operators and a certain linear combination of scalars in the vector multiplet. Recall that the vector multiplet contains an $SU(2)_C$ triplet of scalars $\Phi_{\dot{a}\dot{b}} = \Phi_{\dot{b}\dot{a}}$, and using the notation of [38], the following linear combination is annihilated by $\mathcal{Q}^C$:

$$\Phi(\varphi) = \Phi_{\dot{a}\dot{b}}(\varphi) v^{\dot{a}} v^{\dot{b}}, \quad v = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi/2} \\ e^{-i\varphi/2} \end{pmatrix}, \tag{2.3}$$

whenever this operator is inserted along $S^1_\varphi \subset S^3$. On the other hand, (bare) BPS monopole operators are defined as defects imposing special boundary conditions on the gauge field and on $\Phi_{\dot{a}\dot{b}}$. They were first defined for 3D $\mathcal{N}=4$ theories in [58], while the non-supersymmetric version was introduced earlier in [1]. The twisted-translated monopole operators that we study — which are essentially those of [58] undergoing an additional $SU(2)_C$ rotation as we move along S^1_φ — were described in detail in [38]. Their definition is rather intricate, so it

[4] Concretely, τ is the fiber coordinate in an S^1 fibration over the disk D^2, i.e., $S^1_\tau \to S^3 \to D^2$. After conformally mapping to flat space, P_τ would be a rotation that fixes the image of S^1_φ, which is a line.

will be helpful to review it, with an eye toward the additional complications that arise in non-abelian gauge theories.

First recall that in a $U(1)$ gauge theory, a (bare) non-supersymmetric monopole operator is a local defect operator that sources magnetic flux at a point in 3D spacetime. In a non-abelian gauge theory, the quantized charge b is promoted to a matrix, or more precisely, a cocharacter of G (referred to as the GNO charge [59]).[5] A cocharacter is an element of $\mathrm{Hom}(U(1), G)/G \cong \mathrm{Hom}(U(1), \mathbb{T})/\mathcal{W}$. Passing from the element of $\mathrm{Hom}(U(1), \mathbb{T})/\mathcal{W}$ to the map of algebras $\mathbb{R} \to \mathfrak{t}$, we see that cocharacters can also be identified as Weyl orbits in the coweight lattice $\Lambda_w^\vee \subset \mathfrak{t}$ of G, i.e., in the weight lattice of the Langlands dual group ${}^L G$. Since every Weyl orbit contains exactly one dominant weight (lying in the fundamental Weyl chamber), it is conventional to label monopole charges by dominant weights of ${}^L G$ [53]. Let $b \in \mathfrak{t}$ be such a dominant weight of ${}^L G$. Then a bare monopole operator is defined by a sum over $\mathcal{W}b$, the Weyl orbit of b, of path integrals with singular boundary conditions defined by elements of $\mathcal{W}b$. Specifically, the insertion of a twisted-translated monopole operator at a point $\varphi \in S^1_\varphi$ is defined by the following singular boundary conditions for $F_{\mu\nu}$ and $\Phi_{\dot{a}\dot{b}}$:

$$*F \sim b\frac{y_\mu \mathrm{d}y^\mu}{|y|^3}, \quad \Phi_{\dot{1}\dot{1}} = -(\Phi_{\dot{2}\dot{2}})^\dagger \sim -\frac{b}{2|y|}e^{-i\varphi}, \quad \Phi_{\dot{1}\dot{2}} \sim 0, \tag{2.4}$$

where it is understood that one must compute not a single path integral, but rather a sum of path integrals over field configurations satisfying (2.4) with b ranging over the full Weyl orbit of a given dominant weight. Here, "$\sim$" means "equal up to regular terms" and y^μ are Riemann normal coordinates centered at the monopole insertion point. The origin of (2.4) is that twisted-translated monopoles are chiral with respect to the $\mathcal{N} = 2$ subalgebra defined by the polarization vector in (2.3) at any given φ. This requires that the real scalar in the $\mathcal{N} = 2$ vector multiplet diverge as $\frac{b}{2|y|}$ near the monopole [58] and results in nontrivial profiles for the $\mathcal{N} = 4$ vector multiplet scalars near the insertion point.

We denote such twisted-translated monopole operators by $\mathcal{M}^b(\varphi)$, or simply $\mathcal{M}^b$. The $\mathcal{Q}^C$ cohomology, in addition to $\mathcal{M}^b(\varphi)$ and (gauge-invariant polynomials in) $\Phi(\varphi)$, contains monopole operators dressed by polynomials $P(\Phi)$, or dressed monopoles, which we denote as $\left[P(\Phi)\mathcal{M}^b\right]$. Note that because monopoles are really given by sums over Weyl orbits, the

[5] This is a more refined notion than the topological charge labeled by $\pi_1(G)$ (when it exists): such charges correspond to global symmetries of the Coulomb branch whose conserved currents in the UV are the abelian field strengths and which may be enhanced in the IR.

notation $[P(\Phi)\mathcal{M}^b]$ is not merely a product of $P(\Phi)$ and $\mathcal{M}^b$, but rather:

$$\left[P(\Phi)\mathcal{M}^b\right] = \frac{1}{|\mathcal{W}_b|}\sum_{\mathrm{w}\in\mathcal{W}} P(\Phi^{\mathrm{w}}) \times (\text{insertion of a charge } \mathrm{w}\cdot b \text{ monopole singularity})\,, \quad (2.5)$$

where Φ^{w} means that as we sum over the Weyl orbit, we act on the $P(\Phi)$ insertion as well. Because $\mathcal{M}^b$ breaks the gauge group at the insertion point down to the subgroup $G_b \subset G$ that preserves b, $P(\Phi)$ must be invariant under the G_b action.[6] Also, to avoid overcounting, we have to divide by the order of the stabilizer of b in $\mathcal{W}$.[7]

At this point, we pause to discuss a few subtleties inherent to the above definition. They are important for precise understanding and, ultimately, to perform computations correctly, but may be skipped on first reading.

First, let us ask ourselves what exactly Φ^{w} is. After all, the Weyl group acts canonically on the Cartan subalgebra $\mathfrak{t}$, but does not have a natural action on the full Lie algebra $\mathfrak{g}$ where Φ is valued. Indeed, from the identification $\mathcal{W} = N(\mathbb{T})/Z(\mathbb{T})$, any Weyl group element is interpreted as an element of the normalizer $N(\mathbb{T}) \subset G$ of the maximal torus $\mathbb{T}$, up to an element of the centralizer $Z(\mathbb{T}) \subset G$. On $\mathfrak{t}$ the centralizer $Z(\mathbb{T})$ acts trivially, but it certainly acts non-trivially on the full algebra, making the action of $\mathcal{W}$ on $\mathfrak{g}$ ambiguous. However, the action of $\mathcal{W}$ on a G_b-invariant polynomial $P(\Phi)$ is nevertheless unambiguous. To understand this, note that the magnetic charge $b \in \mathfrak{t}$ is obviously preserved by $Z(\mathbb{T})$, so the group G_b includes $Z(\mathbb{T})$ as a subgroup. In particular, it means that $P(\Phi)$ is $Z(\mathbb{T})$-invariant, and hence the action of $\mathrm{w} \in N(\mathbb{T})/Z(\mathbb{T})$ on $P(\Phi)$ is unambiguous – this is the action that appears in (2.5).

This is not the only subtlety to take care of. It is also worth noting that the Weyl group action on the dressing factor is different from the action on b. The fundamental reason for this is that Φ represents a "non-defect" observable (given by an insertion of fundamental fields in the path integral), while b characterizes the defect, namely it describes the strength of the monopole singularity that plays a role of boundary conditions for the fundamental fields. In Appendix B.1 of [38], it was explained how symmetries act on observables of these two types (it was emphasized for global symmetries there, but the argument is exactly the same for

[6] As we will see, after the localization, Φ will take values in the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$, in which case the G_b-invariance of $P(\Phi)$ boils down to the $\mathcal{W}_b$-invariance, where $\mathcal{W}_b$ is the Weyl group of G_b. But then, because (2.5) includes summation over the Weyl orbit of $\mathcal{W}$, there is no real need to require $\mathcal{W}_b$-invariance of $P(\Phi)$, as it will be automatically averaged over the subgroup $\mathcal{W}_b \subset \mathcal{W}$ upon this summation. Therefore, later on, when we write formulas in terms of $\Phi \in \mathfrak{t}_{\mathbb{C}}$, we can insert arbitrary polynomials $P(\Phi)$ in $[P(\Phi)\mathcal{M}^b]$.

[7] The factor $|\mathcal{W}_b|^{-1}$ only appears when we sum over elements of $\mathcal{W}$, while equations written directly in terms of a sum over the Weyl orbit do not need such a factor.

gauge symmetries): the actions are inverse of each other. Namely, in our case, both b and Φ are elements of $\mathfrak{g}$, and if the gauge symmetry acts on Φ by U, that is $\Phi \mapsto U\Phi U^{-1}$, then it acts on b by U^{-1}, that is $b \mapsto U^{-1}bU$. The monopole singularity is labeled by $b \in \mathfrak{t}$, and the Weyl group has a natural action on it coming from the identification $\mathcal{W} = N(\mathbb{T})/Z(\mathbb{T})$. The dressing factor is a G_b-invariant polynomial $P(\Phi)$, that is also acted on by $\mathcal{W}$, as we just explained. If we act on b by $\mathrm{w} \in \mathcal{W}$, that is $b \mapsto \mathrm{w} \cdot b$, then we should act by w^{-1} on $P(\Phi)$, that is $P(\Phi^{\mathrm{w}}) = \mathrm{w}^{-1} \cdot P(\Phi)$. After restricting Φ to take values in $\mathfrak{t}_{\mathbb{C}}$ (which happens after localization and gauge fixing), it is convenient to note that $\mathcal{W}$ acts on $\mathfrak{t}$ by orthogonal matrices, and hence the left action by w^{-1} is the same as the right action by w. This provides a convenient way to perform actual calculations: the Weyl group acts from the left on $b \in \mathfrak{t}$ and from the right on $\Phi \in \mathfrak{t}$, once we represent them as a column (b^i) and a row (Φ^i), respectively, in some orthonormal basis of $\mathfrak{t}$.

Another convention that we choose to follow is that by b in $[P(\Phi)\mathcal{M}^b]$ we normally mean some weight of ${}^LG$ within the given Weyl orbit, not necessarily the dominant one. Whenever we label monopoles by dominant weights, we explicitly say so. The polynomial $P(\Phi)$ appearing inside the square brackets, $[P(\Phi)\mathcal{M}^b]$, is always the one attached to the charge b singularity (whether or not b is dominant), while the Weyl-transformed singularities $\mathrm{w} \cdot b$ are multiplied by Weyl-transformed polynomials, as in the equation (2.5).

In this paper, we develop methods for computing correlation functions of dressed monopole operators of the form (2.5). There are several techniques that we combine in order to achieve our results: cutting and gluing techniques [45, 60], localization, and algebraic consistency of the resulting OPE. In what follows, we describe each of them and what role they play in the derivation.

2.2 Gluing Formula

The cutting and gluing property [45, 60] holds in local quantum field theory, and it has already been applied to the abelian version of our problem in [38]. This is also one of the key ingredients in the non-abelian generalization here. We can motivate its application as follows. As explained in [38], only a very restricted class of configurations of twisted CBOs on S^3 is amenable to a direct localization computation. A less direct approach is to endow the path integral on S^3 with extra structure by dividing it into path integrals on two open halves. These path integrals individually prepare states in the Hilbert space of the theory on S^2. The advantage of this procedure is that it allows for operator insertions within S^3 to be implemented by acting on these boundary states with operators on their associated

Hilbert spaces.

Specifically, the round S^3 is glued from two hemispheres, HS^3_+ and HS^3_-, and we need to know how this is represented at the level of quantum field theories living on them. Recall that gluing corresponds to taking $\langle\Psi_-|\Psi_+\rangle$, where $|\Psi_+\rangle \in \mathcal{H}_{S^2}$ and $\langle\Psi_-| \in \mathcal{H}^\vee_{S^2}$ are states generated at the boundaries of the two hemispheres. Furthermore, in Lagrangian theories with no more than two derivatives, this operation is represented by the integral over the space of polarized boundary conditions [45, 60] for a choice of polarization on $\mathcal{P}(S^2)$, the phase space associated with $S^2 = \partial HS^3$. For a special choice of supersymmetry-preserving polarization, this integral can be localized to the finite-dimensional subspace of half-BPS boundary conditions of a certain type, which results in a simple gluing formula [38, 45]:

$$\langle\Psi_-|\Psi_+\rangle = \frac{1}{|\mathcal{W}|}\sum_{B\in\Lambda^\vee_w}\int_{\mathfrak{t}} \mathrm{d}^r\sigma\, \mu(\sigma,B)\,\langle\Psi_-|\sigma,B\rangle\langle\sigma,B|\Psi_+\rangle. \tag{2.6}$$

Here, the integration goes over the Cartan $\mathfrak{t}\subset\mathfrak{g}$, $\Lambda^\vee_w \subset \mathfrak{t}$ is the coweight lattice, $\mu(\sigma,B)$ is the gluing measure given by the one-loop determinant on S^2,

$$\begin{aligned}
\mu(\sigma,B) &= Z^{\text{c.m.}}_{\text{one-loop}}(\sigma,B) Z^{\text{v.m.}}_{\text{one-loop}}(\sigma,B)\mathcal{J}(\sigma,B),\\
Z^{\text{v.m.}}_{\text{one-loop}}(\sigma,B)\mathcal{J}(\sigma,B) &= \prod_{\alpha\in\Delta^+}(-1)^{\alpha\cdot B}\left[\left(\frac{\alpha\cdot\sigma}{r}\right)^2 + \left(\frac{\alpha\cdot B}{2r}\right)^2\right],\\
Z^{\text{c.m.}}_{\text{one-loop}}(\sigma,B) &= \prod_{w\in\mathcal{R}}(-1)^{\frac{|w\cdot B|-w\cdot B}{2}}\frac{\Gamma\left(\frac{1}{2}+iw\cdot\sigma+\frac{|w\cdot B|}{2}\right)}{\Gamma\left(\frac{1}{2}-iw\cdot\sigma+\frac{|w\cdot B|}{2}\right)},
\end{aligned} \tag{2.7}$$

and $\langle\Psi_-|\sigma,B\rangle$, $\langle\sigma,B|\Psi_+\rangle$ are the hemisphere partition functions with prescribed boundary conditions determined by $\sigma\in\mathfrak{t}$ and $B\in\Lambda^\vee_w\subset\mathfrak{t}$.[8] We think of $\langle\Psi_-|\sigma,B\rangle$, $\langle\sigma,B|\Psi_+\rangle$ as wavefunctions on $\mathfrak{t}\times\Lambda^\vee_w$: they are elements of the appropriate functional space, such as $L^2(\mathfrak{t}\times\Lambda^\vee_w)$, a precise identification of which is not important. The boundary conditions parametrized by σ, B are half-BPS boundary conditions on bulk fields preserving 2D $(2,2)$ SUSY on S^2, namely an $\mathfrak{su}(2|1)$ subalgebra of $\mathfrak{s}$ containing $\mathcal{Q}^C$. In terms of the on-shell components of the multiplets $\mathcal{H} = (q_a, \widetilde{q}^a, \psi_{\alpha\dot{a}}, \widetilde{\psi}_{\alpha\dot{a}})$ and $\mathcal{V} = (A_\mu, \Phi_{\dot{a}\dot{b}}, \lambda_{\alpha a\dot{a}})$, as well as the variables $q_\pm = q_1 \pm iq_2$, $\widetilde{q}_\pm = \widetilde{q}_1 \pm i\widetilde{q}_2$, these boundary conditions are given by:

$$q_+\big| = \widetilde{q}_-\big| = \left(\mathcal{D}_\perp q_- + \frac{\Phi_{\dot{1}\dot{1}}-\Phi_{\dot{2}\dot{2}}}{2}q_-\right)\Big| = \left(\mathcal{D}_\perp \widetilde{q}_+ + \frac{\Phi_{\dot{1}\dot{1}}-\Phi_{\dot{2}\dot{2}}}{2}\widetilde{q}_+\right)\Big| = 0,$$

[8] In (2.7), $\mathcal{J}$ is a standard Vandermonde determinant and we have omitted an overall power of r from the logarithmic running of the 2D FI parameters.

$$
\begin{gathered}
(\psi_{\dot{1}}-\sigma_3\psi_{\dot{2}})\big| = (\widetilde{\psi}_{\dot{1}}+\sigma_3\widetilde{\psi}_{\dot{2}})\big| = 0, \\
A_{\parallel}\big| = \pm\frac{B}{2}(\sin\theta-1)\mathrm{d}\tau, \quad \frac{\Phi_{\dot{1}\dot{1}}+\Phi_{\dot{2}\dot{2}}}{2i}\big| = \frac{B}{2r}, \quad \Phi_{\dot{1}\dot{2}} = \frac{\sigma}{r}, \\
(\lambda_{1\dot{2}}-i\lambda_{2\dot{2}}+\sigma_3(\lambda_{1\dot{1}}-i\lambda_{2\dot{1}}))\big| = (\lambda_{1\dot{2}}+i\lambda_{2\dot{2}}-\sigma_3(\lambda_{1\dot{1}}+i\lambda_{2\dot{1}}))\big| = 0.
\end{gathered} \tag{2.8}
$$

Note that such boundary conditions impose the magnetic flux determined by $B \in \Lambda_w^\vee$ through the boundary S^2. We thus could alternatively think of B as a cocharacter, i.e., the full Weyl orbit $\mathcal{W}B$, in which case the sum in (2.6) would run over the set of cocharacters (allowed magnetic charges) $\Gamma_m = \Lambda_w^\vee/\mathcal{W}$. In such a case, the boundary conditions above would have to be understood in the same way as the definition of the monopole operator: one would have to evaluate the hemisphere partition function for every element of the Weyl orbit $\mathcal{W}B \subset \Lambda_w^\vee$, and sum the results. We find it more convenient to treat B as an element of $\Lambda_w^\vee$, in which case we simply sum over $B \in \Lambda_w^\vee$ in the gluing formula (and there is no need for a separate sum over the Weyl reflections).

The gluing formula (2.6) is valid as long as the states $\Psi_\pm$ are supersymmetric, i.e., annihilated by $\mathcal{Q}^C$ [38,45]. This is true for the state generated at the boundary of the empty hemisphere, and remains valid if we start inserting $\mathcal{Q}^C$-closed observables inside. Such insertions will modify the hemisphere partition function, and can be represented as certain operators acting on the empty hemisphere partition function. In this paper we are only concerned with local observables: they were described above as gauge-invariant polynomials in $\Phi(\varphi)$ and dressed monopole operators. Such local observables form an OPE algebra $\mathcal{A}_C$ that we are interested in, which is going to be the quantization of the Coulomb branch. Therefore, all we need to do is find how $\Phi(\varphi)$ and dressed monopoles act on the hemisphere partition function.

2.3 Input from Localization

An important step is to compute the hemisphere partition function with insertions of local $\mathcal{Q}^C$-closed observables. As explained in [38], because correlation functions do not depend on the positions of the insertions, we can move them all to the tip of the hemisphere and replace them by an equivalent composite operator located there. In the abelian case, the GNO charge of the twisted CBO at the tip was equal to the sum of GNO charges of all insertions, while in the non-abelian case, it is determined by taking tensor products of representations of ${}^LG$. It suffices to consider a bare monopole at the tip, as it is trivial to include insertions of (gauge-invariant monomials in) the scalar $\Phi(\varphi)$ anywhere along S^1_φ.

A way to compute the hemisphere partition function is using supersymmetric localization. In fact, half of the computation that we need has already been performed in [38], whose conventions we closely follow. Recall that the round sphere is parametrized by $0 \leq \theta \leq \pi/2$, $0 \leq \varphi \leq 2\pi$ and $-\pi \leq \tau \leq \pi$, and S^1_φ is located at $\theta = \pi/2$ where the τ-circle shrinks. The sphere is cut into two hemisphere along the S^2 located at $\varphi = 0$ and $\varphi = \pm\pi$. It may also be convenient to use spherical coordinates (η, ψ, τ), which are related to the "fibration" coordinates (θ, φ, τ) by

$$(\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi) = (\sin\eta\sin\psi, -\sin\eta\cos\psi, \cos\eta), \quad \tau = \tau, \tag{2.9}$$

where $\eta, \psi \in [0, \pi]$. In terms of such coordinates, the cut is located at $\eta = \pi/2$.

We place the monopole of charge b at $\theta = \pi/2$, $\varphi = \pi/2$, which is the tip of the hemisphere, by imposing (2.4) there. In spherical coordinates, the monopole insertion point is $\eta = 0$. We also impose conditions (2.8) at the boundary of the hemisphere. The BPS equations that follow from $\mathcal{Q}^C$ can be conveniently written in terms of:

$$R \equiv \sin\theta, \quad \Phi_r \equiv \mathrm{Re}(Re^{i\varphi}\Phi_{\dot{1}\dot{1}}), \quad \Phi_i \equiv \mathrm{Im}(Re^{i\varphi}\Phi_{\dot{1}\dot{1}}), \tag{2.10}$$

and take the form:

$$\begin{gathered}
[\Phi_{\dot{1}\dot{2}}, \Phi_i] = [\Phi_{\dot{1}\dot{2}}, \Phi_r] = 0, \\
D_{12} = \mathrm{Re}(D_{11}) = 0, \quad \mathrm{Im}(D_{11}) = -\tfrac{1}{r}\Phi_{\dot{1}\dot{2}}, \\
\mathcal{D}_\mu\Phi_{\dot{1}\dot{2}} = \mathcal{D}_\tau\Phi_i = 0, \quad \mathcal{D}_\tau\Phi_r = ir[\Phi_r, \Phi_i], \\
R\mathcal{D}_R\Phi_i + \mathcal{D}_\varphi\Phi_r = 0, \quad R(1-R^2)\mathcal{D}_R\Phi_r - \mathcal{D}_\varphi\Phi_i = 0, \\
F_{\mu\nu} = \sqrt{g}\epsilon_{\mu\nu\rho}\kappa_\rho\mathcal{D}^\rho\Phi_r, \quad \text{where } \kappa_\rho = \left(1, 1, \tfrac{1}{\sin^2\theta}\right)_\rho.
\end{gathered} \tag{2.11}$$

In the last equation, the index ρ is summed over, and indices are raised and lowered using the metric in [38]. They have a straightforward (non-bubbling) solution that only exists if the boundary (flux) coweight B matches one of the coweights in the Weyl orbit corresponding to the monopole charge. In other words, if the monopole's dominant coweight is b, then the straightforward solution exists iff $B = wb$ for some $w \in \mathcal{W}$. The solution has vanishing fields in the hypermultiplet, as well as vanishing fermions in the vector multiplet, while bosons in the vector multiplet take the form:

$$D_{12} = 0, \quad \Phi_{\dot{1}\dot{2}} = irD_{11} = irD_{22} = \frac{1}{r}\sigma \in \mathfrak{t},$$

$$\Phi_{\dot{1}\dot{1}} = \Phi_{\dot{2}\dot{2}} = \frac{iB}{2r\sin\eta} = \frac{iB}{2r\sqrt{\cos^2\theta + \sin^2\theta\cos^2\varphi}},$$
$$A^{\pm} = -\frac{B}{2}(\cos\psi \mp 1)\mathrm{d}\tau = \frac{B}{2}\left(\frac{\sin\theta\cos\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}} \pm 1\right)\mathrm{d}\tau, \tag{2.12}$$

where A^- is defined everywhere on the hemisphere except the interval $\pi/2 \le \varphi \le \pi$ at $\theta = \pi/2$; similarly A^+ is defined everywhere except $0 \le \varphi \le \pi/2,\ \theta = \pi/2$. Here D_{ab} are auxiliary fields from the vector multiplet.

The above straightforward solution is a direct generalization of the abelian one from [38], therefore (2.12) can be called the "abelian solution." Indeed, since $B \in \mathfrak{t}$, we see that only components valued in the maximal torus of the gauge group have vevs. It is known, however, that in the non-abelian case, the equations (2.11) might have additional loci of solutions. They correspond to screening effects that go by the name of "monopole bubbling" [46]. One notices that close to the special circle $\theta = \pi/2$, the last equation in (2.11) becomes the Bogomolny equation, and the bubbling loci in the moduli spaces of Bogomolny equations have been an area of active study. We will discuss this in more detail soon, but for now let us focus on (2.12).

The abelian solution (2.12) has a feature that all fields with non-trivial vevs on the localization locus are vector multiplet fields valued in $\mathfrak{t}$. In other words, the vevs look as if the gauge group were actually $\mathbb{T}$, the maximal torus of G. This is essentially how "abelianization" of [53] makes appearance in our approach.

Note that since the Yang-Mills action is $\mathcal{Q}^C$-exact [37,38], one can use it for the localization and to compute the determinants, i.e., simply go to the weak-coupling limit $g_{\mathrm{YM}} \to 0$. The action (with properly included boundary terms, such that the sum of the bulk and boundary pieces is $\mathcal{Q}^C$-exact [38]) vanishes on the localization locus, and it only remains to compute one-loop determinants in the background of (2.12).

The action for hypermultiplets in the background of (2.12) becomes quadratic, so there is no need to localize them separately, – one can directly integrate them out. Furthermore, this action is simply that of the free hypermultiplets coupled to the $\mathbb{T}$-valued gauge background. Each representation $\mathcal{R}$ of G gives a set of abelian charges under $\mathbb{T}$ given by the weights $w \in \mathcal{R}$. Therefore, we can borrow the corresponding one-loop determinant from our previous work [38] where the abelian case was studied:

$$Z_{1-\mathrm{loop}}^{\mathrm{hyper}} = \prod_{w\in\mathcal{R}} \frac{1}{r^{\frac{|w\cdot B|}{2}}} \frac{\Gamma\left(\frac{1+|w\cdot B|}{2} - iw\cdot\sigma\right)}{\sqrt{2\pi}}\,. \tag{2.13}$$

The only novelty in the computation of non-abelian one-loop determinants is that vector multiplets contribute: we need to include contributions of W-bosons and gaugini. An indirect derivation of these determinants will be presented in Section 2.6. The answer is given by:

$$Z^{\text{vec}}_{1-\text{loop}} = \prod_{\alpha\in\Delta} r^{\frac{|\alpha\cdot B|}{2}} \frac{\sqrt{2\pi}}{\Gamma\left(1+\frac{|\alpha\cdot B|}{2} - i\alpha\cdot\sigma\right)} \,. \tag{2.14}$$

Therefore, the contribution from the abelian solution to the hemisphere partition function with a monopole labeled by a coweight $b \in \Lambda^\vee_w \subset \mathfrak{t}$ inserted at the tip is given by:

$$Z(b;\sigma,B) = \sum_{b'\in\mathcal{W}b} \delta_{B,b'} \frac{\prod_{w\in\mathcal{R}} \frac{1}{\sqrt{2\pi} r^{\frac{|w\cdot b'|}{2}}} \Gamma\left(\frac{1+|w\cdot b'|}{2} - iw\cdot\sigma\right)}{\prod_{\alpha\in\Delta} \frac{1}{\sqrt{2\pi} r^{\frac{|\alpha\cdot b'|}{2}}} \Gamma\left(1+\frac{|\alpha\cdot b'|}{2} - i\alpha\cdot\sigma\right)} \equiv \sum_{b'\in\mathcal{W}b} Z_0(b';\sigma,B), \tag{2.15}$$

where $\delta_{B,b'}$ enforces flux conservation: the flux sourced by the monopole equals the flux exiting through S^2. We have introduced the notation Z_0 for an "incomplete" partition function that does not include a sum over the Weyl orbit of b. Such a quantity does not represent a physical monopole operator, but it will prove to be convenient in the following sections.[9]

In general, Z as given above is not the full answer, because there are contributions from additional loci in the localization computation. We now discuss them.

2.4 Monopole Bubbling

Close to the monopole insertions, our BPS equations behave as Bogomolny equations on $\mathbb{R}^3$ with a monopole singularity at the origin. Such equations are known to have "screening solutions" in addition to the simple abelian "Dirac monopole" solution described in the previous subsection. The main property of such solutions is that while at the origin of $\mathbb{R}^3$ they have a monopole singularity characterized by $b \in \Lambda^\vee_w$, at infinity they behave as Dirac monopoles of different charges $v \in \Lambda^\vee_w$. It is also known that such solutions only exist for $|v| < |b|$ such that v is a weight in a representation determined by the highest weight b (in which case v is said to be "associated to" b, sometimes written simply as $v < b$). Let $\mathcal{M}(b,v)$ denote the moduli space of such screening solutions. For given b and v, let ρ be the length scale over which the screening takes place. It is one of the moduli for solutions of the Bogomolny equations, and taking $\rho \to 0$ corresponds to going to the boundary of $\mathcal{M}(b,v)$.

[9] We do not keep careful track of the overall sign of the hemisphere wavefunction, as it cancels in the gluing formula.

In this limit, the solution approaches a Dirac monopole of charge v everywhere on $\mathbb{R}^3$ except for an infinitesimal neighborhood of the origin where the non-abelian screening takes place. This solution can be thought of as a singular (Dirac) monopole screened by coincident and infinitesimally small smooth ('t Hooft-Polyakov) monopoles; the latter have GNO charges labeled by coroots. It is natural to suppose that such solutions also exist on S^3: while at finite ρ they are expected to receive $1/r$-corrections compared to the flat-space case, in the $\rho \to 0$ limit they should be exactly the same bubbling solutions as on $\mathbb{R}^3$.

Notice that our general BPS equations require solutions to be abelian away from the insertion point. However, within the radius ρ, the screening solutions to the Bogomolny equations are essentially non-abelian. Therefore, smooth screening solutions corresponding to generic points of $\mathcal{M}(b, v)$ cannot give new solutions to the BPS equations. However, boundary components of $\mathcal{M}(b, v)$ where $\rho \to 0$ can give new, singular solutions to the BPS equations, which fail to be abelian only at the insertion point of the monopole operator. They should therefore be taken into account in the localization computation. Since such a solution behaves as an abelian Dirac monopole of charge v everywhere except at the insertion point, it is convenient to factor out $Z(v; \sigma, B)$ computed in the previous subsection, and to say that the full contribution from the bubbling locus "$b \to v$" is given roughly by

$$Z_{\text{mono}}(b, v; \sigma, B) Z(v; \sigma, B), \tag{2.16}$$

where Z_{mono} characterizes the effect of monopole bubbling. We call it the bubbling factor. In fact, such a simple presentation is not quite correct, and we need to be more precise here. Recall that the monopole insertion is not just defined by a single singular boundary condition (2.4): rather, one sums over the Weyl orbit of such singular boundary conditions. Therefore, the above expression is expected to have sums over such orbits both for b and v. A more general expectation, which turns out to be correct, is that the contribution of the bubbling locus takes the form:

$$\sum_{\substack{b' \in \mathcal{W}b \\ v' \in \mathcal{W}v}} Z^{\text{ab}}_{\text{mono}}(b', v'; \sigma, B) Z_0(v'; \sigma, B), \tag{2.17}$$

where, as before, b and v are understood to be (dominant) coweights representing magnetic charges, and we sum over their Weyl orbits. The quantity appearing in this equation,

$$Z^{\text{ab}}_{\text{mono}}(b', v'; \sigma, B), \tag{2.18}$$

is called the "abelianized bubbling factor." It depends on coweights $b', v' \in \Lambda_w^\vee \subset \mathfrak{t}$ rather than on cocharacters, while physical answers in the full non-abelian theory depend on cocharacters and thus always include sums over Weyl orbits. The abelianized bubbling factors introduced here prove to be of great importance for the formalism of this paper, and later on we will provide more rigorous evidence for their relevance based purely on group theory arguments rather than the heuristic path-integral-inspired explanation of this section. Note also that the abelianized bubbling factors are expected to behave under Weyl reflections in the following way:[10]

$$Z_{\rm mono}^{\rm ab}({\rm w}\cdot b, {\rm w}\cdot v; {\rm w}\cdot\sigma, {\rm w}\cdot B) = Z_{\rm mono}^{\rm ab}(b, v; \sigma, B),\ \ \forall {\rm w}\in\mathcal{W}. \tag{2.19}$$

Now we can write the complete answer for the hemisphere partition function:

$$\langle\sigma, B|\Psi_b\rangle = Z(b;\sigma,B) + \sum_{|v|<|b|}\sum_{\substack{b'\in\mathcal{W}b\\ v'\in\mathcal{W}v}} Z_{\rm mono}^{\rm ab}(b', v';\sigma, B) Z_0(v';\sigma, B), \tag{2.20}$$

where Ψ_b represents the state generated at the boundary of the hemisphere with a physical monopole of charge b inserted at the tip. Here the first sum goes over dominant coweights v satisfying $|v| < |b|$, while the second sum goes over the corresponding Weyl orbits.

The localization approach to the computation of $Z_{\rm mono}(b, v; \sigma, B)$ is quite technical, having been a subject of several works in the past [47–49], and more recently [50, 52]. In the current paper we do not attempt a direct computation of $Z_{\rm mono}(b, v; \sigma, B)$ or $Z_{\rm mono}^{\rm ab}(b, v; \sigma, B)$. Instead, we describe a roundabout way to find it from the algebraic consistency of our formalism. We find that the $Z_{\rm mono}^{\rm ab}(b, v; \sigma, B)$ are always given by certain rational functions, but we do not need to assume anything about their form.

2.5 Shift Operators

In this section, we derive how insertions of local $\mathcal{Q}^C$-closed observables are represented by operators acting on the hemisphere wavefunction (up to the so far unknown bubbling factors). The easiest ones are polynomials in $\Phi(\varphi)$. Just like in [38], we can think of them as entering the hemisphere either through the North pole $\varphi = 0$, $\theta = \pi/2$, or through the South pole $\varphi = \pi$, $\theta = \pi/2$. Then we simply substitute solution (2.12) into the definition of $\Phi(\varphi)$ either

[10] Here, all variables take values in $\mathfrak{t}$, so the action of $\mathcal{W}$ is unambiguous.

for $0 < \varphi < \pi/2$ or for $\pi/2 < \varphi < \pi$. We find that for the North pole:

$$\Phi(\varphi = 0) = \frac{1}{2}(\Phi_{\dot{1}\dot{1}} + 2\Phi_{\dot{1}\dot{2}} + \Phi_{\dot{2}\dot{2}}) = \frac{1}{r}\left(\sigma + \frac{i}{2}B\right), \tag{2.21}$$

and similarly for the South pole:

$$\Phi(\varphi = \pi) = \frac{1}{2}(-\Phi_{\dot{1}\dot{1}} + 2\Phi_{\dot{1}\dot{2}} - \Phi_{\dot{2}\dot{2}}) = \frac{1}{r}\left(\sigma - \frac{i}{2}B\right). \tag{2.22}$$

This operator simply measures values of σ and B away from the monopole insertion, and bubbling is accounted for trivially. In particular, in the unbubbled locus, B evaluates to b, while for the bubbling loci it evaluates to the corresponding $B = v$. Thus we conclude that $\Phi(\varphi)$ is represented by the following North and South pole operators:

$$\Phi_N = \frac{1}{r}\left(\sigma + \frac{i}{2}B\right) \in \mathfrak{t}_{\mathbb{C}}, \quad \Phi_S = \frac{1}{r}\left(\sigma - \frac{i}{2}B\right) \in \mathfrak{t}_{\mathbb{C}}, \tag{2.23}$$

where B should be thought of as measuring $B \in \Lambda_w^\vee$ at the boundary S^2, i.e., it multiplies the wavefunction $\Psi(\sigma, B)$ by B, and thus is simply a diagonal multiplication operator.

It is also not too hard to obtain generalizations of abelian shift operators from [38] that represent insertions of non-abelian monopoles. From the structure of the partition functions above, it is clear that they take the following form:

$$\mathcal{M}^b = \sum_{b' \in \mathcal{W}b} M^{b'} + \sum_{|v|<|b|} \sum_{\substack{b' \in \mathcal{W}b \\ v' \in \mathcal{W}v}} Z^{\rm ab}_{\rm mono}(b', v'; \sigma, B) M^{v'}. \tag{2.24}$$

Here M^b is an abelianized (non-Weyl-averaged) shift operator which represents insertion of a bare monopole singularity characterized by the coweight (not cocharacter!) b, and which ignores bubbling phenomena. Inclusion of abelianized bubbling coefficients $Z^{\rm ab}_{\rm mono}$ takes care of the screening effects, and summing over Weyl orbits corresponds to passing to cocharacters, i.e., true physical magnetic charges.

The expression (2.24) is evident from the structure of the hemisphere partition function with a monopole inserted, as described in the previous subsections. Indeed, away from the monopole insertion, its effect must be represented by a sum over bubbling sectors, and within each bubbling sector, the contribution must take the form of a sum over the Weyl reflections of the basic contribution. The expression (2.24), in fact, represents nothing else but the abelianization map proposed in [53]: the full non-abelian operator $\mathcal{M}^b$ is written in terms of the abelianized monopoles M^b acting on $\Psi(\sigma, B)$, wavefunctions on $\mathfrak{t} \times \Lambda_w^\vee$.

It remains to determine expressions for M^b acting on wavefunctions $\Psi(\sigma, B)$. Just like in [38], there are separate sets of operators that implement insertions through the North and South poles, which generate isomorphic algebras, and they are uniquely determined by the following set of consistency conditions:

1) They should shift the magnetic flux at which $\Psi(\sigma, B)$ is supported by $b \in \Lambda_w^\vee$.

2) Commutativity with Φ at the opposite pole, i.e., $[M_N^b, \Phi_S] = [M_S^b, \Phi_N] = 0$.

3) Commutativity with another monopole at the opposite pole, i.e., $[M_N^b, M_S^{b'}] = 0$.

4) When acting on the vacuum (empty hemisphere) wavefunction, the result should agree with (2.15).

This set of conditions determines the North shift operator to be

$$M_N^b = \frac{\prod_{w\in\mathcal{R}} \left[\frac{(-1)^{(w\cdot b)_+}}{r^{|w\cdot b|/2}} \left(\frac{1}{2} + irw\cdot\Phi_N\right)_{(w\cdot b)_+} \right]}{\prod_{\alpha\in\Delta} \left[\frac{(-1)^{(\alpha\cdot b)_+}}{r^{|\alpha\cdot b|/2}} \left(ir\alpha\cdot\Phi_N\right)_{(\alpha\cdot b)_+} \right]} e^{-b\cdot(\frac{i}{2}\partial_\sigma + \partial_B)}, \tag{2.25}$$

where $(a)_+ = a$ if $a \geq 0$ and $(a)_+ = 0$ otherwise, and notation $(x)_n$ stands for the Pochhammer symbol $\Gamma(x+n)/\Gamma(x)$. Also, $x \cdot y$ represents the canonical pairing $\mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$. The South pole operator is

$$M_S^b = \frac{\prod_{w\in\mathcal{R}} \left[\frac{(-1)^{(-w\cdot b)_+}}{r^{|w\cdot b|/2}} \left(\frac{1}{2} + irw\cdot\Phi_S\right)_{(-w\cdot b)_+} \right]}{\prod_{\alpha\in\Delta} \left[\frac{(-1)^{(-\alpha\cdot b)_+}}{r^{|\alpha\cdot b|/2}} \left(ir\alpha\cdot\Phi_S\right)_{(-\alpha\cdot b)_+} \right]} e^{b\cdot(\frac{i}{2}\partial_\sigma - \partial_B)}. \tag{2.26}$$

By counting powers of r^{-1} in the general expression (2.25), we find that the dimension of a charge-b monopole is

$$\Delta_b = \frac{1}{2}\left(\sum_{w\in\mathcal{R}} |w\cdot b| - \sum_{\alpha\in\Delta} |\alpha\cdot b| \right). \tag{2.27}$$

This dimension formula will come in handy later.

The shift operators satisfy an important multiplication property, which later on will allow to generate monopoles of arbitrary charge starting from a few low charge monopoles:

$$M_N^{b_1} \star M_N^{b_2} = P_{b_1,b_2}(\Phi) M_N^{b_1+b_2}, \quad \text{for dominant } b_1 \text{ and } b_2, \tag{2.28}$$

and similarly for the South pole operators, where $P_{b_1,b_2}(\Phi)$ is some polynomial in Φ. We use $\star$ to denote products *as operators* (in particular, shift operators act on the Φ-dependent prefactors in $M_{N,S}$), emphasizing that they form an associative non-commutative algebra. In fact, (2.28) holds slightly more generally than for dominant weights: if Δ_+ is some choice of positive roots (determined by a hyperplane in $\mathfrak{t}^*$), then (2.28) holds whenever $(b_1 \cdot \alpha)(b_2 \cdot \alpha) \geq 0$ for all $\alpha \in \Delta_+$. Property (2.28) ensures that in the product of two physical bare monopoles, the highest-charge monopole appears without denominators. If in addition, b_1 and b_2 satisfy property that $(b_1 \cdot w)(b_2 \cdot w) \geq 0$ for all matter weights $w \in \mathcal{R}$, then a stronger equality holds:

$$M_N^{b_1} \star M_N^{b_2} = M_N^{b_1+b_2}. \tag{2.29}$$

Finally, for general b_1 and b_2, we have:

$$M_N^{b_1} \star M_N^{b_2} = \frac{\prod_{w\in\mathcal{R}}(-iw\cdot\Phi_N)^{(w\cdot b_1)_+ + (w\cdot b_2)_+ - (w\cdot(b_1+b_2))_+}}{\prod_{w\in\Delta}(-i\alpha\cdot\Phi_N)^{(\alpha\cdot b_1)_+ + (\alpha\cdot b_2)_+ - (\alpha\cdot(b_1+b_2))_+}} M_N^{b_1+b_2} + O(1/r). \tag{2.30}$$

These are precisely the abelian chiral ring relations of [53].

In addition to defining $M_{N,S}$, we also need to properly define dressed monopoles, which is an important question, especially due to the interplay with bubbling. It is discussed in Section 3. Before going into that, let us first compare our results to supersymmetric indices in four dimensions, which also provide a way to derive the vector multiplet one loop determinant.

2.6 Reduction of Schur Index

Our setup has a natural uplift to a supersymmetric index of 4D $\mathcal{N}=2$ theories on $S^3 \times S^1$. The operators constructed from $\Phi(\varphi)$ lift to supersymmetric Wilson loops wrapping the S^1, while monopole operators correspond to supersymmetric 't Hooft loops on S^1. Certain questions relevant to this 4D setup have been studied in the literature in great detail, and in particular we can use them to determine the unbubbled partition functions. By shrinking the S^1 factor, we can use the known 4D answers as a way to derive the unbubbled one-loop determinants mentioned in the previous subsections. Doing this for the bubbling contributions is more subtle and will be discussed later in this paper, where we find agreement with our method of deriving bubbling terms in cases where the 4D results are known. For simplicity, let us first set the radius r of S^3 to 1, and let us denote the circumference of S^1 by β. To restore r, we should simply send $\beta \to \beta/r$ in all formulas.

Since we already know the one-loop determinant for hypermultiplets, it only remains to determine the vector multiplet contribution, and this can be done in a theory with any conveniently chosen matter content. We can always pick matter content in such a way that both the 4D $\mathcal{N}=2$ and the 3D $\mathcal{N}=4$ theories are conformal. The corresponding 4D index is known as the Schur index. The Schur index in 4D is defined as

$$\mathcal{I}(p) = \mathrm{Tr}_{\mathcal{H}_{S^3}}(-1)^F p^{E-R} \tag{2.31}$$

where the trace is taken over the Hilbert space of the 4d theory on S^3, and R is the Cartan of the $\mathfrak{su}(2)$ R-symmetry, normalized so that the allowed charges are quantized in half-integer units. In the path integral description, $\mathcal{I}(p)$ evaluated when $p = e^{-\beta}$ is given by an $S^3 \times S^1$ partition function, with S^1 of circumference β, and with the R-symmetry twist by $e^{\beta R}$ as we go once around the S^1. This $S^1 \times S^3$ partition function is invariant under all 4d superconformal generators that commute with $E - R$, or in other words that have $E = R$. One can easily list these generators and check that they form an $\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)$ superalgebra. Thus, the superconformal index (2.31) is invariant under $\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)$. It is also invariant under all continuous deformations of the superconformal theory: in particular, it is independent of g_{YM} and can be computed at weak coupling.

One can additionally insert an 't Hooft loop of GNO charge b (taken to be a dominant coweight) wrapping S^1 at one pole of S^3 and the oppositely-charged loop at the opposite pole of S^3. The answer for this modified index in a general 4D $\mathcal{N}=2$ gauge theory, up to a sign and ignoring the bubbling effect, can be found in Eq. (3.44) of [49]:[11]

$$\mathcal{I}_b(p) = \frac{1}{|\mathcal{W}_b|}\int \left(\prod_{i=1}^{\mathrm{rank}(G)} \frac{d\lambda_i}{2\pi}\right) \left(\prod_{\alpha\in\Delta}\left(1 - e^{i\alpha\cdot\lambda} p^{\frac{|\alpha\cdot b|}{2}}\right)\right) \mathrm{P.E.}[I_v(e^{i\lambda_i}, p)]\, \mathrm{P.E.}[I_h(e^{i\lambda_i}, p)] \tag{2.32}$$

where Δ is the set of all roots, I_v is the contribution from the $\mathcal{N}=2$ vector, and I_h is the contribution from the $\mathcal{N}=2$ hyper in representation $\mathcal{R}$,

$$\begin{aligned} I_v(e^{i\lambda_i}, p) &= -2 \sum_{\alpha\in\mathrm{adj}} \frac{p^{1+\frac{|\alpha\cdot b|}{2}}}{1-p} e^{i\alpha\cdot\lambda}\,, \\ I_h(e^{i\lambda_i}, p) &= \sum_{w\in\mathcal{R}} \frac{p^{\frac{1}{2}+\frac{|w\cdot b|}{2}}}{1-p}\left(e^{iw\cdot\lambda} + e^{-iw\cdot\lambda}\right), \end{aligned} \tag{2.33}$$

[11]We set $\eta_a = 1$ and $x = \sqrt{p}$ in Eq. (3.44) of [49].

and P.E. is the plethystic exponential defined as $\mathrm{P.E.}[f(x_i)] = \exp\left[\sum_{n=1}^{\infty}\frac{f(x_i^n)}{n}\right]$. Using the identity $\exp\left[-\sum_{n=1}^{\infty}\frac{a^n}{n(1-q^n)}\right] = (a;q)$, where $(a;q) \equiv \prod_{n=0}^{\infty}(1-aq^n)$ is the q-Pochhammer symbol, we can rewrite $\mathcal{I}_b(p)$ as

$$\mathcal{I}_b(p) = \frac{(p;p)^{2\mathrm{rank}(G)}}{|\mathcal{W}_b|}\int_{-\pi}^{\pi}\left(\prod_{i=1}^{\mathrm{rank}(G)}\frac{\mathrm{d}\lambda_i}{2\pi}\right)\frac{\prod_{\alpha\in\Delta}\left[\left(1-e^{i\alpha\cdot\lambda}p^{\frac{|\alpha\cdot b|}{2}}\right)(e^{i\alpha\cdot\lambda}p^{1+\frac{|\alpha\cdot b|}{2}};p)^2\right]}{\prod_{w\in\mathcal{R}}(e^{iw\cdot\lambda}p^{\frac{1}{2}+\frac{|w\cdot b|}{2}};p)(e^{-iw\cdot\lambda}p^{\frac{1}{2}+\frac{|w\cdot b|}{2}};p)}\,. \quad (2.34)$$

We are interested in determining the 3D hemisphere partition function. One way to obtain it is to use the results of [45] to first extract the 4D half-index from (2.34), and then reduce it to the 3D hemisphere partition function. Alternatively (and this is how we proceed), we first reduce the index (2.34) to 3D to find the S^3 partition function, and then use the gluing formula from Section 2.2 to recover the hemisphere partition function as the square root of the absolute value of integrand. One can then fix signs by consistency with gluing.

To reduce (2.34) down to three dimensions, we have to take the $\beta \to 0$ limit, where in addition to $p = e^{-\beta}$, we have to scale the integration variable accordingly:

$$\lambda = \beta\sigma \in \mathfrak{t}\,, \quad (2.35)$$

so that the angular variable λ (parametrizing the maximal torus $\mathbb{T} \subset G$) "opens up" into an affine variable $\sigma \in \mathfrak{t}$. To take the limit, one needs the following identities (see [38]):

$$\frac{1}{(p^x;p)} = e^{\frac{\pi^2}{6\beta}}\beta^{x-\frac{1}{2}}\frac{1}{\sqrt{2\pi}}\Gamma(x)(1+O(\beta))\,, \qquad (p;p) = \sqrt{\frac{2\pi}{\beta}}e^{-\frac{\pi^2}{6\beta}}(1+O(\beta))\,, \quad (2.36)$$

which give

$$\begin{aligned}\mathcal{I}_b \approx\ & \frac{e^{-\frac{\pi^2 r}{3\beta}\left(\dim G-\sum_{I=1}^{N_f}\dim R_I\right)}}{|\mathcal{W}_b|}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{\mathrm{rank}(G)}d\sigma_i\right)\prod_{\alpha\in\Delta^+}\left((\alpha\cdot\sigma)^2+\frac{|\alpha\cdot b|^2}{4}\right)\\ &\times\frac{\prod_{w\in\mathcal{R}}\left|\frac{\beta^{\frac{|w\cdot b|}{2}}}{r^{\frac{|w\cdot b|}{2}}\sqrt{2\pi}}\Gamma\left(\frac{1+|w\cdot b|}{2}-iw\cdot\sigma\right)\right|^2}{\prod_{\alpha\in\Delta}\left|\frac{\beta^{\frac{|\alpha\cdot b|}{2}}}{r^{\frac{|\alpha\cdot b|}{2}}\sqrt{2\pi}}\Gamma\left(1+\frac{|\alpha\cdot b|}{2}-i\alpha\cdot\sigma\right)\right|^2}\,,\end{aligned} \quad (2.37)$$

as $\beta \to 0$. In (2.37) we restored the radius r of S^3 by dimensional analysis.

The exponential prefactor in (2.37) is precisely the Cardy behavior of [61]. In the integrand, we recognize the 1-loop contribution of the hypermultiplet to the S^3 partition function,

$$Z^{\text{hyper}}_{\text{1-loop},S^3}(\sigma) = \prod_{w\in\mathcal{R}} \left| \frac{1}{r^{\frac{|w\cdot b|}{2}}\sqrt{2\pi}} \Gamma\left(\frac{1+|w\cdot b|}{2} - iw\cdot\sigma\right)\right|^2 , \tag{2.38}$$

multiplied by $\beta^{\frac{|w\cdot b|}{2}}$. The remaining factor in the integrand must be proportional to the 1-loop contribution of vector multiplet to the S^3 partition function. Assuming that the 1-loop vector multiplet contribution comes multiplied by $\beta^{-\frac{|\alpha\cdot b|}{2}}$ (by analogy with the hypermultiplet factor), we conclude that it is equal to

$$Z^{\text{vec}}_{\text{1-loop},S^3}(\sigma) = \frac{\prod_{\alpha\in\Delta^+}\left((\alpha\cdot\sigma)^2 + \frac{|\alpha\cdot b|^2}{4}\right)}{\prod_{\alpha\in\Delta}\left|\frac{1}{r^{\frac{|\alpha\cdot b|}{2}}\sqrt{2\pi}}\Gamma\left(1+\frac{|\alpha\cdot b|}{2} - i\alpha\cdot\sigma\right)\right|^2} . \tag{2.39}$$

The S^3 partition function is then given by the expression

$$Z_b = \frac{1}{|\mathcal{W}_b|}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{\text{rank}(G)} d\sigma_i\right) Z^{\text{vec}}_{\text{1-loop},S^3}(\sigma) Z^{\text{hyper}}_{\text{1-loop},S^3}(\sigma) . \tag{2.40}$$

Note that using this method, the overall normalization of Z_b is ambiguous, but we propose the correct expression is given by (2.40). This expression passes the check that, when $b=0$, it reduces to the S^3 partition function derived in [29], namely

$$Z = Z_0 = \frac{1}{|\mathcal{W}|}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{\text{rank}(G)} d\sigma_i\right) \frac{\prod_{\alpha\in\Delta^+} 4\sinh^2(\pi\alpha\cdot\sigma)}{\prod_{w\in\mathcal{R}} 2\cosh(\pi w\cdot\sigma)} . \tag{2.41}$$

What remains to be done is to use (2.38) and (2.39) to infer the hemisphere one-loop determinants given in (2.13) and (2.14). To do so, we use the gluing formula (2.6) as well as the explicit expression for the gluing measure in (2.7). It immediately follows that the hypermultiplet contributes (2.13) to the hemisphere partition function and that the vector multiplet contributes (2.14). The hypermultiplet contribution (2.13) was also previously determined using an explicit computation of the one-loop determinant on the hemisphere in [38]. It would be interesting to carry out the analogous computation for the non-abelian vector multiplet determinant, which we have bypassed by means of the above argument.[12]

[12]Note that the hemisphere and the $\mathcal{Q}^C$-invariant background (2.12) with a monopole at the tip $\eta = 0$

3 Dressing and Abelianized Bubbling

We have derived the structure of bare monopoles, up to the so far unknown bubbling factors. In this section, we extend this to more general dressed monopoles. Recall that the magnetic charge b breaks the gauge group at the insertion point down to G_b, the centralizer of b. One is allowed to dress the monopole operator by some G_b-invariant polynomial $P(\Phi)$ in the variable $\Phi(\varphi)$, as is well-known in the literature [53], see also Section 2.1.2 above.

If $P(\Phi)$ is invariant under the full gauge group G, then it is a valid $\mathcal{Q}^C$-closed observable on its own. This makes definition of the dressed monopole essentially trivial: we simply "collide" two separate observables $P(\Phi)$ and $\mathcal{M}^b$, which within our formalism means multiplying them *as operators* acting on the hemisphere wavefunction. Using the star product notation for such multiplication, we thus have:

$$\left[P(\Phi)\mathcal{M}^b\right] := P(\Phi) \star \mathcal{M}^b. \tag{3.1}$$

When the polynomial $P(\Phi)$ is only invariant under a subgroup G_b, not the full gauge group G, we have to proceed differently as $P(\Phi)$ only makes sense as part of $\left[P(\Phi)\mathcal{M}^b\right]$, not as a separate observable. Had bubbling not been an issue, the solution would be straightforward

preserve an $\mathcal{N}=2$ subalgebra $\mathfrak{su}(2|1)$, generated by what are called $Q_1^\pm$ and $Q_2^\pm$ in [38]. The suggestive form of (2.14) then leads one to wonder whether it can be explained by a Higgsing argument familiar in the study of theories with four supercharges (see, e.g., [62]). Namely, with respect to the aforementioned $\mathcal{N}=2$ subalgebra, the hypermultiplet decomposes into $\mathcal{N}=2$ chiral multiplets of R-charge 1/2 and the vector multiplet decomposes into an $\mathcal{N}=2$ vector multiplet and an adjoint chiral multiplet of R-charge 1. Suppose that one could deform the action in such a way as to accommodate arbitrary R-charge q for the chiral multiplets transforming in representations $\mathcal{R}, \bar{\mathcal{R}}$ of G (as in, e.g., [63,64]) while preserving the $\mathcal{N}=2$ superpotential coupling that descends from the $\Phi^{\dot{a}\dot{b}}\Phi_{\dot{a}\dot{b}}$ term in the $\mathcal{N}=4$ Lagrangian. Then one might expect the corresponding chiral multiplet one-loop determinant on the hemisphere to take the form of a product over weights $w \in \mathcal{R}$ of

$$Z^q_{\text{chiral}}(w\cdot\sigma) \sim \Gamma\left(1-q+\frac{|w\cdot B|}{2}-iw\cdot\sigma\right), \tag{2.42}$$

so that the numerator of $Z_0(b';\sigma,B)$ in (2.15) comes from $Z^{1/2}_{\text{chiral}}(w\cdot\sigma)$ and the denominator from

$$Z_{\text{vector}}(\alpha\cdot\sigma) = \frac{1}{Z^0_{\text{chiral}}(-\alpha\cdot\sigma)}, \tag{2.43}$$

by reflection symmetry of the roots α. Here, (2.43) follows from the Higgs mechanism and $Z_{\text{vector}}(\alpha\cdot\sigma)$ denotes the contribution to the vector multiplet one-loop determinant from a mode in the α-direction.

It would be interesting to make this intuition precise. However, due to our choice of $\mathcal{N}=2$ superalgebra on S^3, ours is not the standard $\mathcal{N}=2$ Coulomb branch localization, where chiral multiplet fields vanish on the localization locus. Indeed, (2.12) implies a nontrivial background for the scalar in the adjoint chiral multiplet (i.e., σ/r) as well as for the scalar in the $\mathcal{N}=2$ vector multiplet (i.e., $-B/r\sin\eta$). In particular, $\sigma\in\mathfrak{t}$ is not the standard Coulomb branch parameter: it labels the scalar zero mode of the adjoint chiral and *not* of the vector.

again: we would simply define $[P(\Phi)\mathcal{M}^b] = |\mathcal{W}_b|^{-1}\sum_{\mathrm{w}\in\mathcal{W}} P(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot b}$. In general, however, the bubbling is present, making such a simple definition incomplete.

For the remainder of this section, we focus on the case of a simple gauge group G. The generalization to the situation where G is a simple factor of a larger gauge group is straightforward: different simple factors couple to each other only through the matter multiplets, and representation-theoretic issues can be addressed for each simple factor separately. The final conclusion of this section—Theorem 1—holds for general G, with the understanding that for non-simple gauge groups, bubbling terms for a monopole operator magnetically charged under one simple factor might depend on scalars Φ from other simple factors as well. From the point of view of a given simple factor G, Φ's valued in other simple factors G' act as twisted masses for G'.

3.1 Dressed Monopoles and Invariant Theory

A general dressed monopole operator takes the form

$$\left[P(\Phi)\mathcal{M}^b\right] = \frac{1}{|\mathcal{W}_b|}\sum_{\mathrm{w}\in\mathcal{W}} P(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot b} + \cdots \tag{3.2}$$

where the ellipses stand for bubbling contributions. It is intuitively clear that such observables constructed for all possible choices of $P(\Phi)$ cannot all be independent: there should exist a minimal set of dressed monopoles, a basis in some sense, from which all other dressed monopoles somehow follow. In this subsection, we make this intuition precise by rigorously proving that for a given magnetic charge b, there exists a set of *primitive* dressed monopoles that accomplish this.

Definition 1: dressed monopoles $\left[P_1(\Phi)\mathcal{M}^b\right], \left[P_2(\Phi)\mathcal{M}^b\right], \ldots, \left[P_p(\Phi)\mathcal{M}^b\right]$ are called primitive (of magnetic charge b) if they form a basis for the (free) module of dressed charge-b monopoles over the ring of G-invariant polynomials. This means that by taking linear combinations

$$\sum_{i=1}^{p} Q_i(\Phi)\star\left[P_i(\Phi)\mathcal{M}^b\right] \tag{3.3}$$

with all possible G-invariant polynomials $Q_i(\Phi)$, we obtain dressed monopoles with all possible leading terms of the form (3.2), and furthermore, p is the minimal number that makes this possible. We will always assume $P_1(\Phi) = 1$, so that the first primitive monopole is the bare monopole itself.

Example *In $SU(2)$ gauge theory, the Weyl group is $\mathbb{Z}_2$, which simply flips $b \to -b$ and $\Phi \to -\Phi \in \mathfrak{t}_{\mathbb{C}}$. Each dressed monopole of charge b takes the form $P(\Phi)M^b + P(-\Phi)M^{-b} + bubbling$. In this case, there are only two primitive dressed monopoles for each b:*

$$\begin{aligned} \mathcal{M}^b &= M^b + M^{-b} + \textit{bubbling}, \\ \left[\Phi\mathcal{M}^b\right] &= \Phi(M^b - M^{-b}) + \textit{bubbling}. \end{aligned} \tag{3.4}$$

By writing:

$$P(\Phi) = \frac{P(\Phi) + P(-\Phi)}{2} + \frac{P(\Phi) - P(-\Phi)}{2\Phi}\Phi, \tag{3.5}$$

it becomes obvious that any other dressed monopole can be defined as:

$$\left[P(\Phi)\mathcal{M}^b\right] := \frac{P(\Phi) + P(-\Phi)}{2} \star \mathcal{M}^b + \frac{P(\Phi) - P(-\Phi)}{2\Phi} \star \left[\Phi\mathcal{M}^b\right]. \tag{3.6}$$

To describe primitive monopoles for general gauge groups, it is enough to focus on the leading term of (3.2), which is what we do in this subsection. Bubbling contributions will be analyzed from this point of view in the next subsection.

The leading term in (3.2) is constructed to be invariant under the Weyl group action. Therefore, we can classify such leading terms by identifying invariants of the Weyl group in the corresponding (reducible) representations of $\mathcal{W}$. Alternatively, we could achieve this by focusing on the dressing factors, and classify polynomials $P(\Phi)$ invariant under G_b. Since after the localization (and gauge fixing), $\Phi \in \mathfrak{t}_{\mathbb{C}}$, it is enough to impose invariance under $\mathcal{W}_b$ (the Weyl group of G_b), and thus dressed monopoles can be classified by $\mathcal{W}_b$-invariant polynomials in Φ.[13] Nevertheless, we find it more convenient to study the invariants of $\mathcal{W}$ directly, and this is the approach we adhere to.

Proposition 1: Let G be a simple gauge group, $\mathcal{W}$ its Weyl group, and b a dominant coweight (a magnetic charge). Then there exists a set of primitive monopoles (of magnetic charge b) $\left[P_1(\Phi)\mathcal{M}^b\right], \left[P_2(\Phi)\mathcal{M}^b\right], \ldots, \left[P_p(\Phi)\mathcal{M}^b\right]$, where $p = |\mathcal{W}b|$ is the size of the Weyl orbit of b.

The remainder of this subsection is devoted to proving this proposition using classical facts from the invariant theory. Less mathematically inclined readers are free to skip it.

Proof. Consider ρ^b – a representation of $\mathcal{W}$ spanned, as a $\mathbb{C}$-linear space, by the Weyl

[13] $\mathcal{W}_b$-invariant polynomials on $\mathfrak{t}$ can be uniquely extended to G_b-invariant polynomials on $\mathfrak{g}$.

orbit of coweight b; we write it in terms of shift operators $M^{\mathrm{w}\cdot b}, \mathrm{w} \in \mathcal{W}$ as:

$$\rho^b := \mathrm{Span}_{\mathbb{C}}\{M^{\mathrm{w}\cdot b} | \mathrm{w} \in \mathcal{W}\}. \tag{3.7}$$

It is reducible and, in particular, contains a trivial subrepresentation spanned by $\sum_{b' \in \mathcal{W}\cdot b} M^{b'}$, which is the simplest invariant corresponding to the bare monopole operator.

The Cartan subalgebra $\mathfrak{t}$ itself is also a $\mathcal{W}$-module: $\mathcal{W}$ acts on it in an irreducible r-dimensional representation, where $r = \mathrm{rank}(G)$. We will denote such a representation simply as $\mathfrak{t}$, and the variable $\Phi = \sum_{a=1}^r \Phi_a h^a$ clearly takes values in this representation.

However, recall from the discussion after (2.5) that when $\mathrm{w} \in \mathcal{W}$ acts on a dressed monopole operator, transforming the weight according to $b \mapsto \mathrm{w}\cdot b$, physics tells us that the dressing factor should be acted on by w^{-1}, i.e., $\Phi \mapsto \Phi^{\mathrm{w}} = \mathrm{w}^{-1}\cdot\Phi$. Since w is represented by an orthogonal matrix on $\mathfrak{t}$, this is the same as to act with w^{T} from the left or with w from the right. This is how one acts in a dual representation; thus in a dressed monopole operator, we think of Φ as transforming in the dual representation $\mathfrak{t}^*$. The dressing factor $P(\Phi)$ entering (3.2), being a polynomial in Φ, transforms in $S\mathfrak{t}^*$, the symmetric algebra of $\mathfrak{t}^*$, or equivalently, the algebra $\mathbb{C}[\mathfrak{t}]$ of polynomial functions on $\mathfrak{t}$. This implies that any dressed monopole operator is determined by an invariant vector inside the following $\mathcal{W}$-module:

$$\mathfrak{R}_b := S\mathfrak{t}^* \otimes \rho^b = \mathbb{C}[\mathfrak{t}] \otimes \rho^b. \tag{3.8}$$

Thus the leading terms in dressed monopoles of charge b are classified by invariants $\mathfrak{R}_b^{\mathcal{W}}$.

Questions of this sort have been studied extensively in the ancient subject of invariant theory (see, for example, [65]). To begin, let us understand the structure of $S\mathfrak{t}^* \cong \mathbb{C}[\mathfrak{t}]$ as a representation of $\mathcal{W}$, in particular its isotypic decomposition. It is well-known that the ring of invariants for reflection groups (such as the Weyl group) has the structure of another polynomial ring (see [65, Part V], in particular [65, Sec.18-1]):

$$\mathbb{C}[\mathfrak{t}]^{\mathcal{W}} \cong \mathbb{C}[f_1, \dots, f_r], \text{ where } r = \dim\mathfrak{t} = \mathrm{rank}(G). \tag{3.9}$$

Where f_i are invariant homogeneous polynomials of fixed degrees d_i satisfying $\prod_{i=1}^r d_i = |\mathcal{W}|$.

Another well-known object is the ring of covariants [65, Part VII], which is defined as follows. Consider an ideal in $\mathbb{C}[\mathfrak{t}]$ generated by non-constant invariant polynomials:

$$I = \left(\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}_{\deg>0}\right) = (f_1, f_2, \dots, f_r). \tag{3.10}$$

The ring of covariants is defined as:

$$\mathbb{C}[\mathfrak{t}]_{\mathcal{W}} = \mathbb{C}[\mathfrak{t}]/I. \tag{3.11}$$

It is again well-known [65, Sec.24-1] that $\mathbb{C}[\mathfrak{t}]_{\mathcal{W}} \cong \mathbb{C}[\mathcal{W}]$, as a $\mathcal{W}$-module, where $\mathbb{C}[\mathcal{W}]$ is the regular representation. Since $\mathcal{W}$ maps I to itself, Maschke's theorem implies that one can find a $\mathcal{W}$-invariant subspace $M_{\mathcal{W}} \subset \mathbb{C}[\mathfrak{t}]$ such that $\mathbb{C}[\mathfrak{t}] \cong I \oplus M_{\mathcal{W}}$, and this $M_{\mathcal{W}} \cong \mathbb{C}[\mathfrak{t}]_{\mathcal{W}} \cong \mathbb{C}[\mathcal{W}]$. Finally, [65, Sec.18-3] implies that $\mathbb{C}[\mathfrak{t}]$ is a free $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$-module generated by the basis of $M_{\mathcal{W}}$, that is $\mathbb{C}[\mathfrak{t}] \cong \mathbb{C}[\mathfrak{t}]^{\mathcal{W}} \otimes_{\mathbb{C}} M_{\mathcal{W}}$. To summarize, the structure of $S\mathfrak{t}^* \cong \mathbb{C}[\mathfrak{t}]$ as a $\mathcal{W}$-module is:

$$\mathbb{C}[\mathfrak{t}] \cong \mathbb{C}[\mathfrak{t}]^{\mathcal{W}} \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{W}], \tag{3.12}$$

where $\mathbb{C}[\mathcal{W}]$ is realized on polynomials from $M_{\mathcal{W}} \subset \mathbb{C}[\mathfrak{t}]$. This also encodes the isotypic decomposition since every m-dimensional irrep of $\mathcal{W}$ appears in $\mathbb{C}[\mathcal{W}]$ precisely m times.

With this knowledge, our representation of interest becomes:

$$\mathfrak{R}_b \cong \mathbb{C}[\mathfrak{t}]^{\mathcal{W}} \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho^b. \tag{3.13}$$

Now the problem of identifying $\mathfrak{R}_b^{\mathcal{W}}$ simplifies substantially,

$$\mathfrak{R}_b^{\mathcal{W}} \cong \mathbb{C}[\mathfrak{t}]^{\mathcal{W}} \otimes_{\mathbb{C}} \left(\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho^b\right)^{\mathcal{W}}, \tag{3.14}$$

namely, we have to find an invariant subspace in $\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho^b$, which is a product of two finite-dimensional representations of $\mathcal{W}$. Any other element of $\mathfrak{R}_b^{\mathcal{W}}$ is obtained by multiplication with invariant polynomials from $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}} = \mathbb{C}[f_1, \dots, f_r]$.

What we have proven so far is the following: $\mathfrak{R}_b^{\mathcal{W}}$ is a free $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$-module, and any $\mathbb{C}$-basis of $\left(\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho^b\right)^{\mathcal{W}}$ gives a $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$-basis of $\mathfrak{R}_b^{\mathcal{W}}$, i.e., a set of primitive dressed monopoles of magnetic charge b.

To compute $\left(\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho^b\right)^{\mathcal{W}}$, we simply have to decompose each of the two representations into irreducible components, and pair up dual representations. Indeed, by Schur's lemma, only tensor products like $V \otimes V^*$, where V is some irrep and V^* is its dual, can contain invariant subspaces. We can also easily find the dimension of $\left(\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho^b\right)^{\mathcal{W}}$. Recall that $\mathbb{C}[\mathcal{W}]$ contains each irreducible representation ρ_i of $\mathcal{W}$ exactly $\dim(\rho_i)$ times, which implies:

$$\left(\mathbb{C}[\mathcal{W}] \otimes_{\mathbb{C}} \rho_i\right)^{\mathcal{W}} \cong \mathbb{C}^{\dim(\rho_i)}. \tag{3.15}$$

Decomposing ρ^b into irreducible components, $\rho^b \cong \oplus_{i\in I(b)}\rho_i$, this obviously gives:

$$\left(\mathbb{C}[\mathcal{W}]\otimes_{\mathbb{C}}\rho^b\right)^{\mathcal{W}} \cong \oplus_{i\in I(b)}\left(\mathbb{C}[\mathcal{W}]\otimes_{\mathbb{C}}\rho_i\right)^{\mathcal{W}} \cong \mathbb{C}^{\dim(\rho^b)}. \tag{3.16}$$

Hence there are exactly $\dim(\rho^b) = |\mathcal{W}b|$ primitive dressed monopoles of charge b. □

Recall that we were actually classifying leading terms in dressed monopoles. Each such leading term is supposed to be extended by the appropriate bubbling contributions to give a complete physical dressed monopole, and primitive monopoles are no exception:

$$\left[P_i(\Phi)\mathcal{M}^b\right] = \frac{1}{|\mathcal{W}_b|}\sum_{\mathrm{w}\in\mathcal{W}} P_i(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot b} + \text{bubbling}, \quad i = 1\dots|\mathcal{W}b|. \tag{3.17}$$

We now turn to the study of these bubbling contributions.

3.2 Abelianized Monopole Bubbling

Suppose we have found a set of polynomials $P_1,\dots,P_{|\mathcal{W}b|}$ such that the dressed monopoles $[P_i(\Phi)\mathcal{M}^b]$ form the primitive set for a given magnetic charge b, in the sense explained in the previous subsection – that is, $|\mathcal{W}_b|^{-1}\sum_{\mathrm{w}\in\mathcal{W}} P_i(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot b}$ for $i = 1\dots|\mathcal{W}b|$ form a basis of $\mathfrak{R}_b^{\mathcal{W}}$ (the space of dressed charge-b monopoles) over $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$ (the algebra of gauge-invariant polynomials in Φ). In this subsection, we would like to show that there exists a special abelianized monopole shift operator $\widetilde{M}^b = M^b + \dots$, such that

$$\left[P_i(\Phi)\mathcal{M}^b\right] = \frac{1}{|\mathcal{W}_b|}\sum_{\mathrm{w}\in\mathcal{W}} P_i(\Phi^{\mathrm{w}})\widetilde{M}^{\mathrm{w}\cdot b}. \tag{3.18}$$

The left-hand side has the following structure: for each i,

$$\left[P_i(\Phi)\mathcal{M}^b\right] = \frac{1}{|\mathcal{W}_b|}\sum_{\mathrm{w}\in\mathcal{W}} P_i(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot b} + \frac{1}{|\mathcal{W}_b|}\sum_{|v|<|b|}\sum_{\mathrm{w}\in\mathcal{W}} V_i^{b\to v}(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot v}. \tag{3.19}$$

Here, the first sum corresponds to the sector with no screening effects, and the remaining terms describe monopole bubbling, with $V_i^{b\to v}$ given by some rational functions of $\Phi \in \mathfrak{t}_{\mathbb{C}}$ that encode the bubbling data (because V_i are not known yet, $|\mathcal{W}_b|^{-1}$ in the second term is optional). By equating the right-hand sides of (3.18) and (3.19), we obtain a system of linear equations on $\widetilde{M}^{\mathrm{w}\cdot b}$, $\mathrm{w} \in \mathcal{W}$:

$$\sum_{\mathrm{w}\in\mathcal{W}} P_i(\Phi^{\mathrm{w}})\widetilde{M}^{\mathrm{w}\cdot b} = \sum_{\mathrm{w}\in\mathcal{W}} P_i(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot b} + \sum_{|v|<|b|}\sum_{\mathrm{w}\in\mathcal{W}} V_i^{b\to v}(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot v}. \tag{3.20}$$

Its solution is going to be the definition of $\widetilde{M}^b$, but first we need to show that such a solution exists, that is the matrix of coefficients $P_i(\Phi^w)$ is non-degenerate. This basically follows from $[P_i(\Phi)\mathcal{M}^b]$ being primitive, and the corresponding proof is given in Appendix C.

The solution to (3.20) takes the form

$$\widetilde{M}^b = M^b + \sum_{|v|<|b|} Z^{\text{ab}}_{b\to v}(\Phi) M^v, \tag{3.21}$$

where the first term has an obvious origin – it is the shift operator that describes the sector without bubbling. Here b is a fixed coweight, whereas the sum in the second term is taken over *all coweights* whose length is less than that of $|b|$.

The functions $Z^{\text{ab}}_{b\to v}(\Phi)$ are some rational functions of $\Phi \in \mathfrak{t}_{\mathbb{C}}$ that encode the bubbling phenomena. They do not have any invariance property under the action of $\mathcal{W}$. We may extend them to non-dominant b by postulating the following transformation property:

$$Z^{\text{ab}}_{\text{w}\cdot b\to \text{w}\cdot v}(\Phi) = Z^{\text{ab}}_{b\to v}(\Phi^{\text{w}}), \tag{3.22}$$

consistent with (2.19). These functions are what we refer to as *abelianized bubbling factors*. Recall that we have previously argued for their existence using heuristic path integral reasoning. We have now rigorously proven their existence by relying solely on group theory.

As also mentioned in Appendix C, the expression for $\widetilde{M}^{\text{w}\cdot b}$ can be obtained from the expression for $\widetilde{M}^b$ by a Weyl reflection:

$$\widetilde{M}^{\text{w}\cdot b} = M^{\text{w}\cdot b} + \sum_{|v|<|b|} Z^{\text{ab}}_{b\to v}(\Phi^{\text{w}}) M^{\text{w}\cdot v} = M^{\text{w}\cdot b} + \sum_{|v|<|b|} Z^{\text{ab}}_{\text{w}\cdot b\to \text{w}\cdot v}(\Phi) M^{\text{w}\cdot v}. \tag{3.23}$$

Having established the existence of such abelianized bubbled monopoles $\widetilde{M}^b$, one can very easily construct arbitrary dressed monopoles. In fact, this proves the following theorem.

Theorem 1: A shift operator describing an arbitrary physical dressed monopole of magnetic charge b can be constructed as

$$\frac{1}{|\mathcal{W}_b|} \sum_{\text{w}\in\mathcal{W}} F(\Phi^{\text{w}}) \widetilde{M}^{\text{w}\cdot b}, \tag{3.24}$$

where $F(\Phi)$ is an arbitrary polynomial in $\Phi \in \mathfrak{t}_{\mathbb{C}}$.

Such an expression will automatically produce, in the leading term, F averaged over $\mathcal{W}_b$, the stabilizer of b in $\mathcal{W}$, as well as generating the appropriate subleading terms describing bubbling.

The abelianized bubbling coefficients prove to be very useful below.

4 Bubbling from Polynomiality

The algebra of quantum Coulomb branch operators, $\mathcal{A}_C$, is believed to be formed by gauge invariant polynomials $P(\Phi)$ in the $\mathcal{Q}^C$-closed variable $\Phi(\varphi)$ and dressed monopole operators $[F(\Phi)\mathcal{M}^b]$, where the dressing factor $F(\Phi)$ is a G_b-invariant polynomial in $\Phi(\varphi)$. Note that the subleading (bubbling) terms in $[F(\Phi)\mathcal{M}^b]$ can involve rational functions of Φ, and only the leading term is built from the polynomial $F(\Phi)$. Such an assumption has also been made in the recent literature on 3d $\mathcal{N}=4$ Coulomb branches [53,66]. One of the reasons to expect this is that vevs of such operators are expected to be algebraic functions on the Coulomb branch, thus it would be unnecessary (and problematic) to introduce poles by choosing $P(\Phi)$ or $F(\Phi)$ rational. The appearance of rational functions in the OPE can also be ruled out using similar reasoning.

In good or ugly theories, we can make this argument slightly more explicit. The Coulomb branch in such theories is expected to be a hyperkähler cone. Furthermore, because conformal dimensions are bounded from below, and there are only finitely many operators below any fixed value of conformal dimension, and because $\frac{1}{r}$ has dimension one, only finitely many operators can appear on the right in any OPE. In particular, this should hold for star products in $\mathcal{A}_C$, which is simply a sector of the OPE algebra in the IR CFT. This excludes denominators of the form $\left(\frac{1}{r}+P(\Phi)\right)^{-1}$, where $P(0)=0$, as such denominators, when expanded in $\frac{1}{r}$, give infinitely many terms. The remaining possibility is to have denominators of the form $1/P(\Phi)$, where $P(0)=0$. But such operators blow up at the origin of the cone: they are not part of the coordinate ring and thus should not appear in the algebra.

In general, it is hard to give a more rigorous argument for polynomiality of observables, especially due to the absence of mathematical definition of QFTs that we are working with. Nevertheless, we proceed under the assumption that polynomiality holds, using the heuristic reasoning and support from the existing literature as a good evidence for it. Furthermore, the results that we are going to describe are in complete agreement with this assumptions, implying that the algebra $\mathcal{A}_C$ constructed to satisfy polynomiality is self-consistent.

One observation is that if we neglect to include bubbling terms in the definition of $[P(\Phi)\mathcal{M}^b]$, then polynomiality in general fails: operator products of such observables produce denominators that do not cancel. Therefore, one role of the bubbling terms is to guarantee polynomiality. In this section, we argue that polynomiality actually fully determines the algebra $\mathcal{A}_C$, up to a natural ambiguity of operator mixing.

4.1 Mixing Ambiguity and Deformation Quantization

In quantum field theories, an arbitrarily chosen basis of observables need not be diagonal with respect to the two-point function, nor does it need to diagonalize the dilatation operator in case of a CFT. Observables can mix with others of the same dimension, and on curved spaces, they can also mix with those of lower dimension, the difference being compensated by powers of background (super)gravity fields. The mixing patterns often depend on the short-distance effects, in particular how we define composite observables, creating ambiguities that have to be resolved in the end by diagonalizing the two-point function.

For our theories on S^3, the Riemann curvature is proportional to $1/r^2$. Mixing with odd powers of $1/r$ might not necessarily be generated by coupling to background SUGRA, but we still include it in the formalism, as it helps with the polynomiality argument in following sections. It could be that there is some other requirement that we can impose along with polynomiality such that we would still determine the bubbling coefficients uniquely up to mixing with only even powers of $1/r$. However, we do not have to do so in this paper: mixing ambiguities can still be resolved in the end. The presence of operator mixing implies that in our problem it is natural to make r-dependent basis changes of the form:

$$\mathcal{O} \mapsto \mathcal{O} + \sum_{n\geq 0} \frac{1}{r^n}\mathcal{O}_n, \tag{4.1}$$

where dimensions should be respected: if $\mathcal{O}$ has dimension Δ, then $\mathcal{O}_n$ should have dimension $\Delta - n$. Other quantum numbers, if present, should also be preserved by such transformations.

One might recognize redefinitions of the form (4.1) as typical "gauge" transformations considered in (equivariant) deformation quantization. In the present context, it was discussed in [36], where the problem was posed for an SCFT in flat space, and transformations of the form (4.1) were less relevant due to the absence of a natural "mixing" parameter (like our $1/r$). In the S^3 setup, however, (4.1) do naturally arise because of mixing. Such transformations have of course first appeared in the deformation quantization literature [67–73], where classification of quantizations often drastically simplifies once the problem is studied modulo (4.1). It is therefore quite reasonable to first solve our problem of constructing $\mathcal{A}_C$ modulo transformations of the form (4.1) (or rather, similar ones defined in the next paragraph). After that, the mixing ambiguities can be resolved: in an SCFT, this can be attained by diagonalizing the two-point function. The diagonalization determines a preferred basis of "SCFT operators" in the algebra $\mathcal{A}_C$. Alternatively, sometimes it might be enough to have an answer given in *some* basis, not necessarily the diagonal one (especially in bad theories, where one cannot straightforwardly compute correlators).

For the study of Coulomb branch operators, transformations of the general form (4.1) might not be the most adequate choice. We wish to think of the leading (i.e., no-bubbling) term of a dressed monopole $[P(\Phi)\mathcal{M}^b]$ as canonically defined, while the subleading, bubbling terms, might be ambiguous. If $P(\Phi)$ has large enough degree, one can find other monopole operators in the theory that have higher magnetic charge but lower dimension. According to (4.1), they can mix with $[P(\Phi)\mathcal{M}^b]$. This can indeed happen in physical operator mixing. However, if we wish to study the structure of monopole operators, such a mixing is too crude as it would alter the leading term of $[P(\Phi)\mathcal{M}^b]$. We therefore define other, less general transformations that respect the GNO charge. Namely, if $\mathcal{O}$ is a monpole operator of GNO charge b, we only consider mixing with operators corresponding to GNO charges v (including zero) such that b can bubble into v. Recall that it means that $|v| < |b|$, and v belongs to the ${}^L G$-representation of highest weight b. Such relation determines a partial order on the set of operators, and we denote by $|\mathcal{O}_n| < |\mathcal{O}|$ the situation when the GNO charge of $\mathcal{O}$ "can bubble" into the GNO charge of $\mathcal{O}_n$. Then we may consider less restrictive transformations:

$$\mathcal{O} \mapsto \mathcal{O} + \sum_{\substack{n\geq 0 \\ |\mathcal{O}_n|<|\mathcal{O}|}} \frac{1}{r^n}\mathcal{O}_n, \tag{4.2}$$

where as before the dimension of $\mathcal{O}_n$ is n units smaller than that of $\mathcal{O}$.

We wish to first study monopole operators modulo such transformations. This means that for a given monopole $[P(\Phi)\mathcal{M}^b]$, the bubbling terms are not uniquely determined. We can shift $[P(\Phi)\mathcal{M}^b]$ by a linear combination of dressed monopoles of lower magnetic charge and lower dimension (differences in dimensions being compensated by powers of $1/r$), still obtaining a valid, though different, definition of a dressed monopole operator. We refer to (4.2) as the mixing ambiguity later on in this paper. The more general mixing (4.1) would only be relevant in an SCFT if we look for an orthonormal basis of observables in the end.

Such shifts significantly alter bubbling coefficients $V_i^{b\to v}(\Phi)$ in the definition of $[P(\Phi)\mathcal{M}^b]$: they can be shifted by polynomials or even by multiples of other bubbling terms, which translates into complicated rational ambiguities of *abelianized* bubbling coefficients $Z^{\rm ab}_{b\to v}(\Phi)$. Any concrete expressions for bubbling coefficients available in the literature always implicitly refer to some choice of basis, thus resolving mixing ambiguity in the algebra of observables. The presence of such ambiguities inherent to $\mathcal{A}_C$ means that there is no chance to determine $\mathcal{A}_C$ simply from polynomiality. In particular, this gives a negative answer to a question raised, e.g., in [66] on whether structural properties of $\mathcal{A}_C$ (polynomiality and gauge-invariance) determine it uniquely. We are going to argue, however, that the next simplest possibility holds: $\mathcal{A}_C$ is uniquely determined by polynomiality precisely up to mixing ambiguities of the form (4.2)! We start by proving this claim in the simplest cases.

4.2 Baby Case: Theories with Minuscule Monopoles

The simplest case is when the algebra $\mathcal{A}_C$ is fully generated by monopole operators in minuscule representations of ${}^LG$. Such monopoles cannot bubble because for minuscule coweights b, there are no v such that both $|v| < |b|$ holds and $b - v$ is a coroot. For such monopole operators, we have the following simple expressions:

$$\left[P(\Phi)\mathcal{M}^b\right] = \frac{1}{|\mathcal{W}_b|} \sum_{\mathrm{w}\in\mathcal{W}} P(\Phi^{\mathrm{w}}) M^{\mathrm{w}\cdot b}. \tag{4.3}$$

Higher-charge monopole operators might contain bubbling terms, but they are easily determined by taking products of lower-charge monopoles. Such cases were previously addressed in the literature using different methods, and essentially comprise the main examples in [53] because abelianization has a simpler structure in this case.

Theories with minuscule generators include those with the gauge group $PSU(N) = SU(N)/\mathbb{Z}_N$, whose Langlands dual is $SU(N)$: the fundamental weights of $SU(N)$ are minuscule and thus cannot bubble. Another example is $U(N)$ gauge theory, since $U(N)$ is self-dual, and its fundamental weights are also minuscule.[14] We will discuss this more in Section 5. Now we move on to the more interesting (and novel) case of theories with no minuscule generators. We start from the lower-rank gauge groups.

4.3 Rank-One Theories

The only rank-one gauge theory that has no minuscule generators is an $SU(2)$ gauge theory. The dual group is $SO(3)$, and thus the lowest monopole operator corresponds to a root, i.e., vector representation of $SO(3)$. In a normalization where weights of $SU(2)$ are half-integer, and products of weights with monopole charges (cocharacters, or dominant coweights) are integers, the minimal monopole has $b = 2$. It can bubble to a zero charge sector, because

[14] One can also use $U(N)$ results to solve the $SU(N)$ theory, even though the latter has no minuscule monopoles. This point will be discussed later.

$0 < |b|$ and $b - 0$ is a root. The abelianized monopole operator takes the form

$$\widetilde{M}^2 = M^2 + Z(\Phi), \tag{4.4}$$

with a single abelianized bubbling term, a function $Z(\Phi)$. Knowledge of $Z(\Phi)$ allows one to construct arbitrary dressed monopole operators of charge 2 and ultimately, by taking star products of the latter, monopoles of arbitrary charge.

For the purpose of generality, we may assume that $SU(2)$ is a simple factor in a larger gauge group $G = SU(2) \times \cdots$. Therefore, we implicitly assume that $Z(\Phi)$ might also depend on scalars Φ valued in other simple factors of G, which from the point of view of a given $SU(2)$ factor play the role of masses.

The basic shift operator of charge $b \in \mathbb{Z}$ is (we work in the North picture from now on, so we drop the "N" subscript):

$$M^b = \frac{\prod_{w\in\mathcal{R}} \left[\frac{(-1)^{(w\cdot b)_+}}{r^{|w\cdot b|/2}} \left(\frac{1}{2} + irw\cdot\Phi_N\right)_{(w\cdot b)_+}\right]}{\prod_{\alpha\in\{+1,-1\}} \left[\frac{(-1)^{(\alpha\cdot b)_+}}{r^{|\alpha\cdot b|/2}} \left(ir\alpha\cdot\Phi_N\right)_{(\alpha\cdot b)_+}\right]} e^{-b\cdot(\frac{i}{2}\partial_\sigma + \partial_B)}. \tag{4.5}$$

By counting powers of r^{-1}, we read off the dimension of a bare monopole of charge b:

$$\Delta_b = \sum_{w\in\mathcal{R}} \frac{|w\cdot b|}{2} - |b|. \tag{4.6}$$

The dressed monopole is constructed as:[15]

$$\begin{aligned} \left[P(\Phi)\mathcal{M}^2\right] &= P(\Phi)\widetilde{M}^2 + P(-\Phi)\widetilde{M}^{-2} \\ &= P(\Phi)M^2 + P(-\Phi)M^{-2} + P(\Phi)Z(\Phi) + P(-\Phi)Z(-\Phi). \end{aligned} \tag{4.7}$$

Since an arbitrary $P(\Phi)$ can be written as (3.5), we clearly see that the primitive dressed monopoles in this case are:

$$\begin{aligned} \mathcal{M}^2 &= M^2 + M^{-2} + Z(\Phi) + Z(-\Phi), \\ \left[\Phi\mathcal{M}^2\right] &= \Phi(M^2 - M^{-2}) + \Phi(Z(\Phi) - Z(-\Phi)). \end{aligned} \tag{4.8}$$

[15] If $Z(\Phi)$ and/or $P(\Phi)$ depend on Φ's valued in other simple factors, we only reverse the sign of Φ valued in $SU(2)$, as we are only concerned with the action of the Weyl group of $SU(2)$ here.

We compute the following star products with the Weyl-invariant polynomial Φ^2:

$$\begin{aligned}\mathcal{M}^2 \star \Phi^2 &= \left[\left(\Phi - \frac{2i}{r}\right)^2 \mathcal{M}^2\right] + \frac{4}{r^2}\left(Z(\Phi) + Z(-\Phi)\right) + \frac{4i}{r}\Phi\left(Z(\Phi) - Z(-\Phi)\right),\\ \left[\Phi\mathcal{M}^2\right] \star \Phi^2 &= \left[\left(\Phi - \frac{2i}{r}\right)^2 \Phi\mathcal{M}^2\right] + \frac{4}{r^2}\Phi\left(Z(\Phi) - Z(-\Phi)\right) + \frac{4i}{r}\Phi^2\left(Z(\Phi) + Z(-\Phi)\right),\end{aligned} \tag{4.9}$$

where the first terms on the right are dressed monopoles with dressing factors $\left(\Phi - \frac{2i}{r}\right)^2$ and $\left(\Phi - \frac{2i}{r}\right)^2\Phi$, respectively. The polynomiality condition implies that the remaining terms must be Weyl-invariant polynomials in $\Phi \in \mathfrak{su}(2)$ (and possibly other simple factors):

$$\begin{aligned}\frac{4}{r^2}\left(Z(\Phi) + Z(-\Phi)\right) + \frac{4i}{r}\Phi\left(Z(\Phi) - Z(-\Phi)\right) &= \frac{1}{r}A_0(\Phi^2) \in \mathbb{C}[\Phi^2],\\ \frac{4}{r^2}\Phi\left(Z(\Phi) - Z(-\Phi)\right) + \frac{4i}{r}\Phi^2\left(Z(\Phi) + Z(-\Phi)\right) &= \frac{1}{r}A_1(\Phi^2) \in \mathbb{C}[\Phi^2].\end{aligned} \tag{4.10}$$

Recall that the operator mixing ambiguity allows one to shift the bubbling factors $Z(\Phi) + Z(-\Phi)$ and $\Phi\left(Z(\Phi) - Z(-\Phi)\right)$ by arbitrary Weyl-invariant polynomials whose degrees are fixed by dimension analysis. Using the freedom to shift $\Phi\left(Z(\Phi) - Z(-\Phi)\right)$, we can make $A_0(\Phi^2)$ vanish. After this, we solve equation (4.10) for $Z(\Phi)$:

$$Z(\Phi) = -\frac{iA_1(\Phi^2)}{8\Phi(\Phi - \frac{i}{r})}. \tag{4.11}$$

We have not yet used the ambiguity to shift $Z(\Phi) + Z(-\Phi)$ by a Weyl-invariant polynomial. Such shifts that do not shift $\frac{1}{r}(Z(\Phi) + Z(-\Phi)) + i\Phi\left(Z(\Phi) - Z(-\Phi)\right)$ (because we have fixed the latter by demanding $A_0(\Phi) = 0$) give the freedom to shift $Z(\Phi)$ by

$$\Delta Z(\Phi) = \frac{\Phi + \frac{i}{r}}{2\Phi}V(\Phi^2), \tag{4.12}$$

with $V(\Phi^2)$ an arbitrary Weyl-invariant polynomial. Adding this ambiguity to (4.11) gives:

$$Z(\Phi) = -i\frac{A_1(\Phi^2) + 4i(\Phi^2 + \frac{1}{r^2})V(\Phi^2)}{8\Phi(\Phi - \frac{i}{r})}. \tag{4.13}$$

For any $A_1(\Phi^2)$, there exists a unique polynomial $V(\Phi^2)$ such that the numerator $A_1(\Phi^2) +$

$4i(\Phi^2+\frac{1}{r^2})V(\Phi^2) \equiv 8ic$ does not depend on $\Phi \in \mathfrak{su}(2)$, where $c \in \mathbb{C}$ is a constant.[16] Therefore, by completely fixing the mixing ambiguity, we find that:

$$Z(\Phi) = \frac{c}{\Phi(\Phi - \frac{i}{r})}. \tag{4.14}$$

It remains to determine c. To this end, we compute the following expression:

$$\mathcal{M}^2 \star \left[\Phi\mathcal{M}^2\right] - \left[(\Phi - 2i/r)\mathcal{M}^2\right] \star \mathcal{M}^2, \tag{4.15}$$

which is supposed to satisfy the polynomiality constraint. This is where the answer starts depending on the precise matter content of the theory, whereas all previous steps were for arbitrary representation $\mathcal{R}$. Assume that the gauge group is precisely $SU(2)$ (with no other simple factors), and that the theory has N_f fundamental and N_a adjoint hypermultiplets. The dimension of a charge-b monopole is

$$\Delta_b = \frac{|b|}{2}N_f + |b|(N_a - 1). \tag{4.16}$$

A straightforward computation with shift operators gives

$$(4.15) = \frac{8ic^2r^3}{(1+r^2\Phi^2)^2} + \left[\frac{1}{2\Phi}\left(\frac{\Phi}{2}+\frac{i}{2r}\right)^{2(N_f-1)}\left(\frac{3i}{2r}+\Phi\right)^{2N_a}\left(\frac{i}{2r}+\Phi\right)^{2N_a} + (\Phi \leftrightarrow -\Phi)\right]. \tag{4.17}$$

At this point, we see that the answer for c depends on whether $N_f \geq 1$ or $N_f = 0$. If $N_f \geq 1$, then the second term on the right is a Weyl-invariant polynomial and the only non-polynomial piece is $\frac{8ic^2r^3}{(1+r^2\Phi^2)^2}$, thus implying that only $c = 0$ is consistent with polynomiality. However, if $N_f = 0$, one finds that the poles at $\Phi = \pm i/r$ (whose presence would violate polynomiality) vanish when

$$c^2 = (2r)^{-4N_a} \Rightarrow c = \pm(2r)^{-2N_a}. \tag{4.18}$$

The sign of c remains undetermined, and further studying consistency of the algebra does not help much: it is consistent for both signs of c. In fact, it is not hard to see that flipping the sign of c has the same effect on the algebra $\mathcal{A}_C$ as flipping the overall sign of $\mathcal{M}^2$, which is simply a change of basis. This, in particular, shows that after performing the Gram-Schmidt

[16] However, it can still depend on Φ valued in other simple factors.

orthogonalization, the algebra is unaffected, and the physical correlation functions do not depend on the sign of c.

We will see soon that, quite curiously, such a sign ambiguity is not present for higher-rank cases. In the present case, there exists a convenient way to fix the sign, which we follow. Notice that a theory with only adjoint matter admits two possible global forms of the gauge group: either $SU(2)$ or $SO(3)$. They differ by the spectrum of allowed monopole operators. While $\mathcal{M}^2$ is the lowest monopole in the $SU(2)$ case, the $SO(3)$ gauge theory also admits $\mathcal{M}^1$. Indeed, the Langlands dual of $SO(3)$ is $SU(2)$, and $\mathcal{M}^1$ is in the fundamental representation. Because $\mathcal{M}^1$ is minuscule, it contains no bubbling term:

$$\begin{aligned} \mathcal{M}^1 &= M^1 + M^{-1}, \\ \left[\Phi\mathcal{M}^1\right] &= \Phi(M^1 - M^{-1}). \end{aligned} \tag{4.19}$$

We can then define $\mathcal{M}^2 = \mathcal{M}^1 \star \mathcal{M}^1$ and $[\Phi\mathcal{M}^2] = [\Phi\mathcal{M}^1] \star \mathcal{M}^1$, and calculate the bubbling term generated in this way. This gives the following value of c for the $SO(3)$ gauge theory:

$$c = (-4r^2)^{-N_a}. \tag{4.20}$$

One could wonder whether the $SU(2)$ global form corresponds to an opposite sign, but this is not the case. There exists another trick to access bubbling terms in $SU(2)$ (and more generally, in $SU(N)$) gauge theory. It consists of studying the $U(2)$ theory first, and then gauging the $U(1)_{\text{top}}$ symmetry that rotates the dual photon in the diagonal $U(1)$ gauge group (this approach was also used in [74, 75]). In $U(2)$ theory, monopole charges are labeled by two integers $(n, m) \in \mathbb{Z}^2$, and some of them are minuscule. In particular $\mathcal{M}^{(1,0)}$ and $\mathcal{M}^{(-1,0)}$ are minuscule, and their product can be used to determine the non-minuscule $\mathcal{M}^{(1,-1)}$. After gauging $U(1)_{\text{top}}$, the latter becomes $\mathcal{M}^2$ of the $SU(2)$ gauge theory. Proceeding along these lines, gives the same value for c as in (4.20).

So in the end, we find that the bubbling coefficient in $SU(2)$ (or $SO(3)$, when possible) gauge theory with N_f fundemantals and N_a adjoints, up to the operator mixing ambiguity, has the following form:

$$Z(\Phi) = \begin{cases} 0, \text{ if } N_f > 0, \\ \frac{(-4r^2)^{-N_a}}{\Phi(\Phi - \frac{i}{r})}, \text{ if } N_f = 0, \end{cases} \tag{4.21}$$

which is then used to determine:

$$\widetilde{M^2} = M^2 + Z(\Phi). \tag{4.22}$$

Let us now generalize to the case where $SU(2)$ is a simple factor in a larger gauge group, that is, $G = SU(2) \times G'$. Then N_f fundamentals of $SU(2)$ form some generally reducible representation $\mathcal{R}'_f$ of G', while N_a adjoints of $SU(2)$ form another representation $\mathcal{R}'_a$ of G'. This modifies the computation in (4.15) as follows:

$$\begin{aligned}(4.15) = \frac{8ic^2r^3}{(1+r^2\Phi^2)^2} + \Bigg[&\frac{1}{2\Phi\left(\frac{i}{2r}+\frac{\Phi}{2}\right)} \prod_{w\in\mathcal{R}'_f} \left(\left(\frac{i}{2r}+\frac{\Phi}{2}\right)^2 - (w\cdot\Phi')^2 \right) \\ &\times \prod_{w\in\mathcal{R}'_a} \left(\left(\frac{3i}{2r}+\Phi\right)^2 - (w\cdot\Phi')^2 \right) \left(\left(\frac{i}{2r}+\Phi\right)^2 - (w\cdot\Phi')^2 \right) + (\Phi \leftrightarrow -\Phi) \Bigg],\end{aligned} \tag{4.23}$$

where $\Phi \in \mathfrak{t} \subset \mathfrak{su}(2)$, and $\Phi' \in \mathfrak{t}' \subset \mathrm{Lie}(G')$. The cancellation of poles determines c:

$$c = \prod_{w\in\mathcal{R}'_a} \left(-\frac{1}{4r^2} - (w\cdot\Phi')^2 \right) \prod_{w\in\mathcal{R}'_f} (-iw\cdot\Phi'), \tag{4.24}$$

where the sign was fixed by passing to the $U(2)$ theory and applying the "gauging $U(1)_{\text{top}}$" trick. This shows that in a general gauge theory with gauge group $G = SU(2) \times G'$, the abelianized bubbling term for monopoles magnetically charged under the $SU(2)$ factor takes the same form $\frac{c}{\Phi(\Phi-\frac{i}{r})}$ where $\Phi \in \mathfrak{t} \subset \mathfrak{su}(2)$, while c is no longer a constant, but rather a nontrivial function of Φ' from the G' vector multiplets. This last result is enough to study the algebra $\mathcal{A}_C$ and corresponding correlators for arbitrary quivers of $SU(2)$ gauge groups.

4.4 Rank-Two Theories

In this subsection, we repeat the analysis for rank-two gauge groups, namely $SU(3)$, $PSU(3)$, $USp(4) \cong Spin(5)$, $SO(5)$, and G_2, demonstrating how polynomiality determines bubbling coefficients. This will further clarify the general procedure, which was applied to rank-one theories in the previous subsection.

4.4.1 A_2 Theories

Consider the A_2 gauge theories, i.e., those based on either $SU(3)$ or $PSU(3) = SU(3)/\mathbb{Z}_3$ gauge group. The $PSU(3)$ case is trivial as the theory admits monopoles in fundamental representations of the dual group $SU(3)$: such monopoles are minuscule, thus do not bubble, and being the generators, they fully determine the algebra. In the $SU(3)$ gauge theory, however, the monopole charges take values in the weight lattice of $PSU(3)$, which coincides with its root lattice. If α_1 and α_2 are simple roots of $SU(3)$, the coroots $\alpha_1^\vee = 2\alpha_1/(\alpha_1, \alpha_1)$ and $\alpha_2^\vee = 2\alpha_2/(\alpha_2, \alpha_2)$ generate the root lattice of $PSU(3)$, and physical monopole charges (cocharacters) correspond to Weyl orbits in this lattice.

The minimal monopole operator corresponds to the Weyl orbit of $\alpha_1^\vee$, which coincides with the root system of $PSU(3)$. In standard conventions, $\alpha_1^\vee+\alpha_2^\vee$ is the dominant coroot, so we might use it to label the minimal-charge monopole operator $\left[P(\Phi)\mathcal{M}^{\alpha_1^\vee+\alpha_2^\vee}\right]$. In practice, we find it slightly more convenient to still label it by $\alpha_1^\vee$. Such a monopole can only bubble to a trivial representation, since the only weight shorter than $|\alpha_1^\vee|$ is a zero weight, and it belongs to the highest weight representation generated by $\alpha_1^\vee + \alpha_2^\vee$. Therefore, there exists only one bubbling coefficient in this case, $Z(\Phi)$, which determines the abelianized version of the minimal monopole and the dressed monopoles:

$$\begin{aligned}\widetilde{M}^{\alpha_1^\vee} &= M^{\alpha_1^\vee} + Z(\Phi),\\ \left[P(\Phi)\mathcal{M}^{\alpha_1^\vee}\right] &= \sum_{\mathrm{w}\in\mathcal{W}} P(\Phi^{\mathrm{w}})\widetilde{M}^{\mathrm{w}\cdot\alpha_1^\vee}.\end{aligned} \tag{4.25}$$

In the A_2 case, $\Phi = (\Phi_1, \Phi_2)$, and $\mathcal{W} = S_3$; the ring of invariants can be described as

$$\mathbb{C}[\Phi_1,\Phi_2]^{\mathcal{W}} = \mathbb{C}\left[f_1, f_2\right],\ \text{where } f_1 = \Phi_1^2+\Phi_2^2,\ f_2 = \Phi_2(\Phi_2^2 - 3\Phi_1^2). \tag{4.26}$$

There are six primitive dressed monopoles of minimal charge that generate the space of dressed monopoles (of minimal charge) as a $\mathbb{C}[\Phi_1,\Phi_2]^{\mathcal{W}}$-module. They can be chosen as:

$$\begin{aligned}&\mathcal{M}^{\alpha_1^\vee}, \quad \left[\Phi_1\mathcal{M}^{\alpha_1^\vee}\right], \quad \left[\Phi_1^2\mathcal{M}^{\alpha_1^\vee}\right],\\ &\left[\Phi_1^3\mathcal{M}^{\alpha_1^\vee}\right], \quad \left[\Phi_1^4\mathcal{M}^{\alpha_1^\vee}\right], \quad \left[\Phi_1^5\mathcal{M}^{\alpha_1^\vee}\right].\end{aligned} \tag{4.27}$$

The next step, just like in the rank one case, is to compute star product of these with the lowest invariant polynomial $\Phi_1^2+\Phi_2^2$ (often referred to as the quadratic Casimir in physics literature). A straightforward computation for a general dressed $\left[P(\Phi)\mathcal{M}^{\alpha_1^\vee}\right]$ gives:

$$\begin{aligned}\left[P(\Phi)\mathcal{M}^{\alpha_1^\vee}\right]\star(\Phi_1^2+\Phi_2^2) = &\left[((\Phi_1 - 2i/r)^2+\Phi_2^2)P(\Phi)\mathcal{M}^{\alpha_1^\vee}\right]\\ &+\sum_{\mathrm{w}\in\mathcal{W}}\left(\frac{4}{r^2}+\frac{4i}{r}\Phi_1^{\mathrm{w}}\right)P(\Phi^{\mathrm{w}})Z(\Phi^{\mathrm{w}}).\end{aligned} \tag{4.28}$$

The last term must be a Weyl-invariant polynomial for all possible polynomials P. It is enough to impose this requirement for $P = 1, \Phi_1, \Phi_1^2, \ldots, \Phi_1^5$. Recall that

$$\left[\Phi_1^k \mathcal{M}^{\alpha_1^\vee}\right] = \sum_{\mathrm{w}\in\mathcal{W}} (\Phi_1^{\mathrm{w}})^k M^{\mathrm{w}\cdot\alpha_1^\vee} + V_k(\Phi), \text{ where } V_k(\Phi) = \sum_{\mathrm{w}\in\mathcal{W}} (\Phi_1^{\mathrm{w}})^k Z(\Phi^{\mathrm{w}}). \tag{4.29}$$

We see that the last term in (4.28) for $P = 1, \Phi_1, \Phi_1^2, \ldots, \Phi_1^5$ is simply:

$$\sum_{\mathrm{w}\in\mathcal{W}} \left(\frac{4}{r^2} + \frac{4i}{r}\Phi_1^{\mathrm{w}}\right) (\Phi_1^{\mathrm{w}})^k Z(\Phi^{\mathrm{w}}) = \frac{4}{r^2} V_k(\Phi) + \frac{4i}{r} V_{k+1}(\Phi), \quad k = 0, \ldots, 5. \tag{4.30}$$

The right-hand side of each of these equations should be a Weyl-invariant polynomial. This linear system can be solved for $Z(\Phi)$, but we will do better if we first use the operator mixing freedom. Recall that $V_0, \ldots V_5$, being the bubbling terms in $\mathcal{M}^{\alpha_1^\vee}, \ldots, \left[\Phi_1^5 \mathcal{M}^{\alpha_1^\vee}\right]$, can be shifted by Weyl-invariant polynomials in Φ_1, Φ_2 (and r^{-1}). Using such shifts of V_1 through V_5, we can make the right-hand sides of the first five equations in (4.30) vanish, while the sixth one should be a Weyl-invariant polynomial $A(\Phi)$. In other words, we obtain:

$$\begin{aligned} \frac{4}{r^2} V_k(\Phi) + \frac{4i}{r} V_{k+1}(\Phi) &= 0 \qquad (k = 0, \ldots, 4), \\ \frac{4}{r^2} V_5(\Phi) + \frac{4i}{r} V_6(\Phi) &= \frac{1}{r} A(\Phi). \end{aligned} \tag{4.31}$$

Inserting this into (4.30), we solve this linear system for $Z(\Phi)$ to find:

$$Z(\Phi) = -\frac{2iA(\Phi)}{3\Phi_1(\Phi_1 - i/r)(3\Phi_1^4 - 10\Phi_1^2\Phi_2^2 + 3\Phi_2^4)}. \tag{4.32}$$

The only freedom that we have not used so far is that of shifting V_0 by a Weyl-invariant polynomial $F(\Phi) = F(f_1, f_2)$. To preserve the condition that $\frac{4}{r^2} V_k(\Phi) + \frac{4i}{r} V_{k+1}(\Phi) = 0$ for $k = 0, \ldots, 4$, such shifts should be accompanied by $V_1 \to V_1 + \frac{i}{r} F$, $V_2 \to V_2 + \left(\frac{i}{r}\right)^2 F$, $\ldots$, $V_5 \to V_5 + \left(\frac{i}{r}\right)^5 F$. Solving another linear system, it is easy to find that such shifts are traced back to the following shift in $Z(\Phi)$:

$$\Delta Z(\Phi) = \frac{F(\Phi)(\Phi_1 + i/r)(16/r^4 + (\Phi_1^2 - 3\Phi_2^2)^2 + 8(\Phi_1^2 + 3\Phi_2^2)/r^2)}{6\Phi_1(3\Phi_1^4 - 10\Phi_1^2\Phi_2^2 + 3\Phi_2^4)}. \tag{4.33}$$

Shifting Z by such an expression is equivalent to shifting $A(\Phi)$ by $\Delta A(\Phi) = (i/4)F(\Phi)(\Phi_1^2 + 1/r^2)(16/r^4 + (\Phi_1^2 - 3\Phi_2^2)^2 + 8(\Phi_1^2 + 3\Phi_2^2)/r^2)$. Writing this in terms of $f_1 = \Phi_1^2 + \Phi_2^2$ and

$f_2 = \Phi_2(\Phi_2^2 - 3\Phi_1^2)$, we have:

$$\Delta A(\Phi) = \frac{i}{4}\left((f_1 + 1/r^2)(f_1 + 4/r^2)^2 - f_2^2\right) F(f_1, f_2). \tag{4.34}$$

Recalling that $A(\Phi)$, a Weyl-invariant polynomial, is also a polynomial in f_1 and f_2, we can use such shifts to almost eliminate the f_2-dependence of A. Indeed, an arbitrary polynomial $A(f_1, f_2)$ can be represented as

$$\frac{i}{4}\left((f_1 + 1/r^2)(f_1 + 4/r^2)^2 - f_2^2\right) F(f_1, f_2) + R(f_1, f_2), \tag{4.35}$$

where the remainder $R(f_1, f_2)$ is at most linear in f_2. The part proportional to $F(f_1, f_2)$ can be eliminated by (4.34), and thus we are left with A at most linear in f_2. We have completely used the mixing freedom, and found that the abelianized bubbling term takes the form:

$$Z(\Phi) = \frac{R_1(f_1) + f_2 R_2(f_1)}{\Phi_1(\Phi_1 - i/r)(3\Phi_1^4 - 10\Phi_1^2\Phi_2^2 + 3\Phi_2^4)},\ f_1 = \Phi_1^2 + \Phi_2^2,\ f_2 = \Phi_2(\Phi_2^2 - 3\Phi_1^2). \tag{4.36}$$

We have reached the limits of what can be done based on the gauge group only. The concrete expressions for the polynomials $R_1(f_1)$ and $R_2(f_2)$ depend on the matter content, as in the rank-one case. For simplicity, let us consider only the case of an $SU(3)$ vector multiplet coupled to N_f fundamental flavors. We then compute the following star product:

$$\mathcal{M}^{\alpha_1^\vee} \star \left[\Phi_1 \mathcal{M}^{\alpha_1^\vee}\right] - \left[(\Phi_1 - 2i/r)\mathcal{M}^{\alpha_1^\vee}\right] \star \mathcal{M}^{\alpha_1^\vee} = \left[P(\Phi)\mathcal{M}^{\alpha_1^\vee}\right] + R(\Phi). \tag{4.37}$$

The combination above is devised in such a way that $\mathcal{M}^{2\alpha_1^\vee}$ does not show up on the right. The monopole $\mathcal{M}^{\alpha_2^\vee + 2\alpha_1^\vee}$ would be present for more general matter representations (e.g., if we included adjoint matter), but does not show up in our case either, which is why the theory with only fundamental matter is somewhat simpler. Here, $P(\Phi)$ is some polynomial dressing factor, while $R(\Phi)$ is supposed to be a Weyl-invariant polynomial.

The expressions for P and R are lengthy, so we do not provide them here for brevity. Polynomiality of $P(\Phi)$—that is, the cancellation of poles—determines the unknown terms $R_1(f_1)$ and $R_2(f_1)$. We find the following:

$$\begin{cases} \text{For even } N_f\text{: } R_2(f_1) = 0,\ R_1(f_1) = 4\left(\frac{-i}{2\sqrt{3}}\right)^{N_f}\left(\frac{4}{r^2} + f_1\right)\left(\frac{1}{r^2} + f_1\right)^{\frac{N_f}{2}}, \\ \text{For odd } N_f\text{: } R_1(f_1) = 0,\ R_2(f_1) = 4\left(\frac{-i}{2\sqrt{3}}\right)^{N_f}\left(\frac{1}{r^2} + f_1\right)^{\frac{N_f - 1}{2}}. \end{cases} \tag{4.38}$$

As in the rank-one case, this can be generalized to a gauge group $G = SU(3) \times G'$ and $SU(3)$-valued monopoles. If the N_f fundamentals of $SU(3)$ form a representation $\mathcal{R}'$ of G', then the no-pole condition encodes polynomials $R_1(f_1)$ and $R_2(f_1)$ in the following form:

$$R_1(x^2 - r^{-2}) + x(x^2 + 3r^{-2})R_2(x^2 - r^{-2}) = 4(x^2 + 3r^{-2}) \prod_{w \in \mathcal{R}'} \left(-\frac{i}{2\sqrt{3}}x - iw \cdot \Phi' \right). \quad (4.39)$$

Φ' here corresponds to scalars from the G' vector multiplet. This formula reproduces (4.38) if we take $\mathcal{R}' = \mathbb{C}^{N_f}$ to be a trivial representation of G', that is all weights w to be zero.

This final result allows to study quivers of $SU(3)$ groups in which every gauge node only couples to fundamental matter; it also allows to include masses by treating Φ' as background.

Higher magnetic charges. Finally, we would like to explain how to construct monopoles of other magnetic charges. This was obvious in the theory with $PSU(3)$ gauge group: the dual group was $SU(3)$, so both fundamental representations of $SU(3)$ gave allowed monopole charges. Their tensor products could generate an arbitrary representation of $SU(3)$. In the case of $SU(3)$ gauge theory, things are slightly more involved, but still tractable.

So far, we have derived an expression for $Z(\Phi)$, which is enough to build a dressed monopole $[P(\Phi_1, \Phi_2)\mathcal{M}^{\alpha_1^\vee}]$ corresponding to the Weyl orbit of $\alpha_1^\vee$, with arbitrary polynomial P. Is it enough to construct all allowed monopoles in the theory? After all, the coroots take values in a two-dimensional lattice spanned by $\alpha_1^\vee, \alpha_2^\vee$, and merely on the representation-theoretic grounds, one cannot construct all representations labeled by dominant weights in this lattice just from tensor products of the adjoint representation. However, by taking star products of dressed monopoles $[P(\Phi_1, \Phi_2)\mathcal{M}^{\alpha_1^\vee}]$, one can actually generate everything else.

From (2.27), we see that in an $SU(3)$ theory with N_f fundamentals, the dimensions of three lowest bare monopoles are:

$$\Delta_{\alpha_1^\vee} = N_f - 4, \quad \Delta_{2\alpha_1^\vee} = 2N_f - 8, \quad \Delta_{\alpha_2^\vee + 2\alpha_1^\vee} = 2N_f - 6. \quad (4.40)$$

If we take a product of two bare monopoles $\mathcal{M}^{\alpha_1^\vee}$, because $2\Delta_{\alpha_1^\vee} < \Delta_{\alpha_2^\vee + 2\alpha_1^\vee}$, we cannot generate $\mathcal{M}^{\alpha_2^\vee + 2\alpha_1^\vee}$ on the right. However, it can appear if we compensate for the mismatch in dimensions by dressing monopoles with the appropriate number of Φ's. For example, the star product

$$\left[\Phi_1(\Phi_1 - 2i/r)\mathcal{M}^{\alpha_1^\vee}\right] \star \mathcal{M}^{\alpha_1^\vee} - \left[\Phi_1 \mathcal{M}^{\alpha_1^\vee}\right] \star \left[\Phi_1 \mathcal{M}^{\alpha_1^\vee}\right] \quad (4.41)$$

has $\mathcal{M}^{\alpha_2^\vee + 2\alpha_1^\vee}$ as a leading term, and can therefore serve as a definition of $\mathcal{M}^{\alpha_2^\vee + 2\alpha_1^\vee}$. Similarly taking products of monopoles dressed by higher-degree polynomials, we can obtain dressed versions of $\mathcal{M}^{\alpha_2^\vee+2\alpha_1^\vee}$. Having constructed in this way both $\mathcal{M}^{\alpha_1^\vee}$, $\mathcal{M}^{\alpha_2^\vee+2\alpha_1^\vee}$ and their dressed versions, we can generate all other allowed monopoles.

4.4.2 $B_2 \cong C_2$ Theories

There are two compact rank-two gauge groups that correspond to the $B_2 \cong C_2$ Lie algebra: $USp(4) \cong Spin(5)$ and $SO(5) \cong USp(4)/\mathbb{Z}_2$, which are Langlands dual to each other. The group $USp(4)$ is often called $Sp(2)$, but we will use the former notation. The root lattice is generated by the short simple root α and the long simple root β. In our conventions, we write them in Cartesian coordinates as $\alpha = (1,0)$ and $\beta = (-1,1)$. The coroots are $\alpha^\vee = 2\alpha = (2,0)$ and $\beta^\vee = \beta = (-1,1)$, so that $\alpha^\vee$ is a long coroot.

$SO(5)$ **gauge theory.** First consider the $SO(5)$ gauge theory. The monopoles are labeled by (Weyl orbits of) the dominant weights of the dual group $USp(4)$, whose root lattice is generated by $\alpha^\vee$ and $\beta^\vee$. The group $USp(4)$ has two fundamental representations: the four-dimensional (the defining representation) and the five-dimensional (the vector representation of $SO(5) = USp(4)/\mathbb{Z}_2$). The four-dimensional representation has weights:

$$\omega_1^\vee = \frac{1}{2}\alpha^\vee = (1,0);\ \omega_1^\vee + \beta^\vee = (0,1);\ \omega_1^\vee - \alpha^\vee = (-1,0);\ \omega_1^\vee - \alpha^\vee - \beta^\vee = (0,-1). \quad (4.42)$$

This representation is minuscule, hence the smallest monopole of the model does not bubble:

$$\left[P(\Phi)\mathcal{M}^{\omega_1^\vee}\right] = \sum_{\mathrm{w}\in\mathcal{W}} P(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot\omega_1^\vee}. \quad (4.43)$$

The five-dimensional representation of $USp(4)$ is not minuscule; it has weights:

$$\omega_2^\vee = \alpha^\vee+\beta^\vee = (1,1);\ \omega_2^\vee-\alpha^\vee = \beta^\vee = (-1,1);\ -\beta^\vee = (1,-1);\ -\beta^\vee-\alpha^\vee = (-1,-1);\ (0,0). \quad (4.44)$$

We see that the charge-$\omega_2^\vee$ monopole can bubble to the zero magnetic charge. Therefore, the abelianized monopole takes the form:

$$\widetilde{M}^{\omega_2^\vee} = M^{\omega_2^\vee} + Z(\Phi). \quad (4.45)$$

This $Z(\Phi)$ can be deduced by computing star products involving only the minimal monopole $[P(\Phi)\mathcal{M}^{\omega_1^\vee}]$. On the other hand, it can also be found using our polynomiality approach (which is really a different application of the same idea, namely consistency of the OPE algebra). Let us determine it using such an approach – both for practice, and because it will soon be useful for the study of the $USp(4)$ gauge theory.

The charge-$\omega_2^\vee$ dressed monopoles are constructed as:

$$\left[P(\Phi)\mathcal{M}^{\omega_2^\vee}\right] = \frac{1}{2}\sum_{\mathrm{w}\in\mathcal{W}} P(\Phi^{\mathrm{w}})\widetilde{M}^{\mathrm{w}\cdot\omega_2^\vee}. \tag{4.46}$$

As before, $\Phi = (\Phi_1, \Phi_2) \in \mathfrak{t}_{\mathbb{C}}$. The Weyl group is $D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$, and the ring of invariants is:

$$\mathbb{C}[\Phi_1, \Phi_2]^{\mathcal{W}} = \mathbb{C}[f_1, f_2], \text{ where } f_1 = \Phi_1^2 + \Phi_2^2,\ f_2 = \Phi_1^2\Phi_2^2. \tag{4.47}$$

Notice that $\omega_2^\vee$ is preserved by the subgroup of $\mathcal{W}$ that switches $\Phi_1 \leftrightarrow \Phi_2$, which explains $\frac{1}{2}$ in (4.46). Therefore such monopoles can only be dressed by polynomials symmetric under $\Phi_1 \leftrightarrow \Phi_2$ (this happens automatically once we apply (4.46)). Dressing by a symmetric polynomial only depends on the symmetric part of $Z(\Phi_1, \Phi_2)$. Therefore, it is enough to assume $Z(\Phi_1, \Phi_2) = Z(\Phi_2, \Phi_1)$.

Since the Weyl orbit of $\omega_2^\vee$ has four elements, there are four primitive dressed monopoles that generate all dressed charge-$\omega_2^\vee$ monopoles as a $\mathbb{C}[\Phi_1, \Phi_2]^{\mathcal{W}}$-module. Choose them as:

$$\mathcal{M}^{\omega_2^\vee},\ \left[(\Phi_1+\Phi_2)\mathcal{M}^{\omega_2^\vee}\right],\ \left[(\Phi_1+\Phi_2)^2\mathcal{M}^{\omega_2^\vee}\right],\ \left[(\Phi_1+\Phi_2)^3\mathcal{M}^{\omega_2^\vee}\right]. \tag{4.48}$$

The next step is to compute their star product with f_1. For arbitrary $P(\Phi)$, we find:

$$\begin{aligned}\left[P(\Phi)\mathcal{M}^{\omega_2^\vee}\right] \star (\Phi_1^2+\Phi_2^2) - \left[\left(\left(\Phi_1 - \frac{i}{r}\right)^2 + \left(\Phi_2 - \frac{i}{r}\right)^2\right) P(\Phi)\mathcal{M}^{\omega_2^\vee}\right] \\ = \sum_{\mathrm{w}\in\mathcal{W}} \left(\frac{1}{r^2} + \frac{i}{r}(\Phi_1^{\mathrm{w}} + \Phi_2^{\mathrm{w}})\right) P(\Phi^{\mathrm{w}})Z(\Phi^{\mathrm{w}}).\end{aligned} \tag{4.49}$$

We require that the second line be a polynomial, and do so for $P = 1, \Phi_1+\Phi_2, (\Phi_1+\Phi_2)^2$ and $(\Phi_1+\Phi_2)^3$. Using the notation similar to the $SU(3)$ case:

$$\left[(\Phi_1+\Phi_2)^k\mathcal{M}^{\omega_2^\vee}\right] = \frac{1}{2}\sum_{\mathrm{w}\in\mathcal{W}}(\Phi_1^{\mathrm{w}}+\Phi_2^{\mathrm{w}})^k\widetilde{M}^{\mathrm{w}\cdot\omega_2^\vee}+V_k(\Phi), \text{ where } V_k(\Phi) = \frac{1}{2}\sum_{\mathrm{w}\in\mathcal{W}}(\Phi_1^{\mathrm{w}}+\Phi_2^{\mathrm{w}})^k Z(\Phi^{\mathrm{w}}), \tag{4.50}$$

we identify the last term in (4.49) for $P(\Phi) = (\Phi_1+\Phi_2)^k$ as $\frac{2}{r^2}V_k(\Phi) + \frac{2i}{r}V_{k+1}(\Phi)$, which has to be a Weyl-invariant polynomial. Using the operators mixing freedom to shift V_1, V_2 and

V_3, we make the first three of these polynomials vanish, except for the last one:

$$\begin{aligned}
&\frac{2}{r^2}V_k(\Phi)+\frac{2i}{r}V_{k+1}(\Phi)=0 \qquad (k=0,1,2),\\
&\frac{2}{r^2}V_3(\Phi)+\frac{2i}{r}V_4(\Phi)=\frac{1}{r}A\left(\Phi_1^2+\Phi_2^2,\Phi_1^2\Phi_2^2\right)\in\mathbb{C}\left[\Phi_1^2+\Phi_2^2,\Phi_1^2\Phi_2^2\right].
\end{aligned} \tag{4.51}$$

Using the expressions $V_k(\Phi)=\sum_{\mathrm{w}\in\mathcal{W}}(\Phi_1^{\mathrm{w}}+\Phi_2^{\mathrm{w}})^k Z(\Phi^{\mathrm{w}})$, we solve this system of four equations (under the assumption that $Z(\Phi_1,\Phi_2)=Z(\Phi_2,\Phi_1)$) to find:

$$Z(\Phi)=-\frac{iA\left(\Phi_1^2+\Phi_2^2,\Phi_1^2\Phi_2^2\right)}{32\Phi_1\Phi_2(\Phi_1+\Phi_2)(\Phi_1+\Phi_2-i/r)}. \tag{4.52}$$

The next step is to fix the remaining mixing freedom, which allows for shifts of V_0 by Weyl-invariant polynomials $F\left(\Phi_1^2+\Phi_2^2,\Phi_1^2\Phi_2^2\right)$, namely $V_0\to V_0+F$. To preserve the form of equations (4.51), we also shift $V_1\to V_1+\frac{i}{r}F$, $V_2\to V_2+\left(\frac{i}{r}\right)^2F$, and $V_3\to V_3+\left(\frac{i}{r}\right)^3F$. This can be solved for the corresponding shift $\Delta Z(\Phi)$ of $Z(\Phi)$:

$$\Delta Z(\Phi)=-\frac{(\Phi_1+\Phi_2+i/r)(\Phi_1-\Phi_2+i/r)(\Phi_1-\Phi_2-i/r)F\left(\Phi_1^2+\Phi_2^2,\Phi_1^2\Phi_2^2\right)}{16\Phi_1\Phi_2(\Phi_1+\Phi_2)}. \tag{4.53}$$

Comparing with (4.52), we see that such shifts are equivalent to shifting A by

$$\Delta A=\left(\frac{2}{r^4}+\frac{4f_1}{r^2}+f_1^2-4f_2\right)F[f_1,f_2], \tag{4.54}$$

where $f_1=\Phi_1^2+\Phi_2^2$ and $f_2=\Phi_1^2\Phi_2^2$. Because the expression in parentheses is no more than linear in f_2, such shifts can completely eliminate the f_2-dependence from A. Indeed, for an arbitrary polynomial $A(f_1,f_2)$, we can find a unique $F(f_1,f_2)$ such that $A(f_1,f_2)+\Delta A(f_1,f_2)$ depends only on f_1. This fully fixes the mixing freedom, and in the end we have:

$$Z(\Phi)=\frac{a\left(\Phi_1^2+\Phi_2^2\right)}{\Phi_1\Phi_2(\Phi_1+\Phi_2)(\Phi_1+\Phi_2-i/r)}. \tag{4.55}$$

Determining a requires computing an appropriate star product. The answer depends on the matter content, and in this case it is not too hard to include both N_f five-dimensional flavors of $SO(5)$ and N_a adjoint flavors. We consider the following star product of minimal monopoles, which is enough to generate the next-to-minimal monopole of charge $\omega_2^\vee$:

$$\mathcal{M}^{\omega_1^\vee}\star\left[\Phi_1^3\mathcal{M}^{\omega_1^\vee}\right]-\left[(\Phi_1-i/r)^3\mathcal{M}^{\omega_1^\vee}\right]\star\mathcal{M}^{\omega_1^\vee}. \tag{4.56}$$

The reason we have to include Φ_1^3 can be seen from dimensions of the monopoles:

$$\Delta_{\omega_1^\vee} = N_f + 3(N_a - 1), \quad \Delta_{\omega_2^\vee} = 2N_f + 4(N_a - 1). \tag{4.57}$$

Only with the insertion of (at least) Φ_1^3 do we find that the dimension of (4.56), given by $2\Delta_{\omega_1^\vee} + 3 = 2N_f + 6N_a - 3$, is larger than $\Delta_{\omega_2^\vee}$ for all values of N_a, thus allowing the monopole of charge $\omega_2^\vee$ to appear on the right. It indeed appears, bare for $N_a = 0$ and dressed for $N_a \neq 0$. We subtract it from the above star product and look at the free (charge zero) term, demanding its polynomiality. This determines $a\,(\Phi_1^2 + \Phi_2^2)$. For brevity, we do not present the cumbersome intermediate formulas and only give the final answer:

$$a\,(x) = -\left(\frac{x}{2} + \frac{1}{4r^2}\right)^{N_f} \left(-\frac{x}{8r^2} - \frac{1}{16r^4}\right)^{N_a}. \tag{4.58}$$

This expression determines $Z(\Phi)$, and we can construct arbitrary dressed monopoles of charge $\omega_2^\vee$. Having the two monopoles corresponding to fundamental weights of $USp(4)$, we can construct arbitrary monopoles in the $SO(5)$ gauge theory. Notice that it was clear from the beginning that the charge-$\omega_1^\vee$ monopole suffices to generate the algebra.

Like in all cases so far, it is not hard to generalize to a non-simple gauge group $G = SO(5) \times G'$, assuming that N_f fundamentals of $SO(5)$ form a representation $\mathcal{R}_f'$ of G', while N_a adjoints transform in $\mathcal{R}_a'$ of G'. Modifying the above calculation appropriately gives:

$$a\,(x) = -\prod_{w\in\mathcal{R}_f'} \left(\frac{x}{2} + \frac{1}{4r^2} - (w\cdot\Phi')^2\right) \prod_{w\in\mathcal{R}_a'} \left[\left(-\frac{1}{4r^2} - (w\cdot\Phi')^2\right)\left(\frac{x}{2} + \frac{1}{4r^2} - (w\cdot\Phi')^2\right)\right]. \tag{4.59}$$

Here, as before, Φ' is valued in the G' vector multiplets. This answer for $a(x)$ of course reduces to (4.58) when all weights in $\mathcal{R}_f'$ and $\mathcal{R}_a'$ vanish. As usual, Φ' plays the role of mass matrix if we treat G' as a global symmetry.

$USp(4)$ **gauge theory.** Consider the $USp(4)$ gauge theory. It has the same simple roots α, β and simple coroots $\alpha^\vee$, $\beta^\vee$ as in the $SO(5)$ case. Only the lattice of allowed weights of matter representations is different.

The dual group is $SO(5)$, and it has no minuscule representations. The minimal monopole has charge $\omega_2^\vee$, same as the next-to-minimal monopole of the $SO(5)$ gauge theory. It is defined by the same equations (4.45) and (4.46). Further steps about the ring of invariants, the primitive dressed monopoles, and ultimately the answer (4.55) are applicable to the $USp(4)$

case as well – they do not depend on the global form of the gauge group. To proceed, we need to find the polynomial a entering the abelianized bubbling coefficient (4.55). This step depends on the matter content, and hence on the global form of the gauge group.

Let us consider for simplicity a theory which only has matter in N_4 copies of the four-dimensional representation of $USp(4)$. We compute the star product

$$\mathcal{M}^{\omega_2^\vee} \star \left[(\Phi_1+\Phi_2)\mathcal{M}^{\omega_2^\vee}\right] - \left[(\Phi_1+\Phi_2-2i/r)\mathcal{M}^{\omega_2^\vee}\right] \star \mathcal{M}^{\omega_2^\vee}. \tag{4.60}$$

Charges $2\omega_2^\vee$ and $2\omega_1^\vee$ cancel from the result. Polynomiality of the rest requires $a(x)$ to be a constant, and determines its square. We describe the answer in a more general case of $G = USp(4)\times G'$, assuming that N_4 fundamentals of $USp(4)$ transform in $\mathcal{R}'_4$ of G':

$$a = -\prod_{w\in\mathcal{R}'_4}(-iw\cdot\Phi'), \tag{4.61}$$

which determines the bubbling coefficient

$$Z(\Phi) = \frac{a}{\Phi_1\Phi_2(\Phi_1+\Phi_2)(\Phi_1+\Phi_2-i/r)}. \tag{4.62}$$

The situation here is reminiscent of the $SU(2)$ case: without an extra group G', all weights w vanish, and we find that $a=0$. In other words, in the $USp(4)$ gauge theory with $N_4>0$, in the absence of extra gaugings and masses, the bubbling coefficient of the minimal monopole can be removed using mixing. If $N_4=0$, then $a=-1$, where the sign was chosen to agree with the $SO(5)$ answer: in the $N_4=0$ case, the matter content (its absence) allows for both $USp(4)$ and $SO(5)$ gauge groups, and we can determine the $USp(4)$ answer from the $SO(5)$ answer (theories only differ by a $\mathbb{Z}_2$ gauging). For other values of N_4, we picked the sign of a at random since it does not affect the algebra $\mathcal{A}_C$, up to basis changes.

Finally, let us add that at this point, using some physics intuition, we can easily guess the answer for $a(x)$ in a more general case when we have matter in a representation $[\mathbf{4}\otimes\mathcal{R}'_4]\oplus[\mathbf{5}\otimes\mathcal{R}'_5]\oplus[\mathbf{adj}\otimes\mathcal{R}'_a]$ of the gauge group $G=USp(4)\times G'$. Here $\mathbf{4}$, $\mathbf{5}$ and $\mathbf{adj}$ are the four-dimensional, the five-dimensional, and the adjoint representations of $USp(4)$, and $\mathcal{R}'_4$, $\mathcal{R}'_5$, $\mathcal{R}'_a$ are some representations of G'. We have seen before that contributions of different matter multiplets enter the answer for $a(x)$ multiplicatively. This makes sense from the localization point of view as well: after all, the bubbling terms are given by certain one-loop determinants around fixed points in the bubbling loci. One-loop determinants of various matter multiplets contribute multiplicatively. So it is natural to expect that the answer in such a more general case should be given by

$$a(x) = -\prod_{w\in\mathcal{R}'_4}(-iw\cdot\Phi')\prod_{w\in\mathcal{R}'_5}\left(\frac{x}{2}+\frac{1}{4r^2}-(w\cdot\Phi')^2\right)$$
$$\times\prod_{w\in\mathcal{R}'_a}\left(-\frac{1}{4r^2}-(w\cdot\Phi')^2\right)\left(\frac{x}{2}+\frac{1}{4r^2}-(w\cdot\Phi')^2\right), \tag{4.63}$$

where we borrowed contributions of **5** and **adj** from the subsection on the $SO(5)$ case, as the theory with only these types of matter allows for both the $SO(5)$ and $USp(4)$ gauge groups.

4.4.3 G_2 Theories

The remaining rank-two simple gauge group is G_2. It has only one compact form, which is of course centerless and Langlands dual to itself, meaning that we do not have to study various cases as before. We describe the root system Δ of G_2 in Cartesian coordinates such that the short simple root is $\alpha = (1,0)$ and the long simple root is $\beta = (-\frac{3}{2}, \frac{\sqrt{3}}{2})$. The corresponding coroots are $\alpha^\vee = 2\alpha = (2,0)$ and $\beta^\vee = \frac{2}{3}\beta = (-1, \frac{\sqrt{3}}{3})$, which are now long and short respectively, and generate the root system $\Delta^\vee$ of the dual G_2. It is convenient to describe $\Delta^\vee$ in terms of another pair of simple coroots, which we define as $\alpha_{\text{mon}} = \alpha^\vee + 2\beta^\vee = (0, \frac{2}{\sqrt{3}})$ and $\beta_{\text{mon}} = -2\alpha^\vee - 3\beta^\vee = (-1, -\sqrt{3})$, where now α_{mon} is short and β_{mon} is long.

The smallest irreducible representation is 7-dimensional: its weights are given by a zero weight $(0,0)$ and the six short roots in Δ, namely α and its Weyl images. Because of the zero weight, the representation is not minuscule. The next irreducible representation is the 14-dimensional adjoint representation. These **7** and **14** are fundamental representations, but we will only call **7** the fundamental, while **14** is referred to as the adjoint.

The minimal monopole charge is described by the non-zero weights in **7** of the dual G_2, i.e., by the Weyl orbit of the short coroot α_{mon} (or equivalently $\beta^\vee$ which belongs to the same Weyl orbit). Because **7** is not minuscule, it can bubble into the identity:

$$\widetilde{M}^{\alpha_{\text{mon}}} = M^{\alpha_{\text{mon}}} + Z(\Phi), \tag{4.64}$$

and the physical dressed monopole of minimal charge is defined by:

$$[P(\Phi)\mathcal{M}^{\alpha_{\text{mon}}}] = \frac{1}{2}\sum_{\mathrm{w}\in\mathcal{W}} P(\Phi^{\mathrm{w}})\widetilde{M}^{\mathrm{w}\cdot\alpha_{\text{mon}}}. \tag{4.65}$$

The Weyl group is $\mathcal{W} = D_6 = \mathbb{Z}_6 \rtimes \mathbb{Z}_2$, the group of symmetries of a hexagon. As usual,

$\Phi = (\Phi_1, \Phi_2) \in \mathfrak{t}_{\mathbb{C}}$, and the ring of invariants is:

$$\mathbb{C}[\Phi_1, \Phi_2]^{\mathcal{W}} = \mathbb{C}[f_1, f_2], \text{ where } f_1 = \Phi_1^2 + \Phi_2^2,\ f_2 = \Phi_2^2(\Phi_2^2 - 3\Phi_1^2)^2. \tag{4.66}$$

Because the Weyl orbit of $\alpha_{\rm mon}$ has size 6, there are six primitive dressed monopoles:

$$\begin{gathered} \mathcal{M}^{\alpha_{\rm mon}},\ [\Phi_2\mathcal{M}^{\alpha_{\rm mon}}],\ \left[\Phi_2^2\mathcal{M}^{\alpha_{\rm mon}}\right], \\ \left[\Phi_2^3\mathcal{M}^{\alpha_{\rm mon}}\right],\ \left[\Phi_2^4\mathcal{M}^{\alpha_{\rm mon}}\right],\ \left[\Phi_2^5\mathcal{M}^{\alpha_{\rm mon}}\right]. \end{gathered} \tag{4.67}$$

The next steps are exactly the same as before, namely we compute the star product:

$$\begin{aligned} [P(\Phi)\mathcal{M}^{\alpha_{\rm mon}}] \star (\Phi_1^2 + \Phi_2^2) - \left[\left(\Phi_1^2 + \left(\Phi_2 - \frac{2i}{r\sqrt{3}}\right)^2\right) P(\Phi)\mathcal{M}^{\alpha_{\rm mon}}\right] \\ = \frac{1}{2}\sum_{\rm w\in\mathcal{W}} \left(\frac{4}{3r^2} + \frac{4i}{r\sqrt{3}}\Phi_2^{\rm w}\right) P(\Phi^{\rm w}) Z(\Phi^{\rm w}), \end{aligned} \tag{4.68}$$

and demand polynomiality for $P = 1, \Phi_2, \ldots, \Phi_2^5$. Also, because $\alpha_{\rm mon}$ is preserved by the Weyl reflection $(\Phi_1, \Phi_2) \to (-\Phi_1, \Phi_2)$, it is enough to consider only dressing by polynomials invariant under such a reflection (which also explains $\frac{1}{2}$ in the definition of the monopole). Therefore, one can assume from the beginning that

$$Z(-\Phi_1, \Phi_2) = Z(\Phi_1, \Phi_2). \tag{4.69}$$

Polynomiality of the last term in (4.68) and the operator mixing freedom almost completely determine $Z(\Phi)$. To avoid repetition, we simply give the final answer:

$$Z(\Phi) = \frac{A(\Phi_1^2 + \Phi_2^2)}{\Phi_2(\Phi_2\sqrt{3} - \frac{i}{r})(3\Phi_1^4 - 10\Phi_1^2\Phi_2^2 + 3\Phi_2^4)}, \tag{4.70}$$

Finally, to determine the polynomial A, we compute another star product:

$$\mathcal{M}^{\alpha_{\rm mon}} \star [\Phi_2\mathcal{M}^{\alpha_{\rm mon}}] - \left[\left(\Phi_2 - \frac{2i}{r\sqrt{3}}\right)\mathcal{M}^{\alpha_{\rm mon}}\right] \star \mathcal{M}^{\alpha_{\rm mon}}. \tag{4.71}$$

At this point, we limit ourselves to the theory with N_f seven-dimensional flavors of G_2. In such a case, higher magnetic charges $2\alpha_{\rm mon}$ and $\beta_{\rm mon}$ cancel from the above expression. The monopole of charge $2\alpha_{\rm mon}$ cancels because the expression (4.71) is specifically constructed to ensure its cancellation, while charge $\beta_{\rm mon}$ cannot appear due to dimensional reasons.

Dimensions of the lowest monopoles are

$$\Delta_{\alpha_{\rm mon}} = 2N_f - 6, \quad \Delta_{\beta_{\rm mon}} = 4N_f - 10. \tag{4.72}$$

The dimension of (4.71) is $2\Delta_{\alpha_{\rm mon}} + 1 < \Delta_{\beta_{\rm mon}}$, so the monopole of dimension $\Delta_{\beta_{\rm mon}}$ indeed cannot appear on the right. However, the dressed monopole of charge $\alpha_{\rm mon}$ appears, and demanding polynomiality of its dressing factor determines A as:

$$A(x) = \frac{16}{3\sqrt{3}} \left(\frac{x}{4} + \frac{1}{12r^2} \right)^{N_f} . \tag{4.73}$$

It is also not too hard to generalize to the case of a gauge group $G = G_2 \times G'$, assuming that N_f fundamentals of G_2 transform in a representation $\mathcal{R}'$ of G':

$$A(x) = \frac{16}{3\sqrt{3}} \prod_{w \in \mathcal{R}'} \left(\frac{x}{4} + \frac{1}{12r^2} - (w \cdot \Phi')^2 \right) . \tag{4.74}$$

4.5 General Case

The detailed exploration of the lower-rank theories in the above subsections should give the reader a sense of what the polynomiality-based computations look like. Further, it shows a clear pattern and allows to formulate a strategy that should work for general gauge groups.

First of all, one has to identify the set of minimal monopoles that are expected to generate the algebra. They can either be minuscule or bubble into the charge-zero sector (we call it "bubbling into the identity"). They cannot bubble into smaller non-zero charges, as that would contradict minimality. If all of them are minuscule, we are done: it only remains to make sure that they indeed generate everything, and determine the relations.

If there is a minimal monopole of charge ω that is not minuscule, it can bubble into the identity, and we should determine the corresponding abelianized bubbling factor $Z(\Phi)$. First, we use invariant theory to identify the set of primitive dressed monopoles of charge ω. Then we compute their star products with the quadratic Casimir $f_1 = \sum_{i=1}^{r} \Phi_i^2$. By demanding polynomiality of the answer and using the operator mixing freedom, we almost completely determine the bubbling factor $Z(\Phi)$, up to an unknown Weyl-invariant polynomial A. These steps clearly work in an arbitrary gauge theory. The next step is the most challenging one: we need to construct a star product that determines the unknown polynomial A. We have seen that at this step, sometimes A is uniquely determined, and sometimes it is only determined up to a sign, which is a harmless ambiguity that can be related to a change of basis in $\mathcal{A}_C$.

We consider the above procedure as strong evidence that polynomiality fully determines the algebra $\mathcal{A}_C$ (if not a proof, at a physical level of rigor). It would still be desirable to find a more elegant and mathematically illuminating way to reach this conclusion.

5 Applications and Examples

We now demonstrate the applications of our shift operator formalism in a number of simple examples. More elaborate examples can be found in the appendices.

5.1 Chiral Rings and Coulomb Branches

In the commutative limit ($r \to \infty$), the quantum algebra $\mathcal{A}_C$ reduces to the Coulomb branch chiral ring. Because finite-r computations, as shown above, allow to determine bubbling coefficients and thus $\mathcal{A}_C$ in any theory, this provides a simple way to construct Coulomb branches even when other approaches face difficulties. However, finite-r computations can be very hard, so it is important to first develop the commutative version of shift operators. This is the subject of this section, and the answer take the form of abelianization as in [53].

We begin by noting that the shift operator M_N^b from (2.25) has a well-defined $r \to \infty$ limit. First, because the operator $e^{-b\cdot(\frac{i}{2}\partial_\sigma+\partial_B)}$ acts on Φ by a shift

$$\Phi \mapsto \Phi - \frac{i}{r}b, \tag{5.1}$$

this shift vanishes in the $r \to \infty$ limit, and $e^{-b\cdot(\frac{i}{2}\partial_\sigma+\partial_B)}$ no longer acts on Φ-dependent terms. Instead, it effectively turns into a generator of the group ring $\mathbb{C}[\Lambda_w^\vee]$ associated to the lattice of coweights (considered as an abelian group). Such generators, denoted by $e[b]$, are subject to the relations:

$$e[b_1]e[b_2] = e[b_1+b_2]. \tag{5.2}$$

Next, we observe that a Φ-dependent rational prefactor in the definition (2.25) of M_N^b also has a well-defined $r \to \infty$ limit. If we denote the commutative limit of M_N^b by v^b, we find the following expression for it:[17]

$$v^b = \frac{\prod_{w\in\mathcal{R}}(-iw\cdot\Phi)^{(w\cdot b)_+}}{\prod_{\alpha\in\Delta}(-i\alpha\cdot\Phi)^{(\alpha\cdot b)_+}}e[b]. \tag{5.3}$$

[17]The $(w \cdot b)_+$ in the exponent is not a typo. It was previously a lower index of a Pochhammer symbol, but in the commutative limit, it turns into a power.

Recall that this expression also includes the case when some matter multiplets have masses, in which case $\mathcal{R}$ is considered as a representation of both gauge and flavor groups, and some Φ's are vevs of the background vector multiplets (that is, masses). Note that (5.3) immediately implies (2.30).

This (5.3) is precisely as in [53], showing that we indeed recover their abelianization map in the $r \to \infty$ limit. An extra bonus that we have in our formalism is that the abelianized bubbling coefficients of Sections 3 and 4 are known, and also have the well-defined $r \to \infty$ limit. Introducing a notation:

$$z_{b\to v}(\Phi) = \lim_{r\to\infty} Z^{\text{ab}}_{b\to v}(\Phi), \tag{5.4}$$

and another notation for commuting abelianized monopole shift operator:

$$\widetilde{v}^b \equiv \lim_{r\to\infty} \widetilde{M}^b = v^b + \sum_{|u|<|b|} z_{b\to u}(\Phi) v^u, \tag{5.5}$$

we conclude that commuting versions of general physical dressed monopoles are given by:

$$[P(\Phi)V^b] = \frac{1}{|\mathcal{W}_b|} \sum_{\text{w}\in\mathcal{W}} P_i(\Phi^{\text{w}}) \widetilde{v}^{\text{w}\cdot b}. \tag{5.6}$$

Let us consider a few examples of Coulomb branches determined using this technique.

5.1.1 $SU(2)$ with N_f Fundamentals and N_a Adjoints

In Section 4.3, we have shown that in the $SU(2)$ gauge theory with $N_f > 1$ fundamentals and any number N_a of adjoints, the abelianized bubbling coefficient $Z^{\text{ab}}_{2\to 0}(\Phi)$ is polynomial, and hence, up to operator mixing, we can take $Z^{\text{ab}}_{2\to 0}(\Phi) = 0$. The same is then true for its $r \to \infty$ limit, $z_{2\to 0}(\Phi) = 0$.

When $N_f = 0$, the bubbling term is a non-trivial rational function,

$$Z_{2\to 0}(\Phi) = \frac{(-4r^2)^{-N_a}}{\Phi(\Phi - \frac{i}{r})}, \tag{5.7}$$

however we see that the $r \to \infty$ limit is zero, unless $N_a = 0$. Hence we can again take $z_{2\to 0}(\Phi) = 0$, except in a pure gauge theory, which will be treated separately.

Since Cartan is one-dimensional, we write the Cartan-valued Φ simply as a complex number. The two primitive monopoles of minimal charge $b = 2$ in the commuting limit take the form:

$$v^2 + v^{-2} = \left(-i\frac{\Phi}{2}\right)^{N_f} (i\Phi)^{2(N_a-1)}(e[2] + (-1)^{N_f} e[-2]),$$
$$\Phi(v^2 - v^{-2}) = \Phi\left(-i\frac{\Phi}{2}\right)^{N_f} (i\Phi)^{2(N_a-1)}(e[2] - (-1)^{N_f} e[-2]). \tag{5.8}$$

In addition, we have the variable Φ^2. Define:

$$\mathcal{U} = 2^{N_f-1}(v^2 + v^{-2}), \qquad \mathcal{V} = -i2^{N_f-1}\Phi(v^2 - v^{-2}), \qquad \mathcal{W} = \Phi^2. \tag{5.9}$$

The only relation between these variables follows from $e[2]e[-2] = 1$ and takes the form

$$\mathcal{V}^2 + \mathcal{U}^2\mathcal{W} = \mathcal{W}^{N_f+2N_a-1}, \tag{5.10}$$

which is the defining equation of a $D_{N_f+2N_a}$ singularity. From the equation (4.16), the dimension of the lowest monopole operator is $\Delta_2 = N_f + 2N_a - 2$. We see that the theory is good whenever $N_f + 2N_a > 2$. Precisely at such values, (5.10) determines a cone. For $N_f + 2N_a = 2$, $\mathcal{U}$ has dimension (or rather R-charge) zero, while for $N_f + 2N_a = 1$, that is $N_a = 0$ and $N_f = 1$, the monopole has negative R-charge, – in both of these cases, the theory is bad and (5.10) is not a cone.

It is also straightforward to include masses by turning on background vevs for flavor symmetry. In such a case, the bubbling term remains non-trivial in the $r \to \infty$ limit, as we know from (4.24), and is given by:

$$z_{2\to 0}(\Phi) = \frac{\prod_{a=1}^{N_a}(-M_a^2)\prod_{i=1}^{N_f}(-iM_i)}{\Phi^2}, \tag{5.11}$$

where M_a and M_i are masses of the adjoint and fundamental hypers respectively. Expressions for commuting shift operators are also modified (as follows from coupling to the background multiplet):

$$v^2 = \frac{\prod_{i=1}^{N_f}\left(-i\frac{\Phi}{2} - iM_i\right)\prod_{a=1}^{N_a}(i\Phi + iM_a)^2}{(i\Phi)^2} e[2],$$
$$v^{-2} = \frac{\prod_{i=1}^{N_f}\left(i\frac{\Phi}{2} - iM_i\right)\prod_{a=1}^{N_a}(-i\Phi + iM_a)^2}{(i\Phi)^2} e[-2]. \tag{5.12}$$

Using variables

$$\mathcal{U} = 2^{N_f-1}\left(v^2 + v^{-2} + 2z_{2\to 0}(\Phi)\right), \qquad \mathcal{V} = -i2^{N_f-1}\Phi\left(v^2 - v^{-2}\right), \qquad \mathcal{W} = \Phi^2 \tag{5.13}$$

and the relation $e[2]e[-2] = 1$, we find:

$$\mathcal{V}^2\mathcal{W} + \left(\mathcal{U}\mathcal{W} - \prod_{a=1}^{N_a}\left(-M_a^2\right)\prod_{i=1}^{N_f}(-2iM_i)\right)^2 = \prod_{i=1}^{N_f}\left(\mathcal{W} - 4M_i^2\right)\prod_{a=1}^{N_a}\left(\mathcal{W} - M_a^2\right)^2, \tag{5.14}$$

which at $N_a = 0$ agrees with the result in [74] found by gauging $U(1)_{\text{top}}$ of the $U(2)$ theory.

5.1.2 Pure $SU(2)$

For a pure $SU(2)$ gauge theory, the bubbling term in the commutative limit is

$$z_{2\to 0}(\Phi) = \frac{1}{\Phi^2}, \tag{5.15}$$

so the abelianized shift operators are:

$$\widetilde{v}^{\pm 2} = v^{\pm 2} + \frac{1}{\Phi^2}. \tag{5.16}$$

The primitive monopoles take the form:

$$\begin{aligned} \widetilde{v}^2 + \widetilde{v}^{-2} &= -\frac{1}{\Phi^2}(e[2] + e[-2]) + \frac{2}{\Phi^2}, \\ \Phi(\widetilde{v}^2 - \widetilde{v}^{-2}) &= -\frac{1}{\Phi}(e[2] - e[-2]). \end{aligned} \tag{5.17}$$

If we define:

$$\mathcal{U} = \frac{1}{2}(\widetilde{v}^2 + \widetilde{v}^{-2}), \qquad \mathcal{V} = \frac{1}{2}\Phi(\widetilde{v}^2 - \widetilde{v}^{-2}), \qquad \mathcal{W} = \Phi^2, \tag{5.18}$$

we find that $e[2]e[-2] = 1$ implies the relation:

$$\mathcal{V}^2 = \mathcal{U}^2\mathcal{W} - 2\mathcal{U}, \tag{5.19}$$

which does not belong to the series (5.10) and agrees with [74].

5.1.3 G_2 with N_f Fundamentals

To demonstrate effectiveness of our formalism, we now discuss theory with the gauge group G_2 and N_f hypermultiplets in a seven-dimensional fundamental representation of G_2. Recall from Section 4.4.3 that the lattice of coweights is generated by a short coroot α_{mon} and a long coroot β_{mon}. At zeor magnetic charge, there are two Casimir invariants:

$$f_1 = \Phi_1^2 + \Phi_2^2, \quad f_2 = \Phi_2^2(\Phi_2^2 - 3\Phi_1^2)^2, \tag{5.20}$$

and at magnetic charge α_{mon}, there are six primitive dressed monopoles, which in the commutative limit give six primitive commutative monopoles:

$$\begin{aligned} m_0 &= [V^{\alpha_{\text{mon}}}], \quad m_1 = [\Phi_2 V^{\alpha_{\text{mon}}}], \quad m_2 = [\Phi_2^2 V^{\alpha_{\text{mon}}}], \\ m_3 &= [\Phi_2^3 V^{\alpha_{\text{mon}}}], \quad m_4 = [\Phi_2^4 V^{\alpha_{\text{mon}}}], \quad m_5 = [\Phi_2^5 V^{\alpha_{\text{mon}}}]. \end{aligned} \tag{5.21}$$

In Section 4.4.3, we found the abelianized factor $Z(\Phi)$ for "$\alpha_{\text{mon}} \to 0$" bubbling. Its $r \to \infty$ limit is:

$$z(\Phi) = \frac{4^{2-N_f}(\Phi_1^2 + \Phi_2^2)^{N_f}}{9\Phi_2^2(3\Phi_1^4 - 10\Phi_1^2\Phi_2^2 + 3\Phi_2^4)}. \tag{5.22}$$

First we observe that by taking the following products:

$$\begin{aligned} m_1^2 - m_2 m_0, \quad m_2 m_1 - m_3 m_0, \quad m_3 m_1 - m_4 m_0 \\ m_4 m_1 - m_5 m_0, \quad m_4 m_2 - m_5 m_1, \quad m_4 m_3 - m_5 m_2, \end{aligned} \tag{5.23}$$

we can obtain all six primitive dressed monopoles of magnetic charge β_{mon}. Because α_{mon} and β_{mon} are fundamental coweights, they obviously generate the rest of charges. Furthermore, (5.23) implies that monopoles of charge β_{mon} are generated from those of charge α_{mon}. Therefore, six monopoles $m_0, \ldots, m_5$ and two Casimirs f_1 and f_2 generate the full chiral ring.

It remains to determine relations. They follow from the relations in $\mathbb{C}[\Lambda_w^\vee]$:

$$\begin{aligned} e[\alpha_{\text{mon}}]e[-\alpha_{\text{mon}}] &= 1, \\ e[\beta_{\text{mon}} + \alpha_{\text{mon}}]e[-\beta_{\text{mon}} - \alpha_{\text{mon}}] &= 1, \\ e[\beta_{\text{mon}} + 2\alpha_{\text{mon}}]e[-\beta_{\text{mon}} - 2\alpha_{\text{mon}}] &= 1, \\ e[\beta_{\text{mon}} + \alpha_{\text{mon}}]e[\alpha_{\text{mon}}] - e[\beta_{\text{mon}} + 2\alpha_{\text{mon}}] &= 0. \end{aligned} \tag{5.24}$$

These relations can be seen to easily follow from linear dependences between the short (co)roots of G_2. Even more, they generate a complete set of relations in $\mathbb{C}[\Lambda_w^\vee]$: short (co)roots generate the full (co)weight lattice, and relations between the short (co)roots determine everything.

Using the definition (5.3) of commuting shift operators, incorporating the abelianized bubbling factor (5.22) according to (5.5) and (5.6), and using the relations (5.24), one can derive the relations

$$L_i = [P_i(\Phi)V^{\alpha_{\text{mon}}}] + F_i(\Phi) \quad (i = 1, 2, 3, 4) \tag{5.25}$$

between the chiral ring generators, where

$$\begin{aligned}
L_1 &\equiv m_2^2 + \frac{1}{2}m_4 m_0 - \frac{3}{2}m_3 m_1 + \frac{3}{8}(m_1^2 - m_2 m_0) f_1, \\
L_2 &\equiv m_3 m_2 + m_5 m_0 - 2m_4 m_1 + \frac{3}{4}(m_2 m_1 - m_3 m_0) f_1, \\
L_3 &\equiv m_3^2 + \frac{1}{2}m_5 m_1 - \frac{3}{2}m_4 m_2 + \frac{3}{16}(m_3 m_1 - m_4 m_0) f_1 - \frac{9}{64}(m_1^2 - m_2 m_0) f_1^2, \\
L_4 &\equiv 2m_5 m_2 - m_4 m_3 - \frac{1}{16}m_1 m_0 f_2 - \frac{3}{4}(3m_4 m_1 - m_5 m_0) f_1 + \frac{9}{16}(2m_2 m_1 - m_3 m_0) f_1^2,
\end{aligned} \tag{5.26}$$

and P_i and F_i are N_f-dependent polynomials in Φ that can be expressed in terms of known generators. The simplest case is $N_f = 0$, where most of the right-hand sides vanish:

$$([P_i(\Phi)V^{\alpha_{\text{mon}}}] + F_i(\Phi))_{i=1,2,3,4} = \left(0, 0, -\frac{1}{3}m_0, 0\right). \tag{5.27}$$

For $N_f = 1$, the answer is:

$$([P_i(\Phi)V^{\alpha_{\text{mon}}}] + F_i(\Phi))_{i=1,2,3,4} = \left(\frac{1}{3}m_0,\ \frac{2}{3}m_1,\ -\frac{11}{24}f_1 m_0 + \frac{1}{3}m_2,\ f_1 m_1 - \frac{2}{3}m_3\right). \tag{5.28}$$

Note that we have not checked whether this is a complete set of equations, i.e., whether the Coulomb branch is a complete intersection (though it must be possible to check this from a more careful analysis of relations). However, these equations are locally independent, so it is at least a local complete intersection.

5.2 Quantized Chiral Rings

Having explained how our formalism can be used to derive the Coulomb branch chiral rings of the gauge theories we study, let us now turn to a more refined observable: the OPE of the Coulomb branch 1D sector. As explained in [36, 37], the OPE can be interpreted as a non-commutative star product on the chiral ring that reduces to the regular product between the corresponding holomorphic functions as we take $r \to \infty$. This star product has the interpretation as a quantization of the ring of holomorphic functions on the Coulomb branch, with $1/r$ serving as the quantization parameter. In particular, the terms appearing multiplied by a single power of $1/r$ in the OPE are interpreted as the Poisson bracket of the holomorphic functions from the $r \to \infty$ limit.

5.2.1 $SU(2)$ with N_f Fundamentals and N_a Adjoints

To illustrate that the OPE indeed gives more information than the chiral ring, let us present an example where distinct 3D theories have the same Coulomb branch chiral ring but different star products. Such an example was in fact already encountered in Section 5.1.1: it is the $SU(2)$ gauge theory with N_f fundamental and N_a adjoint hypermultiplets. Indeed, in the previous section we showed that the Coulomb branch is a $D_{N_f+2N_a}$ singularity, so it depends only on the combination $n = N_f + 2N_a$. Let us now show that the OPE does not depend only on this combination, and so for any fixed n we obtain $\lceil n/2 \rceil$ distinct quantizations of the ring of holomorphic functions on the cone over the D_n singularity.

We restrict to the case $N_f + 2N_a > 2$ where the theory is good and to $N_f > 0$ where all bubbling coefficients can be set to zero. The operators of dimension $\Delta_{\mathcal{U}} = N_f + 2N_a - 2$, $\Delta_{\mathcal{V}} = N_f + 2N_a - 1$, and $\Delta_{\mathcal{W}} = 2$ whose flat-space limits are given in (5.13) are

$$\mathcal{U} = 2^{N_f-1}(M^2 + M^{-2}), \qquad \mathcal{V} = -i2^{N_f-1}\Phi(M^2 - M^{-2}), \qquad \mathcal{W} = \Phi^2. \tag{5.29}$$

Using the corresponding shift operators obtained from (4.5), we then find

$$\mathcal{V}^2 + \mathcal{U} \star \mathcal{W} \star \mathcal{U} = P(\mathcal{W}) + \frac{2}{r}\mathcal{U} \star \mathcal{V}\,, \tag{5.30}$$

where all products are understood to be star products and $P(\mathcal{W}) = \mathcal{W}^{N_f+2N_a-1} + O(1/r)$ is the following polynomial in $\mathcal{W}$:

$$P(\mathcal{W}) \equiv \frac{\left(\sqrt{\mathcal{W}} + \frac{2i}{r}\right)\left(\sqrt{\mathcal{W}} + \frac{i}{r}\right)^{2(N_f-1)}\left[\left(\sqrt{\mathcal{W}} + \frac{i}{2r}\right)\left(\sqrt{\mathcal{W}} + \frac{3i}{2r}\right)\right]^{2N_a}}{2\sqrt{\mathcal{W}}} + (i \leftrightarrow -i) \tag{5.31}$$

(despite appearances, this expression is indeed a polynomial). To leading order in $1/r$, we reproduce (5.10). We can also compute various OPEs such as the antisymmetrized OPEs of the Coulomb branch chiral ring generators (5.29):

$$
\begin{aligned}
[\mathcal{U},\mathcal{W}]_\star &= \frac{4}{r}\mathcal{V} - \frac{4}{r^2}\mathcal{U}\,, \\
[\mathcal{V},\mathcal{W}]_\star &= -\frac{4}{r}\mathcal{W}\star\mathcal{U} - \frac{4}{r^2}\mathcal{V}\,, \\
[\mathcal{U},\mathcal{V}]_\star &= -\frac{2}{r}\mathcal{U}^2 + Q(\mathcal{W})\,,
\end{aligned}
\tag{5.32}
$$

where $Q(\mathcal{W})$ is a polynomial in $\mathcal{W}$ given by:

$$
Q(\mathcal{W}) \equiv \frac{i\left(\sqrt{\mathcal{W}}+\frac{i}{r}\right)^{2(N_f-1)}\left[\left(\sqrt{\mathcal{W}}+\frac{i}{2r}\right)\left(\sqrt{\mathcal{W}}+\frac{3i}{2r}\right)\right]^{2N_a}}{2\sqrt{\mathcal{W}}} + (i \leftrightarrow -i) \tag{5.33}
$$

(this expression is again a polynomial in $\mathcal{W}$ despite its appearance).

We see that (5.30) and (5.32) do not depend only on the combination $N_f + 2N_a$ that determines the Coulomb branch, thus providing an example of different quantizations of the same chiral ring.[18] Note however that the $1/r$ terms in (5.32), like the chiral ring relation (5.10), do depend only on $N_f + 2N_a$, and thus the Poisson structure on $D_{N_f+2N_a}$ is the same for all of the distinct quantizations.

For other examples where our formalism can be used to determine the quantization of the Coulomb branch chiral ring, see Appendix D.

5.2.2 G_2 with N_f Fundamentals

Let us make a few comments on a theory with gauge group G_2, since we had it as one of the examples earlier. At the very least, the same two Casimirs and six primitive monopoles of minimal charge α_{mon} are expected to generate the non-commutative algebra $\mathcal{A}_C$:

$$
\begin{aligned}
&f_1 = \Phi^2 + \Phi_2^2, \quad f_2 = \Phi_2^2(\Phi_2^2 - 3\Phi_1^2)^2, \\
&m_0 = \mathcal{M}^{\alpha_{\text{mon}}}, \quad m_1 = [\Phi_2\mathcal{M}^{\alpha_{\text{mon}}}], \quad m_2 = [\Phi_2^2\mathcal{M}^{\alpha_{\text{mon}}}], \\
&m_3 = [\Phi_2^3\mathcal{M}^{\alpha_{\text{mon}}}], \qquad m_4 = [\Phi_2^4\mathcal{M}^{\alpha_{\text{mon}}}], \qquad m_5 = [\Phi_2^5\mathcal{M}^{\alpha_{\text{mon}}}].
\end{aligned}
\tag{5.34}
$$

They satisfy the same relations as in (5.25), (5.26), with the left-hand side written in terms of the star-product and right-hand side receiving $1/r$ corrections.

[18] One may ask, however, whether a change of basis for the generators $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ could render (5.30) and (5.32) dependent only on $N_f + 2N_a$. For changes of basis where we only allow ourselves to redefine each operator by adding operators of strictly lower dimension multiplied by appropriate factors of $1/r$, it is impossible to make (5.30) and (5.32) only depend on $N_f + 2N_a$.

However, quite curiously, one can find simple relations that identify a much smaller set of generators of $\mathcal{A}_C$ as a non-commutative algebra, or alternatively (but not equivalently in general), as the commutative Poisson algebra. Namely, we find that:

$$m_i \star f_1 - f_1 \star m_i = -\frac{4i}{r\sqrt{3}} m_{i+1} - \frac{4}{3r^2} m_i, \quad i = 0, \ldots, 5, \tag{5.35}$$

implying that it is enough to have f_1, f_2 and m_0 to generate the rest of the algebra through star-products. The above equation also implies the Poisson bracket $\{m_i, f_1\} = -\frac{4i}{\sqrt{3}} m_{i+1}$. In order to compute star products, we must use the bubbling factor derived in Section 4.4.3.

5.3 Correlation Functions and Mirror Symmetry

We now demonstrate the utility of the shift operator formalism for computing correlation functions of twisted CBOs, with applications to non-abelian 3D mirror symmetry [76–78]. We start with the general setup describing the computation of correlation functions, and then we give an example.

5.3.1 General setup

The three ingredients for computing correlation functions are: the vacuum hemisphere wavefunction, the gluing measure, and the shift operators. The vacuum hemisphere wavefunction $\Psi_0(\sigma, B)$ (where σ is valued in the Cartan of $\mathfrak{g}$ and B in the coweight lattice) can be read off from (2.15) by setting $b = 0$:

$$\Psi_0(\sigma, B) \equiv Z_0(\vec{0}; \sigma, B) = \delta_{B,\vec{0}} \frac{\prod_{w\in\mathcal{R}} \frac{1}{\sqrt{2\pi}} \Gamma(\frac{1}{2} - iw\cdot\sigma)}{\prod_{\alpha\in\Delta} \frac{1}{\sqrt{2\pi}} \Gamma(1 - i\alpha\cdot\sigma)}\,, \tag{5.36}$$

where $\mathcal{R}$ denotes the weights of the hypermultiplet representation $\mathcal{R}$ of G and Δ denotes the roots of G. The gluing measure $\mu(\sigma, B)$ is as in (2.7), namely

$$\mu(\sigma, B) = \prod_{\alpha\in\Delta^+} (-1)^{\alpha\cdot B} \left[\left(\frac{\alpha\cdot\sigma}{r}\right)^2 + \left(\frac{\alpha\cdot B}{2r}\right)^2\right] \prod_{w\in\mathcal{R}} (-1)^{\frac{|w\cdot B| - w\cdot B}{2}} \frac{\Gamma\left(\frac{1}{2} + iw\cdot\sigma + \frac{|w\cdot B|}{2}\right)}{\Gamma\left(\frac{1}{2} - iw\cdot\sigma + \frac{|w\cdot B|}{2}\right)} \tag{5.37}$$

(note that $\prod_{\alpha\in\Delta^+}(-1)^{\alpha\cdot B} = e^{2\pi i\rho\cdot B}$ where ρ is the Weyl vector). The shift operators are given by (3.18) combined with (3.21) (see also Appendix A). Without loss of generality, we work in the North picture, where

$$M_N^b = \frac{\prod_{w\in\mathcal{R}} \left[\frac{(-1)^{(w\cdot b)_+}}{r^{|w\cdot b|/2}} \left(\frac{1}{2} + irw\cdot\Phi_N\right)_{(w\cdot b)_+}\right]}{\prod_{\alpha\in\Delta}\left[\frac{(-1)^{(\alpha\cdot b)_+}}{r^{|\alpha\cdot b|/2}} \left(ir\alpha\cdot\Phi_N\right)_{(\alpha\cdot b)_+}\right]} e^{-b\cdot(\frac{i}{2}\partial_\sigma + \partial_B)}\,, \qquad \Phi_N = \frac{1}{r}\left(\sigma + i\frac{B}{2}\right)\,, \tag{5.38}$$

and therefore drop the N subscripts from now on.

With these ingredients, the matrix model expression for the correlator of twisted CBOs $\mathcal{O}^i(\varphi_i)$, $i = 1,\dots,n$, inserted at points φ_i obeying $0 < \varphi_1 < \cdots < \varphi_n < \pi$, takes the form (see also [38])

$$\langle \mathcal{O}^1(\varphi_1)\cdots\mathcal{O}^n(\varphi_n)\rangle_{S^3} = \frac{1}{|\mathcal{W}|Z_{S^3}}\sum_{\vec{B}}\int d\vec{\sigma}\,\mu(\vec{\sigma},\vec{B})\Psi_0(\vec{\sigma},\vec{B})\widehat{\mathcal{O}}^1\cdots\widehat{\mathcal{O}}^n\Psi_0(\vec{\sigma},\vec{B})\,, \tag{5.39}$$

where $\widehat{\mathcal{O}}^i$ are the shift operators corresponding to $\mathcal{O}^i$ and Z_{S^3} is the vacuum S^3 partition function

$$Z_{S^3} = \frac{1}{|\mathcal{W}|}\sum_{\vec{B}}\int d\vec{\sigma}\,\mu(\vec{\sigma},\vec{B})\Psi_0(\vec{\sigma},\vec{B})^2\,, \tag{5.40}$$

which we need to divide by in order to have a normalized correlator. From here on, we will drop the hats on the shift operators and therefore not make a notational distinction between a shift operator and the twisted CBO that it represents.

5.3.2 Example: $U(N_c)$ with one adjoint and one fundamental hypermultiplet

As a concrete example, let us consider the $U(N_c)$ gauge theory with one adjoint hypermultiplet and one fundamental hypermultiplet. This theory has $\mathcal{N} = 8$ SUSY enhancement and is therefore self-mirror [79]. This theory us ugly in the sense of Gaiotto and Witten [30], so the monopoles of lowest dimension saturate the unitarity bound $\Delta = 1/2$ and generate a free subsector. The S^3 partition function is

$$Z_{S^3} = \frac{1}{N_c!}\left(\int\prod_{I=1}^{N_c} d\sigma_I\right)\frac{\prod_{I<J} 4\sinh^2(\pi(\sigma_I-\sigma_J))}{\prod_{I,J} 2\cosh(\pi(\sigma_I-\sigma_J))(\prod_i 2\cosh(\pi\sigma_I))}\,, \tag{5.41}$$

where $I, J = 1,\dots,N_c$.

The weight lattice $\mathbb{Z}^{N_c}$ is generated by the N_c fundamental weights $(1,\vec{0}),\dots,(\vec{0},1)$, and the $N_c^2 - N_c$ roots are the pairwise differences of these fundamental weights. Since $U(N_c)$ is its own Langlands dual, we can think of $\vec{\sigma}$ and $\vec{B}$ as taking values in $\mathbb{Z}^{N_c}$. The vacuum wavefunction (5.36) in this theory simplifies to

$$\Psi_0(\vec{\sigma},\vec{B}) = \delta_{\vec{B},\vec{0}} \prod_{I=1}^{N_c} \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2} - i\sigma_I\right) \prod_{I,J} \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2} - i\sigma_{IJ}\right) \prod_{I<J} \frac{2\sinh(\pi\sigma_{IJ})}{\sigma_{IJ}}\,, \tag{5.42}$$

where $\sigma_{IJ} \equiv \sigma_I - \sigma_J$, and, omitting factors of r for convenience, the gluing measure is[19]

$$\mu(\vec{\sigma},\vec{B}) = \prod_{I<J}\left(\sigma_{IJ}^2 + \frac{1}{4}B_{IJ}^2\right)\left[\prod_{I=1}^{N_c}(-1)^{(-B_I)_+}\frac{\Gamma(\frac{1+|B_I|}{2} + i\sigma_I)}{\Gamma(\frac{1+|B_I|}{2} - i\sigma_I)}\right]^{N_f}\,. \tag{5.43}$$

The partition function obtained by the gluing formula (5.40) then reproduces (5.41).

The operators of the 1D Higgs branch sector of the same theory can be written as $U(N_c)$ invariant products of anti-periodic adjoint scalars Q and $\widetilde{Q}$. The correlation functions of these operators can be computed using the prescription of [37]. (In particular, see Section 7.3 of [37].) They reduce to calculations in a free theory with 1D propagator

$$\langle {Q_i}^j(\varphi_1){\tilde{Q}_{i'}}{}^{j'}(\varphi_2)\rangle_\sigma = -\delta_{ii'}\delta^{jj'}\frac{\operatorname{sign}(\varphi_{12}) + \tanh(\pi\sigma_{ij})}{8\pi r}e^{-\sigma_{ij}\varphi_{12}}\,, \tag{5.44}$$

where $\sigma_{ij} \equiv \sigma_i - \sigma_j$. Here, $\langle\rangle_\sigma$ denotes an auxiliary correlator from which the full correlator $\langle\rangle$ is obtained by an appropriate integral over σ. In particular, for operators $\mathcal{O}^i$ constructed from Q and $\widetilde{Q}$, the correlation function is

$$\begin{aligned}\langle\mathcal{O}^1(\varphi_1)\cdots\mathcal{O}^n(\varphi_n)\rangle = \frac{1}{Z_{S^3}N_c!}\left(\int\prod_{i=1}^{N_c}d\sigma_i\right)\frac{\prod_{i<j}4\sinh^2(\pi(\sigma_i-\sigma_j))}{\prod_{i,j}2\cosh(\pi(\sigma_i-\sigma_j))(\prod_i 2\cosh(\pi\sigma_i))} \\ \times\langle\mathcal{O}^1(\varphi_1)\cdots\mathcal{O}^n(\varphi_n)\rangle_\sigma\,,\end{aligned} \tag{5.45}$$

where $\langle\mathcal{O}^1(\varphi_1)\cdots\mathcal{O}^n(\varphi_n)\rangle_\sigma$ is computed using Wick contractions with the propagator (5.44).

5.3.3 $U(2)$ with $N_a = N_f = 1$

The Coulomb branch chiral ring operators of lowest dimension are $\mathcal{M}^{(\pm1,0)}$ (with $\Delta = 1/2$) and $\operatorname{tr}\Phi$, $\mathcal{M}^{\pm(1,1)}$, $\mathcal{M}^{(1,-1)}$, $\mathcal{M}^{(\pm2,0)}$ (with $\Delta = 1$). Particular linear combinations of these operators comprise the chiral ring generators, namely $\mathcal{M}^{\pm(1,0)}$, $\mathcal{M}^{\pm(1,1)}$, and $-\mathcal{M}^{(1,-1)}$ −

[19] The adjoint hyper contributes a sign $(-1)^{|B_{IJ}|} = (-1)^{(B_{IJ})_+ + (-B_{IJ})_+}$ to the $\prod_{I<J}$, which cancels with the $(-1)^{B_{IJ}}$ in (2.7).

$2i\,\mathrm{tr}\,\Phi$. They satisfy the single relation

$$[(\mathcal{M}^{(-1,0)})^2 - 4\mathcal{M}^{(-1,-1)}][(\mathcal{M}^{(1,0)})^2 - 4\mathcal{M}^{(1,1)}] = (-\mathcal{M}^{(1,-1)} - 2i\,\mathrm{tr}\,\Phi)^2. \tag{5.46}$$

The products in this equation are commutative chiral ring products, not star products. Thus the Coulomb branch factorizes into free and interacting sectors as $\mathrm{Sym}^2(\mathbb{C}^2) \cong \mathbb{C}^2 \times (\mathbb{C}^2/\mathbb{Z}_2)$.

By matching all two- and three-point functions of these lowest-dimension twisted CBOs and HBOs across mirror symmetry, computed within their respective 1D topological sectors, we can derive the mirror map (see Appendix E)

$$\frac{1}{(4\pi)^{1/2}}\mathcal{M}^{(\mp1,0)} \leftrightarrow \mathrm{tr}\,Q, \mathrm{tr}\,\tilde{Q}, \tag{5.47}$$

$$\frac{1}{4\pi}\mathcal{M}^{(\mp2,0)} \leftrightarrow (\mathrm{tr}\,Q)^2, (\mathrm{tr}\,\tilde{Q})^2, \tag{5.48}$$

$$\frac{1}{2\pi}\mathcal{M}^{\mp(1,1)} \leftrightarrow \mathrm{tr}\,Q^2, \mathrm{tr}\,\tilde{Q}^2, \tag{5.49}$$

$$\frac{1}{4\pi}\left(\mathcal{M}^{(1,-1)} - \frac{1}{r}\right) \leftrightarrow \mathrm{tr}\,Q\,\mathrm{tr}\,\tilde{Q}, \tag{5.50}$$

$$-\frac{i}{4\pi}\,\mathrm{tr}\,\Phi \leftrightarrow \mathrm{tr}\,Q\tilde{Q}. \tag{5.51}$$

The operators on the LHS in the Coulomb branch TQFT have precisely the same correlation functions as the operators on the RHS in the Higgs branch TQFT.

A direct way of deriving the mirror map is as follows. First, match certain "basic" operators by computing their correlation functions. Next, generate composite operators from these basic operators via the star product and use the fact that the structure of the star product is the same on both sides to deduce the map between these composite operators (whose one-point functions can then be matched, as a further consistency check; in our basis, mixing with the identity renders one-point functions nonzero). This point of view emphasizes that the shift operators themselves, which generate the star product via composition, are more fundamental than the correlators that they compute in that one can write all correlators as expectation values of composite operators obtained via the OPE. For an illustration of this procedure, see Appendix E.

5.3.4 $U(N_c)$ with $N_a = N_f = 1$

We do not study the case $N_c > 2$ in detail, but let us point out that the mirror map in this case takes

$$\frac{1}{(4\pi)^{1/2}}\mathcal{M}^{(-1,\vec{0})} \leftrightarrow \operatorname{tr} Q, \quad \frac{1}{(4\pi)^{1/2}}\mathcal{M}^{(1,\vec{0})} \leftrightarrow \operatorname{tr} \tilde{Q}, \tag{5.52}$$

with normalizations being fixed by the two-point functions

$$\frac{1}{4\pi}\langle\mathcal{M}^{(-1,\vec{0})}(\varphi_1)\mathcal{M}^{(1,\vec{0})}(\varphi_2)\rangle = \langle\operatorname{tr} Q(\varphi_1)\operatorname{tr}\tilde{Q}(\varphi_2)\rangle = -\frac{N_c \operatorname{sign}\varphi_{12}}{8\pi r}. \tag{5.53}$$

By taking star products, it also follows that the suitably normalized monopoles $\mathcal{M}^{(\mp p,\vec{0})}$ (which can bubble) map to $(\operatorname{tr} Q)^p$ and $(\operatorname{tr}\tilde{Q})^p$. More generally, we expect the Higgs branch operators $\operatorname{tr} Q$, $\operatorname{tr}\tilde{Q}$ to map to monopole operators of GNO charge $(\mp 1, \vec{0})$ in SQCD, but for $N_f > 1$, the insertion on the Coulomb branch side does not simplify so easily, and correlators on the Higgs branch side also become difficult to compute.

Going beyond the free sector, it is natural to conjecture that the monopoles $\mathcal{M}^{(\mp\vec{1}_p,\vec{0}_{N_c-p})}$ (which do not bubble) map to $\operatorname{tr} Q^p$ and $\operatorname{tr}\tilde{Q}^p$ for $p = 1, \ldots, N_c$, although we have been unable to demonstrate this analytically. These monopoles are special for several reasons. First, assuming the correctness of the stated map, they correspond to all of the independent traces of powers of Q and $\tilde{Q}$ individually. Second, it seems that they comprise the minimal set of bare monopoles needed to generate all other bare monopoles via star products.

6 Discussion

6.1 Summary

This work ties various loose ends together. On one hand, it extends the formalism of shift operators for Coulomb branch operators [38] to arbitrary non-abelian 3D $\mathcal{N} = 4$ gauge theories with hypermultiplet matter. On the other hand, by doing so, it provides an alternative approach to the abelianization description of the Coulomb branch [53]. In particular, it can be seen as a derivation of the latter from first principles. In addition, our approach allows us to compute correlation functions of Coulomb branch operators in cases where the IR fixed points are SCFTs, thus providing natural choices of bases that relate the non-commutative star product algebra $\mathcal{A}_C$ of Coulomb branch operators to these correlation functions. The relation between $\mathcal{A}_C$ and correlation functions seems to become transparent only when quantizing the Coulomb branche by placing $\mathcal{N} = 4$ theory on a sphere rather than by studying it in an Ω-background: the latter route to quantization has a less straightforward connection to SCFT operators. Finally, on our way to achieving these goals, we gained a better understanding of monopole bubbling phenomena, which are crucial nonperturbative effects in the description of magnetic defects. Our approach to bubbling is of purely algebraic nature, based on symmetries and algebraic consistency of the OPE. It avoids the technicalities of the direct computations of bubbling that have been performed previously [47, 48, 50, 52], therefore serving as a good check and testing ground for them.

While the focus of this paper was mostly on developing the general formalism, we also provide a few applications and examples. In particular, in Section 4 we derive the "abelianized bubbling coefficients" for a large family of rank-one and rank-two gauge theories, which can then be used to extract the data on Coulomb branch operators of these theories (including the algebra $\mathcal{A}_C$ and correlators) in a completely straightforward and algorithmic fashion. We then illustrate it in Section 5 by a few examples. While abelian examples from our previous paper [38] provide quantizations of A_N singularities, in Section 5 here we start with an example of $SU(2)$ gauge theory with fundamental and adjoint matter resulting in (non-equivalent) quantizations of the D_N singularity. For illustration purposes, we also apply our formalism to the G_2 gauge theory, as in this case no other techniques are available. Finally, we illustrate the point about correlation functions by computing them using our approach and matching them across mirror symmetry when possible. Further applications are gathered in appendices.

6.2 Future Directions and Open Problems

Besides further applications of our formalism to gather more data on various 3D $\mathcal{N} = 4$ gauge theories (in particular SCFTs), or to check or discover new dualities, there are a number of conceptual questions that are interesting avenues for future work:

- It would be interesting to extend our construction to more general gauge theories, namely to gauge theories that also have charged matter in half-hypermultiplets, to gauge theories those that involve both ordinary and twisted multiplets at once, and/or theories with Chern-Simons couplings. Understanding the moduli spaces of vacua, their quantization, and the corresponding correlation functions in such theories, if possible, are among the outstanding questions to address.

- It would be interesting to compare the bubbling terms obtained using our method to those coming from the dimensional reduction of the 4D bubbling terms computed in [48]. We performed a few preliminary comparisons and found that the two agree up to operator mixing and various normalization factors, but a more systematic study is needed. It has been found recently [50, 52] that the results of [48] often involve discrepancies with results obtained using the AGT correspondence, and a fix was proposed. Based on our preliminary checks, it appears that all of the subtleties present in 4D disappear upon reduction to 3D, and it would be nice to understand why.

- It would be interesting to recast our construction of shift operators and bubbling coefficients (or, equivalently, abelianization) in a way that uses the mathematical definition of the Coulomb branch [54–57]. Also, understanding whether the existence of abelianized bubbling terms has any implications for the structure or possible decomposition of monopole moduli spaces could be of interest.[20]

- Lastly, it would be interesting to understand more conceptually whether there exists a relation between quantization via S^3 and quantization via the Ω-background [42–44]. Similar relations are abundant in various dimensions for problems involving a supercharge (equivariant differential) Q such that Q^2 is a vector field with fixed points (see for instance the recent work [80] for the case of isolated fixed points).

- More broadly, our work fits into the larger program of constructing and classifying deformation quantizations arising from the 3D $\mathcal{N} = 4$ theories.[21] While our construction is certainly derived starting from a Lagrangian description, one may wonder if it can possibly be generalized to non-Lagrangian theories (like various classes of SCFTs from [81]), and/or whether Lagrangian theories play a special role in the broader classification program of deformation quantizations.

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A Conventions

Here, we summarize our conventions and notations. Unless otherwise stated, G is assumed to be a simple gauge group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{t}$ its Cartan subalgebra, and $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$ its complexification. The root system is denoted by Δ, the weight lattice by Λ_w, and the coweight lattice (the weight lattice of ${}^LG$) by $\Lambda_w^\vee$. The matter representation is $\mathcal{R} \oplus \overline{\mathcal{R}}$.

The North pole shift operator (2.25) is denoted by M_N^b, while its South pole analog is M_S^b. Sometimes, we simply write M^b, in which case it is assumed to be the North pole operator. Here $b \in \Lambda_w^\vee$ is a coweight of G. The abelianized shift operator is denoted by $\widetilde{M}_N^b$ (with the same remark on N/S):

$$\widetilde{M}^b = M^b + \sum_{|v|<|b|} Z^{\text{ab}}_{b\to v}(\Phi) M^v, \tag{A.1}$$

where the sum is over coweights shorter than b and $Z^{\text{ab}}_{b\to v}(\Phi)$ are abelianized bubbling coefficients. The commutative ($r \to \infty$) limit of the shift operator M_N^b is denoted by v^b, and the same for M_S^b as the N/S distinction disappears in the commutative limit. Similarly, the abelianized bubbling factor in this limit is denoted by $z_{b\to v}(\Phi)$, and the abelianized commutative shift operator is:

$$\widetilde{v}^b = v^b + \sum_{|u|<|b|} z_{b\to u}(\Phi) v^u. \tag{A.2}$$

We deal with a number of objects that involve sums over Weyl orbits. If a quantity $F(b)$ depends on the coweight b, we employ the following convention in summing over its Weyl orbit:

$$\sum_{b' \in \mathcal{W}b} F(b') \equiv \frac{1}{|\mathcal{W}_b|} \sum_{\mathrm{w} \in \mathcal{W}} F(\mathrm{w} \cdot b), \tag{A.3}$$

where $\mathcal{W}_b \subset \mathcal{W}$ is the stabilizer of b. In particular, we use it to define the averaged shift operator, the bare monopole operator and the dressed monopole:

$$\begin{aligned} \overline{M}^b &= \sum_{b' \in \mathcal{W}b} M^{b'}, \\ \mathcal{M}^b &= \sum_{b' \in \mathcal{W}b} \widetilde{M}^{b'} = \frac{1}{|\mathcal{W}_b|} \sum_{\mathrm{w} \in \mathcal{W}} \left(M^{\mathrm{w} \cdot b} + \sum_{|v|<|b|} Z^{\mathrm{ab}}_{b \to v}(\Phi^{\mathrm{w}}) M^{\mathrm{w} \cdot v} \right), \\ [P(\Phi)\mathcal{M}^b] &= \frac{1}{|\mathcal{W}_b|} \sum_{\mathrm{w} \in \mathcal{W}} \left(P(\Phi^{\mathrm{w}}) M^{\mathrm{w} \cdot b} + P(\Phi^{\mathrm{w}}) \sum_{|v|<|b|} Z^{\mathrm{ab}}_{b \to v}(\Phi^{\mathrm{w}}) M^{\mathrm{w} \cdot v} \right). \end{aligned} \tag{A.4}$$

As explained in the main text, $\Phi^{\mathrm{w}} = \mathrm{w}^{-1} \cdot \Phi$, and Φ takes values in $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$. The averaged shift operator $\overline{M}^b$ defined here does not appear in the main text but plays certain role in the appendices. The dressed commuting monopole operator is defined as:

$$[P(\Phi) V^b] = \frac{1}{|\mathcal{W}_b|} \sum_{\mathrm{w} \in \mathcal{W}} P(\Phi^{\mathrm{w}}) \widetilde{v}^{\mathrm{w} \cdot b}. \tag{A.5}$$

Using the transformation property

$$Z^{\mathrm{ab}}_{\mathrm{w} \cdot b \to \mathrm{w} \cdot v}(\Phi) = Z^{\mathrm{ab}}_{b \to v}(\Phi^{\mathrm{w}}), \tag{A.6}$$

we might also introduce

$$Z_{\mathrm{mono}}(b \to v; \Phi) = \sum_{b' \in \mathcal{W}b} Z^{\mathrm{ab}}_{b' \to v}(\Phi), \tag{A.7}$$

so that the bare monopole becomes

$$\mathcal{M}^b = \overline{M}^b + \sum_{|v|<|b|} Z_{\mathrm{mono}}(b \to v; \Phi) M^v. \tag{A.8}$$

In the appendices, we sometimes omit brackets [] around dressed monopoles when no risk of confusion is present.

B Twisted-Translated Operators

Here is a brief, qualitative review of twisted operators and their corresponding topological sectors. Let us first recall the setup in $\mathbb{R}^3$.

In 3D $\mathcal{N}=4$ SCFTs, half-BPS operators are labeled by their charges (Δ, j, j_H, j_C) under the bosonic subalgebra $\mathfrak{so}(3,2)\oplus\mathfrak{su}(2)_H\oplus\mathfrak{su}(2)_C$ of the 3D $\mathcal{N}=4$ superconformal algebra $\mathfrak{osp}(4|4)$. They are Lorentz scalars ($j=0$) and can be classified as either HBOs ($\Delta = j_H$, $j_C = 0$) or CBOs ($\Delta = j_C$, $j_H = 0$), which we write abstractly with $\mathfrak{su}(2)_{H/C}$ spinor indices as $\mathcal{O}_{(a_1\cdots a_{2j_H})}$ and $\mathcal{O}_{(\dot{a}_1\cdots\dot{a}_{2j_C})}$. Hence $\mathfrak{su}(2)_H$ and $\mathfrak{su}(2)_C$ are spontaneously broken on the Higgs and Coulomb branches, respectively. In a Lagrangian theory, the vector multiplet contains adjoint scalars $\Phi_{\dot{a}\dot{b}}$ in the triplet of $\mathfrak{su}(2)_C$ and the hypermultiplet contains scalars $q_a, \tilde{q}_a$ in the doublet of $\mathfrak{su}(2)_H$ and in $\mathcal{R}, \bar{\mathcal{R}}$ of G. Then HBOs are precisely gauge-invariant polynomials in $q_a, \tilde{q}_a$ while CBOs consist of $\Phi_{\dot{a}\dot{b}}$ and (dressed) monopole operators $\mathcal{M}^b_{\dot{a}_1\cdots\dot{a}_{2j_C}}$.

The key fact is that *twisted* HBOs/CBOs, defined as

$$\mathcal{O}(x) = u^{a_1}(x)\cdots u^{a_{2j_H}}(x)\mathcal{O}_{a_1\cdots a_{2j_H}}(x), \quad \mathcal{O}(x) = v^{\dot{a}_1}(x)\cdots v^{\dot{a}_{2j_C}}(x)\mathcal{O}_{\dot{a}_1\cdots\dot{a}_{2j_C}}(x) \tag{B.1}$$

with appropriate position-dependent R-symmetry polarization vectors u and v, have topological correlation functions when the coordinate x is restricted to a line in $\mathbb{R}^3$. This is because they represent equivariant cohomology classes of certain supercharges $\mathcal{Q}^{H/C}\in\mathfrak{osp}(4|4)$. In particular, they are annihilated by $(\mathcal{Q}^{H/C})^2$, and operators $\mathcal{O}(x)$ at different x are related by $\mathcal{Q}^{H/C}$-exact operations called *twisted translations*. Hence the $\mathcal{Q}^{H/C}$-cohomology class of a twisted-translated operator $\mathcal{O}(x)$ is independent of its position x along the line. It follows that each supercharge $\mathcal{Q}^{H/C}$ has an associated 1D topological sector of cohomology classes: the OPE of these twisted HBOs/CBOs is an associative but noncommutative product, since there exists an ordering along the line.

The setup on S^3, where we localize with respect to $\mathcal{Q}^C$, is essentially the same (up to subtleties involving the "branch point" at infinity, discussed at length in [38]): the distinguished line is stereographically mapped to a great circle S^1_φ, so that twisted operators are parametrized by φ rather than x in (B.1), and the deformation parameter r (implicit in the definitions of $\mathcal{Q}^{H/C}$) becomes the S^3 radius. Taking $g_{\text{YM}}\to\infty$ at fixed r gives an SCFT on S^3 whose correlators are equivalent to those of the IR SCFT in flat space by stereographic projection. The non-conformal 3D $\mathcal{N}=4$ superalgebra $\mathfrak{s}$ contains the $\mathfrak{su}(2)_\ell\oplus\mathfrak{su}(2)_r$ isometries of S^3 as well as $\mathfrak{u}(1)_\ell\oplus\mathfrak{u}(1)_r$ R-symmetries. The supercharges $\mathcal{Q}^{H/C}$ each contain terms from both $\mathfrak{su}(2|1)$ factors of $\mathfrak{s}$, as required by the fact that they square to isometries with nontrivial fixed points. The corresponding twisted translations take the form $P_\varphi + R_{H/C} = \{\mathcal{Q}^{H/C}, \ldots\}$. Finally, the embedding of $\mathfrak{s}$ into $\mathfrak{osp}(4|4)$, as well as the polarization vectors in (B.1), are specified by Cartan embeddings of the $\mathfrak{u}(1)$ R-symmetries into $\mathfrak{su}(2)_H$ and $\mathfrak{su}(2)_C$.

These twisted operators are interesting for at least two reasons:

- Their two- and three-point functions fix those of HBOs and CBOs in the full 3D theory, by conformal symmetry and R-symmetry (roughly, conformal symmetry suffices to put any two or three operators on a great circle).

- At any fixed φ, twisted operators in the cohomology of $\mathcal{Q}^{H/C}$ are in one-to-one correspondence with elements of the Higgs/Coulomb branch chiral ring. The R-symmetry polarization vector u or v fixes a complex structure on the corresponding branch, so that the operators are chiral with respect to an $\mathcal{N} = 2$ superconformal subalgebra of $\mathfrak{osp}(4|4)$ whose embedding depends on the vector.

We focus on twisted CBOs representing nontrivial $\mathcal{Q}^C$-cohomology classes, namely: the twisted scalar $\Phi(\varphi) = v^{\dot{a}}(\varphi) v^{\dot{b}}(\varphi) \Phi_{\dot{a}\dot{b}}(\varphi)$, twisted bare monopoles $\mathcal{M}^b(\varphi)$, and twisted dressed monopoles $[P(\Phi)\mathcal{M}^b(\varphi)]$ (composite operators formed by monopoles and scalars).

C Matrix Non-Degeneracy and Abelianized Bubbling

Here we prove that the matrix determining linear system (3.20) is non-degenerate, thus implying that (3.20) has a unique solution.

The Weyl group might not act freely on an orbit of a general (dominant) coweight b, meaning that the size of $\mathcal{W} \cdot b$, equal to $\dim_{\mathbb{C}}(\rho^b)$, is smaller than $|\mathcal{W}|$. Each $\mathrm{w} \cdot b \in \mathcal{W} \cdot$ has a nontrivial stabilizer $\mathrm{St}_{\mathrm{w}\cdot b} \equiv \mathcal{W}_{\mathrm{w}\cdot b} \subset \mathcal{W}$ and, as a result, in equation (3.20), $\widetilde{M}^{\mathrm{w}\cdot b}$ is multiplied by $\sum_{\mathrm{w}' \in \mathrm{St}_{\mathrm{w}\cdot b}} P_i(\Phi^{\mathrm{w}'\mathrm{w}})$. For brevity, let us denote such $P_i(\Phi^{\mathrm{w}})$ averaged by $\mathrm{St}_{\mathrm{w}\cdot b}$ as $\overline{P_i(\Phi^{\mathrm{w}})}$. Finally, let us pick representatives $\mathrm{w}_1, \ldots, \mathrm{w}_{\dim(\rho^b)}$ of classes in $\mathcal{W}/\mathrm{St}_b$, so that the basis of ρ^b is given by $M^b = M^{\mathrm{w}_1\cdot b}, M^{\mathrm{w}_2\cdot b}, \ldots, M^{\mathrm{w}_{\dim(\rho^b)}\cdot b}$ (we will assume that $\mathrm{w}_1 = \mathrm{id}$ represents the trivial class). Now the equation (3.18) can be written in the matrix form as:

$$\begin{pmatrix} \mathcal{M}^b \\ [P_2 \mathcal{M}^b] \\ \cdot \\ \cdot \\ \cdot \\ [P_{\dim(\rho^b)} \mathcal{M}^b] \end{pmatrix} = \mathbf{P} \begin{pmatrix} \widetilde{M}^{\mathrm{w}_1\cdot b} \\ \widetilde{M}^{\mathrm{w}_2\cdot b} \\ \cdot \\ \cdot \\ \cdot \\ \widetilde{M}^{\mathrm{w}_{\dim(\rho^b)}\cdot b} \end{pmatrix}, \tag{C.1}$$

where

$$\mathbf{P} = \begin{pmatrix} \overline{P_1(\Phi^{\mathrm{w}_1})} & \overline{P_1(\Phi^{\mathrm{w}_2})} & \overline{P_1(\Phi^{\mathrm{w}_3})} & \dots & \overline{P_1(\Phi^{\mathrm{w}_{\dim(\rho^b)}})} \\ \overline{P_2(\Phi^{\mathrm{w}_1})} & \overline{P_2(\Phi^{\mathrm{w}_2})} & \overline{P_2(\Phi^{\mathrm{w}_3})} & \dots & \overline{P_2(\Phi^{\mathrm{w}_{\dim(\rho^b)}})} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ \overline{P_{\dim(\rho^b)}(\Phi^{\mathrm{w}_1})} & \overline{P_{\dim(\rho^b)}(\Phi^{\mathrm{w}_2})} & \overline{P_{\dim(\rho^b)}(\Phi^{\mathrm{w}_3})} & \dots & \overline{P_{\dim(\rho^b)}(\Phi^{\mathrm{w}_{\dim(\rho^b)}})} \end{pmatrix}. \tag{C.2}$$

In fact, this matrix is non-degenerate, meaning that its determinant is given by a polynomial in Φ that is not identically zero, as we prove now. By construction, $\sum_{\mathrm{w}\in\mathcal{W}} P_i(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot b}$ for $i = 1\dots\dim(\rho^b)$ form a basis over $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$. This implies that rows of the matrix $\mathbf{P}$ are linearly independent over $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$, i.e., over Weyl-invariant polynomials.

Let us assume that the matrix is nevertheless degenerate: this means that one of the rows, let us say it is the j-th row, is a linear combination of the other rows, with coefficient being some rational, generally non-Weyl-invariant, functions Q_i:

$$\overline{P_j(\Phi^{\mathrm{w}_a})} = \sum_{\substack{i=1..\dim(\rho^b)\\ i\neq j}} Q_i(\Phi)\overline{P_i(\Phi^{\mathrm{w}_a})}, \text{ for all } a = 1\dots\dim(\rho^b). \tag{C.3}$$

This should hold as an identity for all $a = 1\dots\dim(\rho^b)$, with $Q_i(\Phi)$ independent of a. Acting with an element of the Weyl group $\mathrm{w}\in\mathcal{W}$ on (C.3), should give another valid identity for all $a = 1\dots\dim(\rho^b)$. On the other hand, it simply permutes the columns of $\mathbf{P}$, thus it permutes equations in (C.3), at the same time replacing $Q_i(\Phi)$ by $Q_i(\Phi^{\mathrm{w}})$. Doing this for every element of $\mathcal{W}$ and averaging implies that we can replace $Q_i(\Phi)$ in (C.3) by its Weyl-averaged version. So we may assume that Q_i are Weyl-invariant rational functions. Every such Q_i is a ratio,

$$Q_i(\Phi) = \frac{A_i(\Phi)}{B_i(\Phi)}, \tag{C.4}$$

of some polynomials A_i and B_i. Let us consider:

$$D(\Phi) = \prod_{\substack{i=1..\dim(\rho^b)\\ i\neq j}} B_i(\Phi), \tag{C.5}$$

which is the common denominator (not the minimal one) of all Q_i. Even if this D_i is not a

Weyl-invariant polynomial, one can define another polynomial that is Weyl-invariant:

$$D^{\mathcal{W}}(\Phi) = \prod_{\mathrm{w}\in\mathcal{W}} D(\Phi^{\mathrm{w}}). \tag{C.6}$$

If we now multiply relation (C.3) by this $D^{\mathcal{W}}(\Phi)$, we obtain:

$$D^{\mathcal{W}}(\Phi)\overline{P_j(\Phi^{\mathrm{w_a}})} = \sum_{\substack{i=1..\dim(\rho^b)\\ i\neq j}} D^{\mathcal{W}}(\Phi)Q_i(\Phi)\overline{P_i(\Phi^{\mathrm{w_a}})}, \text{ for all } a = 1\ldots\dim(\rho^b). \tag{C.7}$$

This cancels all denominators of Q_i. Furthermore, since both $Q_i(\Phi)$ and $D^{\mathcal{W}}(\Phi)$ are Weyl-invariant, their product $D^{\mathcal{W}}(\Phi)Q_i(\Phi)$ is a Weyl-invariant polynomial. So (C.7) in fact says that rows of $\mathbf{P}$ are linearly dependent over the ring of Weyl-invariant polynomials $\mathbb{C}[\mathfrak{t}]^{\mathcal{W}}$. This is the contradiction, which proves that the matrix $\mathbf{P}$ is non-degenerate.

Having proven that $\mathbf{P}$ is non-degenerate, we can solve (3.18):

$$\begin{pmatrix} \widetilde{M}^{\mathrm{w}_1\cdot b} \\ \widetilde{M}^{\mathrm{w}_2\cdot b} \\ \cdot \\ \cdot \\ \cdot \\ \widetilde{M}^{\mathrm{w}_{\dim(\rho^b)}\cdot b} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \mathcal{M}^b \\ \left[P_2\mathcal{M}^b\right] \\ \cdot \\ \cdot \\ \cdot \\ \left[P_{\dim(\rho^b)}\mathcal{M}^b\right] \end{pmatrix}. \tag{C.8}$$

Since the Weyl group simply permutes columns of $\mathbf{P}$, it is enough to have an expression for $\widetilde{M}^b$, while all other $\widetilde{M}^{\mathrm{w}\cdot b}$ are obtained as Weyl images of such an expression. Remembering that the leading term in $\left[P_i(\Phi)\mathcal{M}^b\right]$ is of the form $\sum_{\mathrm{w}\in\mathcal{W}} P_i(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot b}$, we write the solution as:

$$\widetilde{M}^b = M^b + \sum_{i=1}^{\dim(\rho^b)} (\mathbf{P}^{-1})_1{}^i \sum_{|v|<|b|}\sum_{\mathrm{w}\in\mathcal{W}} V_i^{b\to v}(\Phi^{\mathrm{w}})M^{\mathrm{w}\cdot v}. \tag{C.9}$$

We may introduce a notation $Z^{\mathrm{ab}}_{b\to v}(\Phi)$ for the second term, so the solution takes the form of

$$\widetilde{M}^b = M^b + \sum_{|v|<|b|} Z^{\mathrm{ab}}_{b\to v}(\Phi)M^v. \tag{C.10}$$

Here b is a dominant coweight, whereas the sum is taken over *all coweights* whose length is less than that of $|b|$.

D More (Quantized) Chiral Rings

In this section, we use our formalism to compute the quantized chiral rings of some simple theories.

D.1 SQED_N versus $U(1)$ with One Hyper of Charge N

Let us start by providing another example of a two theories that have the same Coulomb branch, but different quantizations. These theories are SQED_N and $U(1)$ gauge theory with a single hyper of charge N (the $\mathbb{Z}_N$ gauge theory of a free hypermultiplet), which we denote by $U(1)+N$. The CBOs $\mathcal{M}^{\pm 1}, \Phi$ in either theory are represented by the shift operators

$$\mathcal{M}_N^1 = \begin{cases} \dfrac{(-1)^N}{r^{N/2}} \left(\dfrac{1-B}{2} + i\sigma\right)^N e^{-\frac{i}{2}\partial_\sigma - \partial_B} & \text{in } \mathrm{SQED}_N, \\ \dfrac{(-1)^N}{r^{N/2}} \left(\dfrac{1-NB}{2} + iN\sigma\right)_N e^{-\frac{i}{2}\partial_\sigma - \partial_B} & \text{in } U(1)+N, \end{cases} \tag{D.1}$$

as well as by

$$\mathcal{M}_N^{-1} = \frac{e^{\frac{i}{2}\partial_\sigma + \partial_B}}{r^{N/2}}, \quad \Phi_N = \frac{1}{r}\left(\sigma + \frac{i}{2}B\right) \times \begin{cases} 1 & \text{in } \mathrm{SQED}_N, \\ N & \text{in } U(1)+N. \end{cases} \tag{D.2}$$

In both theories, let

$$\mathcal{X} = \frac{1}{(4\pi)^{N/2}}\mathcal{M}^{-1}, \quad \mathcal{Y} = \frac{1}{(4\pi)^{N/2}}\mathcal{M}^{1}, \quad \mathcal{Z} = -\frac{i}{4\pi}\Phi. \tag{D.3}$$

Then we compute that

$$\mathcal{X} \star \mathcal{Y} = \begin{cases} \left(\mathcal{Z} + \dfrac{1}{8\pi r}\right)^N & \text{in } \mathrm{SQED}_N, \\ \displaystyle\prod_{k=0}^{N-1} \left(\mathcal{Z} + \frac{2k+1}{8\pi r}\right) & \text{in } U(1)+N\,, \end{cases} \tag{D.4}$$

where multiplication on the RHS is understood to be $\star$. The two theories have identical Coulomb branches ($\mathbb{C}^2/\mathbb{Z}_N$) and chiral ring relations, as can be seen in the $r \to \infty$ limit of (D.4), but different star products.

D.2 $SU(3)$ with $N_f \geq 0$

Let us describe how to determine the chiral ring of the $SU(3)$ gauge theory with $N_f \geq 0$ flavors of fundamental flavors. Our strategy to determine the dressed monopoles with the proper abelianized bubbling coefficients is to start with a $U(3)$ gauge theory, where the minuscule monopoles and their dressed versions do not bubble, then take star products of these minuscule monopoles and descend to $SU(3)$. From the dimension formula (2.27) for monopoles, we see that this theory is good for $N_f \geq 5$. For comparison, the $U(3)$ theory with the same matter content is ugly for $N_f = 5$ and good for $N_f > 5$.

A set of generators for the Coulomb branch chiral ring of the $SU(3)$ theory has been identified in [82]. Choosing convenient Weyl orderings of the GNO charges, it consists of the (dressed) monopoles

$$\Delta = N_f - 4: \qquad \mathcal{M}^{(1,0,-1)} = \mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(1,0,0)}, \tag{D.5}$$

$$\Delta = 2N_f - 6: \qquad \mathcal{M}^{(2,-1,-1)} = \mathcal{M}^{(0,-1,-1)} \star (\mathcal{M}^{(1,0,0)})^2, \tag{D.6}$$

$$\Delta = 2N_f - 6: \qquad \mathcal{M}^{(1,1,-2)} = (\mathcal{M}^{(0,0,-1)})^2 \star \mathcal{M}^{(1,1,0)}, \tag{D.7}$$

$$\Delta = N_f - 1: \qquad (\Phi_1^3 + \Phi_2^3)\mathcal{M}^{(1,0,-1)} = (\Phi_1^3 + \Phi_2^3)\mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(1,0,0)}, \tag{D.8}$$

$$\Delta = 2N_f - 5: \qquad (\Phi_1 + \Phi_2)\mathcal{M}^{(-1,-1,2)} = (\Phi_1 + \Phi_2)\mathcal{M}^{(-1,-1,0)} \star (\mathcal{M}^{(1,0,0)})^2, \tag{D.9}$$

$$\Delta = 2N_f - 4: \qquad (\Phi_1^2 + \Phi_2^2)\mathcal{M}^{(-1,-1,2)} = (\Phi_1^2 + \Phi_2^2)\mathcal{M}^{(-1,-1,0)} \star (\mathcal{M}^{(1,0,0)})^2, \tag{D.10}$$

$$\Delta = 2N_f - 5: \qquad (\Phi_1 + \Phi_2)\mathcal{M}^{(1,1,-2)} = (\Phi_1 + \Phi_2)\mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(1,1,0)}, \tag{D.11}$$

$$\Delta = 2N_f - 4: \qquad (\Phi_1^2 + \Phi_2^2)\mathcal{M}^{(1,1,-2)} = (\Phi_1^2 + \Phi_2^2)\mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(1,1,0)}, \tag{D.12}$$

which are straightforward to write in terms of non-bubbling monopoles of $U(3)$ (as we have done above), the dressed monopoles

$$\Phi_1\mathcal{M}^{(1,0,-1)}, \quad \Phi_2\mathcal{M}^{(1,0,-1)}, \quad \Phi_1^2\mathcal{M}^{(1,0,-1)}, \quad \Phi_2^2\mathcal{M}^{(1,0,-1)}, \tag{D.13}$$

which are less straightforward to construct from elementary dressed monopoles, and the scalars $\operatorname{tr}\Phi^2$ and $\operatorname{tr}\Phi^3$, where we write $\Phi = \operatorname{diag}(\Phi_1, \Phi_2, -(\Phi_1 + \Phi_2))$ in the reduction to $SU(3)$.[22] Constructing the explicit dressings in (D.13) is a delicate procedure, so we elect to

[22] To obtain this generating set, we consider the $N_R = 0$ case of the results in [82]: (8.17) yields the three bare monopole operators, while (8.25), (8.26), (8.27), (8.28), (8.29) yield their dressed versions; $\operatorname{tr}\Phi^2$ and $\operatorname{tr}\Phi^3$ are the relevant Casimir invariants after passing from $U(3)$ to $SU(3)$ (8.20). See Section 8.4.2 for the Hilbert series.

use the generators

$$\begin{aligned}
\Delta = N_f - 3: &\quad g_1^{(1,0,-1)} \equiv \mathcal{M}^{(0,0,-1)} \star \Phi_1 \mathcal{M}^{(1,0,0)} = \Phi_1 \mathcal{M}^{(1,0,-1)} + P_{N_f-3}(\Phi_1, \Phi_2), && \text{(D.14)}\\
\Delta = N_f - 2: &\quad g_2^{(1,0,-1)} \equiv \mathcal{M}^{(0,0,-1)} \star \Phi_1^2 \mathcal{M}^{(1,0,0)} = \Phi_1^2 \mathcal{M}^{(1,0,-1)} + P_{N_f-2}(\Phi_1, \Phi_2), && \text{(D.15)}\\
\Delta = N_f - 3: &\quad (\Phi_1 + \Phi_2)\mathcal{M}^{(1,0,-1)} = (\Phi_1 + \Phi_2)\mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(1,0,0)}, && \text{(D.16)}\\
\Delta = N_f - 2: &\quad (\Phi_1^2 + \Phi_2^2)\mathcal{M}^{(1,0,-1)} = (\Phi_1^2 + \Phi_2^2)\mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(1,0,0)} && \text{(D.17)}
\end{aligned}$$

in their stead: along with the Casimir invariants, they clearly generate (D.13). The degrees d of the unspecified polynomials P_d in (D.14)–(D.17) follow from the dimension formula (2.27). This basis differs slightly from that of [82] and has the benefit that the corresponding shift operators are easier to construct.

To make sense of the formulas above, first recall that in defining dressed monopoles, the Weyl group actions on the GNO charges of the abelian summands and on the dressing factors are opposite to each other. For example, writing

$$\widetilde{M}^{(1,0,-1)} = M^{(1,0,-1)} + C_{(1,0,-1)\to(0,0,0)}(\Phi_1, \Phi_2, \Phi_3), \tag{D.18}$$

we have

$$\begin{aligned}
P(\Phi_1, \Phi_2, \Phi_3)\mathcal{M}^{(1,0,-1)} &= P(\Phi_1, \Phi_2, \Phi_3)\widetilde{M}^{(1,0,-1)} + P(\Phi_1, \Phi_3, \Phi_2)\widetilde{M}^{(1,-1,0)}\\
&\quad + P(\Phi_2, \Phi_3, \Phi_1)\widetilde{M}^{(-1,1,0)} + P(\Phi_2, \Phi_1, \Phi_3)\widetilde{M}^{(0,1,-1)}\\
&\quad + P(\Phi_3, \Phi_1, \Phi_2)\widetilde{M}^{(0,-1,1)} + P(\Phi_3, \Phi_2, \Phi_1)\widetilde{M}^{(-1,0,1)}\,,
\end{aligned} \tag{D.19}$$

where $\mathcal{W} = S_3$ in our case. A given dressed monopole can also be written in a number of different ways, depending on how the components of the GNO charges are Weyl-ordered: for example, $P(\Phi_1)\mathcal{M}^{(1,0,0)}$, $P(\Phi_2)\mathcal{M}^{(0,1,0)}$, and $P(\Phi_3)\mathcal{M}^{(0,0,1)}$ are all equivalent. Starting with (D.5), we can construct $\mathcal{M}^{(1,0,-1)}$ dressed by any polynomial that is symmetric in the first two arguments by writing

$$P(\Phi_1, \Phi_2, \Phi_3)\mathcal{M}^{(0,0,-1)} \star \mathcal{M}^{(1,0,0)} = \frac{P(\Phi_1, \Phi_2, \Phi_3) + P(\Phi_2, \Phi_1, \Phi_3)}{2}\mathcal{M}^{(1,0,-1)}. \tag{D.20}$$

Similarly, for P symmetric in the last two arguments, the dressing of $\mathcal{M}^{(1,0,-1)}$ by P is given by the leading term of

$$\mathcal{M}^{(0,0,-1)} \star P(\Phi_1, \Phi_2, \Phi_3)\mathcal{M}^{(1,0,0)} \tag{D.21}$$

(as P is now acted on by the shift operators in $\mathcal{M}^{(0,0,-1)}$). Starting with (D.6), we find that

$$P(\Phi_1, \Phi_2, \Phi_3)\mathcal{M}^{(0,-1,-1)} \star (\mathcal{M}^{(1,0,0)})^2 = \frac{P(\Phi_1, \Phi_2, \Phi_3) + P(\Phi_1, \Phi_3, \Phi_2)}{2}\mathcal{M}^{(2,-1,-1)}. \quad \text{(D.22)}$$

Unlike in the case of $\mathcal{M}^{(1,0,-1)}$, this procedure for constructing dressings of $\mathcal{M}^{(2,-1,-1)}$ is completely general because $\mathcal{M}^{(2,-1,-1)}$, like $\mathcal{M}^{(0,-1,-1)}$, is sensitive only to the part of the dressing polynomial that is Weyl-symmetric in the last two arguments. Similar statements hold for $\mathcal{M}^{(1,1,-2)}$. Note that in this theory, all non-bubbling monopoles have nontrivial stabilizers under the Weyl group. In particular, dressing the minuscule monopoles in (D.5) individually is not enough to extract the abelianized bubbling coefficient C in (D.18). However, by dressing both $\mathcal{M}^{(0,0,-1)}$ and $\mathcal{M}^{(1,0,0)}$ at the same time, one can in principle extract C itself (we will not need to do so).

Moving on to the chiral ring, we have listed 14 generators ((D.5)–(D.12), (D.14)–(D.17), and the Casimir invariants), while the Coulomb branch has complex dimension 4. Hence we must find at least 10 relations. If the moduli space is not a complete intersection (meaning that the relations could be redundant at generic points but may all be needed to describe the whole variety), then we will find strictly more than 10 relations. In this case, we cannot read off the degrees of the relations and generators from the Hilbert series.[23] By contrast, the moduli space of U with fundamental hypers is known to be a complete intersection [83]. To proceed, one can write all possible relations according to dimension and solve for the coefficients. Rather than presenting an exhaustive analysis, let us simply determine the lowest-dimension relation that relates operators of different GNO charges. Clearly, this relation has $\Delta = 2N_f - 6$. As a further simplification, we work with the commutative limit of the shift operators (in which the order of the multiplications in (D.5)–(D.12) and (D.14)–(D.17) is immaterial), as our interest is in the chiral ring and not its quantization. Then we find that

$$\begin{aligned} 0 &= \mathcal{M}^{(2,-1,-1)} + \mathcal{M}^{(1,1,-2)} \\ &\quad + (g_1^{(1,0,-1)})^2 - g_1^{(1,0,-1)}[(\Phi_1 + \Phi_2)\mathcal{M}^{(1,0,-1)}] + [(\Phi_1 + \Phi_2)\mathcal{M}^{(1,0,-1)}]^2 \\ &\quad - \frac{1}{2}\operatorname{tr}\Phi^2(\mathcal{M}^{(1,0,-1)})^2 + P(\Phi_1, \Phi_2)\mathcal{M}^{(1,0,-1)} \end{aligned} \quad \text{(D.23)}$$

[23]Indeed, from (8.38) of [82], although we see that the difference between the degrees of the denominator and the numerator of the Hilbert series equals the dimension of the moduli space in this case, the degrees appearing in the denominator only include the dimensions of $\operatorname{tr}\Phi^2$, $\operatorname{tr}\Phi^3$, and the bare monopoles $(1, 0, -1)$, $(1, 1, -2)$, $(2, -1, -1)$; the dimensions of the dressed monopoles are missing.

in the chiral ring, where P is a polynomial of degree $N_f - 2$, expressible in terms of the Casimirs, which we do not write explicitly (note that P necessarily vanishes for $N_f = 0, 1$).

E Correlation Functions

E.1 Mirror Symmetry check for an $\mathcal{N} = 8$ SCFT

In this section we give more details on the derivation the mirror maps (5.47)–(5.52) given in the main text.

When thinking of an $\mathcal{N} = 8$ SCFT in $\mathcal{N} = 4$ language, it is useful to think about the embedding of the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{osp}(4|4)$ into the $\mathcal{N} = 8$ superconformal algebra $\mathfrak{osp}(8|4)$. Consequently, we have [35][24]

$$\mathfrak{so}(8)_R \supset \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2; \tag{E.1}$$

where $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ is the $\mathcal{N} = 4$ R-symmetry and $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ is a flavor symmetry from the $\mathcal{N} = 4$ point of view. For 3D $\mathcal{N} = 8$ theories, the 1D topological theory has an $\mathfrak{su}(2)_F$ symmetry [35], which can be taken to be $\mathfrak{su}(2)_1$ for the Higgs branch TQFT and $\mathfrak{su}(2)_2$ for the Coulomb branch TQFT. To write correlation functions concisely, it is convenient to organize operators in the 1D theory into representations of this $\mathfrak{su}(2)_F$ symmetry and contract their $\mathfrak{su}(2)_F$ indices with $\mathfrak{su}(2)_F$ commuting polarization vectors z^i, $i = 1, 2$, transforming under $\mathfrak{su}(2)_F$ as a doublet. For an operator $\mathcal{O}_{i_1 \cdots i_{2j}}(\varphi)$ transforming in the spin-j representation of $\mathfrak{su}(2)_F$, we define

$$\mathcal{O}(\varphi, z) \equiv \mathcal{O}_{i_1 \ldots i_{2j}}(\varphi) z^{i_1} \cdots z^{i_{2j}} \,. \tag{E.2}$$

We label operators in the 1D theory by $\mathcal{O}^{(\Delta,j)}(\varphi, z)$ and subscripts H, C depending on whether they belong to the Higgs or Coulomb branch TQFT, respectively. The label Δ corresponds to the scaling dimension of the 3D operator that $\mathcal{O}^{(\Delta,j)}$ originates from, and j is the $\mathfrak{su}(2)_F$ spin. We can normalize the two-point functions to be

$$\langle \mathcal{O}^{(\Delta,j)}(\varphi_1, z_1) \mathcal{O}^{(\Delta,j)}(\varphi_2, z_2) \rangle = \langle z_1, z_2 \rangle^{2j} (\operatorname{sign} \varphi_{12})^{2\Delta} \,, \tag{E.3}$$

[24]The subscripts are also called H, C, F, F', respectively, in [84].

with all other two-point functions vanishing, and with where we defined the $\mathfrak{su}(2)_F$ invariant

$$\langle z_A, z_B\rangle \equiv \epsilon_{ij} z_A^i z_B^j\,, \qquad \epsilon^{12} = -\epsilon_{12} = 1\,. \tag{E.4}$$

denoting the $\mathfrak{su}(2)_F$ singlet that can be formed from two polarizations z_A and z_B.

With the normalization (E.3), three-point functions are fixed by the $\mathfrak{su}(2)_F$ symmetry as follows: for spins j_1, j_2, j_3 satisfying the triangle inequality,

$$\begin{aligned}\langle \mathcal{O}^{(\Delta_1,j_1)}(\varphi_1,z_1)\mathcal{O}^{(\Delta_2,j_2)}(\varphi_2,z_2)\mathcal{O}^{(\Delta_3,j_3)}(\varphi_3,z_3)\rangle &= \lambda_{(\Delta_1,j_1),(\Delta_2,j_2),(\Delta_3,j_3)}\\ \times \langle z_1,z_2\rangle^{j_{123}}\langle z_2,z_3\rangle^{j_{231}}\langle z_3,z_1\rangle^{j_{312}}&(\operatorname{sign}\varphi_{12})^{\Delta_{123}}(\operatorname{sign}\varphi_{23})^{\Delta_{231}}(\operatorname{sign}\varphi_{31})^{\Delta_{312}}\,,\end{aligned} \tag{E.5}$$

where $j_{abc} \equiv j_a + j_b - j_c$ (the correlator vanishes otherwise). The sign factors are fixed by conformal symmetry, while $\mathfrak{su}(2)_F$ symmetry requires that the polarizations appear as they do by power counting.

E.1.1 $U(2)$

The 1D Higgs branch theory $U(2)$ in the case $N_c = 2$ was partially analyzed in [37]. For $j = 1/2$ we have

$$\widetilde{\mathcal{O}}_H^{(1/2,1/2)}(\varphi,z) = z^1 \operatorname{tr} Q(\varphi) + z^2 \operatorname{tr}\tilde{Q}(\varphi)\,, \tag{E.6}$$

for $j = 1$, we have two distinct operators

$$\begin{aligned}\widetilde{\mathcal{O}}_{H,1}^{(1,1)}(\varphi,z) &= (z^1)^2(\operatorname{tr} Q)^2(\varphi) + (z^2)^2(\operatorname{tr}\tilde{Q})^2(\varphi) + 2z^1z^2\operatorname{tr} Q\operatorname{tr}\tilde{Q}(\varphi),\\ \widetilde{\mathcal{O}}_{H,2}^{(1,1)}(\varphi,z) &= (z^1)^2\operatorname{tr} Q^2(\varphi) + (z^2)^2\operatorname{tr}\tilde{Q}^2(\varphi) + 2z^1z^2\operatorname{tr} Q\tilde{Q}(\varphi)\,,\end{aligned} \tag{E.7}$$

etc., where the tildes mean that the operators don't necessarily obey (E.3). To find which linear combinations obey (E.3), we should compute the two-point functions using Eq. (5.45) in the main text. Performing the Wick contractions in the theory at fixed σ using (5.44), we have

$$
\begin{aligned}
\langle \operatorname{tr} Q(\varphi_1) \operatorname{tr} \tilde{Q}(\varphi_2) \rangle_\sigma &= -\frac{\operatorname{sign} \varphi_{12}}{4\pi r} \\
\langle (\operatorname{tr} Q)^2(\varphi_1) (\operatorname{tr} \tilde{Q})^2(\varphi_2) \rangle_\sigma &= -2 \langle \operatorname{tr} Q \operatorname{tr} \tilde{Q}(\varphi_1) \operatorname{tr} Q \operatorname{tr} \tilde{Q}(\varphi_2) \rangle_\sigma = \frac{1}{8\pi^2 r^2} \\
\langle \operatorname{tr} Q^2(\varphi_1) \operatorname{tr} \tilde{Q}^2(\varphi_2) \rangle_\sigma &= -2 \langle \operatorname{tr} Q\tilde{Q}(\varphi_1) \operatorname{tr} Q\tilde{Q}(\varphi_2) \rangle_\sigma = \frac{1}{16\pi^2 r^2} \left[1 + \frac{1}{\cosh^2(\pi\sigma_{12})} \right] \\
\langle (\operatorname{tr} Q)^2(\varphi_1) \operatorname{tr} \tilde{Q}^2(\varphi_2) \rangle_\sigma &= \langle \operatorname{tr} Q^2(\varphi_1) (\operatorname{tr} \tilde{Q})^2(\varphi_2) \rangle_\sigma = -2 \langle \operatorname{tr} Q \operatorname{tr} \tilde{Q}(\varphi_1) \operatorname{tr} Q\tilde{Q}(\varphi_2) \rangle_\sigma \\
&= \frac{1}{16\pi^2 r^2} \,.
\end{aligned} \tag{E.8}
$$

Plugging these expressions into (5.45), we obtain that the operators[25]

$$
\begin{aligned}
\mathcal{O}_H^{(1/2,1/2)}(\varphi, z) &= \sqrt{4\pi r}\, \widetilde{\mathcal{O}}_{H,1}^{(1,1)}(\varphi, z), \\
\mathcal{O}_{H,\text{free}}^{(1,1)}(\varphi, z) &= \sqrt{8\pi^2 r^2}\, \widetilde{\mathcal{O}}_{H,1}^{(1,1)}(\varphi, z), \\
\mathcal{O}_{H,\text{int}}^{(1,1)}(\varphi, z) &= \sqrt{24\pi^2 r^2} \left[\widetilde{\mathcal{O}}_{H,2}^{(1,1)}(\varphi, z) - \frac{1}{2} \widetilde{\mathcal{O}}_{H,1}^{(1,1)}(\varphi, z) \right]
\end{aligned} \tag{E.9}
$$

obey (E.3). The subscript "free" or "int" stands for the fact that this theory flows to a product between a free sector and an interacting sector. A similar calculation for the 3-point functions gives

$$
\begin{aligned}
\langle \mathcal{O}_H^{(1/2,1/2)}(\varphi_1, z_1) \mathcal{O}_H^{(1/2,1/2)}(\varphi_2, z_2) \mathcal{O}_{H,\text{free}}^{(1,1)}(\varphi_3, z_3) \rangle &= \sqrt{2} \langle z_2, z_3 \rangle \langle z_3, z_1 \rangle \operatorname{sign}(\varphi_{23}\varphi_{31}) \,, \\
\langle \mathcal{O}_H^{(1/2,1/2)}(\varphi_1, z_1) \mathcal{O}_H^{(1/2,1/2)}(\varphi_2, z_2) \mathcal{O}_{H,\text{int}}^{(1,1)}(\varphi_3, z_3) \rangle &= 0 \,, \\
\langle \mathcal{O}_{H,\text{free}}^{(1,1)}(\varphi_1, z_1) \mathcal{O}_{H,\text{free}}^{(1,1)}(\varphi_2, z_2) \mathcal{O}_{H,\text{free}}^{(1,1)}(\varphi_3, z_3) \rangle &= (\sqrt{2})^3 \langle z_1, z_2 \rangle \langle z_2, z_3 \rangle \langle z_3, z_1 \operatorname{sign}(\varphi_{12}\varphi_{23}\varphi_{31}), \\
\langle \mathcal{O}_{H,\text{int}}^{(1,1)}(\varphi_1, z_1) \mathcal{O}_{H,\text{int}}^{(1,1)}(\varphi_2, z_2) \mathcal{O}_{H,\text{int}}^{(1,1)}(\varphi_3, z_3) \rangle &= \sqrt{6} \langle z_1, z_2 \rangle \langle z_2, z_3 \rangle \langle z_3, z_1 \rangle \operatorname{sign}(\varphi_{12}\varphi_{23}\varphi_{31}) \,.
\end{aligned} \tag{E.10}
$$

On the Coulomb branch side, the monopole operators $\mathcal{M}^{(\pm 1,0)}$ and $\mathcal{M}^{(1,1)}$ don't bubble, so for them $\mathcal{M}$ equals M averaged over the $\mathbb{Z}_2$ Weyl group. With M given in the North

[25] Note that $Z_{S^3} = \frac{1}{32} \int d\sigma_1\, d\sigma_2\, \frac{\sinh^2(\pi\sigma_{12})}{\cosh^2(\pi\sigma_{12}) \cosh(\pi\sigma_1) \cosh(\pi\sigma_2)} = \frac{1}{16\pi}$, and if we denote Z_{S^3} with an extra insertion of $f(\sigma_1, \sigma_2)$ in the integrand by $Z_{S^3}[f(\sigma_1, \sigma_2)]$, we have

$$
Z_{S^3}[\sigma_1\sigma_2] = -\frac{1}{384\pi}, \quad Z_{S^3}[\sigma_1^2 + \sigma_2^2] = \frac{1}{24\pi}, \quad Z_{S^3}\left[\frac{1}{\cosh^2(\pi\sigma_{12})} \right] = \frac{1}{96\pi}.
$$

picture in (5.38), we find, for instance

$$
\begin{aligned}
\frac{1}{4\pi}\langle \mathcal{M}^{(-1,0)}(\varphi_1)\mathcal{M}^{(1,0)}(\varphi_2)\rangle &= -\frac{\operatorname{sign}\varphi_{12}}{4\pi r}\,, \\
\frac{1}{4\pi^2}\langle \mathcal{M}^{(-1,-1)}(\varphi_1)\mathcal{M}^{(1,1)}(\varphi_2)\rangle &= -2\left(-\frac{i}{4\pi}\right)^2 \langle \operatorname{tr}\Phi(\varphi_1)\operatorname{tr}\Phi(\varphi_2)\rangle = \frac{7}{96\pi^2 r^2}\,, \\
\left(\frac{1}{4\pi}\right)^2 \langle \mathcal{M}^{(-2,0)}(\varphi_1)\mathcal{M}^{(2,0)}(\varphi_2)\rangle &= -2\left(\frac{1}{4\pi}\right)^2 \left\langle \left(\mathcal{M}^{(1,-1)} - \frac{1}{r}\right)(\varphi_1)\left(\mathcal{M}^{(1,-1)} - \frac{1}{r}\right)(\varphi_2)\right\rangle \\
&= \frac{1}{8\pi^2 r^2}\,, \\
\left(\frac{1}{4\pi}\right)\left(\frac{1}{2\pi}\right)\langle \mathcal{M}^{(-2,0)}(\varphi_1)\mathcal{M}^{(1,1)}(\varphi_2)\rangle &= -2\left(\frac{1}{4\pi}\right)\left(-\frac{i}{4\pi}\right)\left\langle \left(\mathcal{M}^{(1,-1)} - \frac{1}{r}\right)(\varphi_1)\operatorname{tr}\Phi(\varphi_2)\right\rangle \\
&= \left(\frac{1}{2\pi}\right)\left(\frac{1}{4\pi}\right)\langle \mathcal{M}^{(-1,-1)}(\varphi_1)\mathcal{M}^{(2,0)}(\varphi_2)\rangle = \frac{1}{16\pi^2 r^2}
\end{aligned}
\tag{E.11}
$$

etc. It is straightforward to check that these correlation functions, as well as the various three-point functions, agree precisely with those of the 1D Higgs branch operators given in (5.47)–(5.51), as extracted from the two-point functions of (E.9) and the three-point functions (E.10). This provides a derivation of (5.47)–(5.51).

There is an alternative method of deriving the mirror map. We first record some useful star products of CBOs. The free sector is generated by the $\Delta = 1/2$ operators $\mathcal{M}^{(\pm1,0)}$, whose quadratic star products generate the $\Delta = 1$ operators $\mathcal{M}^{(\pm2,0)}$ and $\mathcal{M}^{(1,-1)}$:

$$(\mathcal{M}^{(\pm1,0)})^2 = \mathcal{M}^{(\pm2,0)}, \tag{E.12}$$

$$\mathcal{M}^{(-1,0)} \star \mathcal{M}^{(1,0)} = \mathcal{M}^{(1,0)} \star \mathcal{M}^{(-1,0)} + \frac{2}{r} = \mathcal{M}^{(1,-1)}. \tag{E.13}$$

On the other hand, the $\Delta = 1$ operators $\operatorname{tr}\Phi$ and $\mathcal{M}^{\pm(1,1)}$ satisfy the quadratic relations

$$\mathcal{M}^{\pm(1,1)} \star \mathcal{M}^{\pm(1,1)} = \mathcal{M}^{\pm(2,2)}, \tag{E.14}$$

$$\mathcal{M}^{\pm(1,1)} \star \mathcal{M}^{\mp(1,1)} = \left(\frac{1}{2r} \pm i\Phi_1\right)\left(\frac{1}{2r} \pm i\Phi_2\right) = \frac{1}{4r^2} \pm \frac{i}{2r}\operatorname{tr}\Phi + \frac{1}{2}[\operatorname{tr}\Phi^2 - (\operatorname{tr}\Phi)^2], \tag{E.15}$$

$$[\operatorname{tr}\Phi, \mathcal{M}^{\pm(1,1)}]_\star = \pm\frac{2i}{r}\mathcal{M}^{\pm(1,1)}, \tag{E.16}$$

where we have defined the commutator $[\cdot,\cdot]_\star$ with respect to the star product. In the mixed sector, we have the relations

$$\mathcal{M}^{\pm(1,1)} \star \mathcal{M}^{(\pm1,0)} = \mathcal{M}^{(\pm1,0)} \star \mathcal{M}^{\pm(1,1)} = \mathcal{M}^{\pm(2,1)}, \tag{E.17}$$

$$\mathcal{M}^{\pm(1,1)} \star \mathcal{M}^{(\mp1,0)} \pm \frac{1}{2r}\mathcal{M}^{(\pm1,0)} = -i[\Phi_2 \mathcal{M}^{(\pm1,0)}], \tag{E.18}$$

$$\mathcal{M}^{(\mp1,0)} \star \mathcal{M}^{\pm(1,1)} \mp \frac{1}{2r}\mathcal{M}^{(\pm1,0)} = -i[\Phi_2 \mathcal{M}^{(\pm1,0)}], \tag{E.19}$$

as well as the miscellaneous relations

$$\mathcal{M}^{(1,-1)} \star \mathcal{M}^{(1,0)} = \mathcal{M}^{(1,0)} \star \mathcal{M}^{(1,-1)} + \frac{2}{r}\mathcal{M}^{(1,0)} = \mathcal{M}^{(2,-1)}, \tag{E.20}$$

$$\mathcal{M}^{(2,0)} \star \mathcal{M}^{(1,0)} = (\mathcal{M}^{(1,0)})^3 = \mathcal{M}^{(3,0)}, \tag{E.21}$$

$$\mathcal{M}^{(1,-1)} \star \mathcal{M}^{(1,-1)} = \mathcal{M}^{(2,-2)} - \frac{2}{r}\mathcal{M}^{(1,-1)}. \tag{E.22}$$

Given the above, we may view the "basic" operators on the Coulomb branch side as $\mathcal{M}^{(\pm1,0)}$ and $\mathcal{M}^{\pm(1,1)}$, for which we have already justified the mirror map in (5.47) and (5.49). The mirror map (5.48) for $\mathcal{M}^{(\pm2,0)}$ then follows from $\operatorname{tr} Q \star \operatorname{tr} Q = (\operatorname{tr} Q)^2$ and (E.12). Next, using Wick contractions to define composite operators, we compute on the Higgs branch side that

$$\operatorname{tr} Q \star \operatorname{tr} \tilde{Q} + \frac{N_c}{8\pi r} = \operatorname{tr} \tilde{Q} \star \operatorname{tr} Q - \frac{N_c}{8\pi r} = \operatorname{tr} Q \operatorname{tr} \tilde{Q} \tag{E.23}$$

for arbitrary N_c, which, in light of (E.13), is consistent with the identification (5.50) for $\mathcal{M}^{(1,-1)}$ when $N_c = 2$. Finally, on the Higgs branch side, we find that

$$\operatorname{tr} Q^2 \star \operatorname{tr} \tilde{Q}^2 + \frac{1}{2\pi r} \operatorname{tr} Q\tilde{Q} = \operatorname{tr} \tilde{Q}^2 \star \operatorname{tr} Q^2 - \frac{1}{2\pi r} \operatorname{tr} Q\tilde{Q} = \operatorname{tr} Q^2 \operatorname{tr} \tilde{Q}^2 + \frac{1}{8\pi^2 r^2}. \tag{E.24}$$

Using (E.15) and (5.49), we deduce the mirror map (5.51) for $\operatorname{tr}\Phi$ as well as

$$\frac{1}{8\pi^2}\left[\operatorname{tr}\Phi^2 - (\operatorname{tr}\Phi)^2 - \frac{1}{2r^2}\right] \leftrightarrow \operatorname{tr} Q^2 \operatorname{tr} \tilde{Q}^2. \tag{E.25}$$

Using (E.25) and the mirror map (5.51), we can further identify what $\operatorname{tr}\Phi^2$ corresponds to: on the Higgs branch side, we compute that

$$\operatorname{tr} Q\tilde{Q} \star \operatorname{tr} Q\tilde{Q} = (\operatorname{tr} Q\tilde{Q})^2 - \frac{1}{16\pi^2 r^2}, \tag{E.26}$$

so in light of $\operatorname{tr}\Phi \star \operatorname{tr}\Phi = (\operatorname{tr}\Phi)^2$ and (E.25), we get that

$$\frac{1}{8\pi^2}\left(\operatorname{tr}\Phi^2 - \frac{3}{2r^2}\right) \leftrightarrow \operatorname{tr}Q^2 \operatorname{tr}\tilde{Q}^2 - 2(\operatorname{tr}Q\tilde{Q})^2. \tag{E.27}$$

One can make further consistency checks of the identifications that we have derived by matching one-point functions of these composite operators. As a consistency check of (5.50), we see from (E.23) that $\langle \operatorname{tr}Q \operatorname{tr}\tilde{Q}\rangle = 0$ for any N_c, so we expect that $\langle \mathcal{M}^{(1,-1)}\rangle = \frac{1}{r}$, which is indeed the case. As a consistency check of (5.51), we have that $\langle \operatorname{tr}Q\tilde{Q}\rangle = 0$, which is consistent with

$$\langle \operatorname{tr}\Phi\rangle = 0, \quad \langle \operatorname{tr}\Phi^2\rangle = \frac{Z_{S^3}[\sigma_1^2 + \sigma_2^2]}{r^2 Z_{S^3}} = \frac{2}{3r^2}, \quad \langle (\operatorname{tr}\Phi)^2\rangle = \frac{Z_{S^3}[(\sigma_1+\sigma_2)^2]}{r^2 Z_{S^3}} = \frac{7}{12r^2}, \tag{E.28}$$

(as follows from (25)) where the $Z_{S^3}[f[\sigma_1, \sigma_2]]$ notation is the same as in Footnote 25. As a consistency check of (E.27), we may use (E.24) and (E.26) to rewrite (E.27) in terms of star products of elementary operators:

$$\frac{1}{8\pi^2}\left(\operatorname{tr}\Phi^2 + \frac{1}{2r^2}\right) = \operatorname{tr}Q^2 \star \operatorname{tr}\tilde{Q}^2 - 2\operatorname{tr}Q\tilde{Q} \star \operatorname{tr}Q\tilde{Q} + \frac{1}{2\pi r}\operatorname{tr}Q\tilde{Q}. \tag{E.29}$$

Taking the expectation value of both sides of (E.29) results in $\langle \operatorname{tr}\Phi^2\rangle = 2/3r^2$, precisely as expected from (E.28).

E.1.2 $U(N_c)$ with $N_a = N_f = 1$

We do not study the case $N_c > 2$ in detail, but let us check that the free sector agrees between the Higgs and Coulomb branch sides. The free sector of $U(N_c)$ with $N_a = N_f = 1$ can be analyzed in the same way for all N_c (compare to the analysis of $U(3)$ with $N_a = N_f = 1$ in [84]). First consider the Higgs branch side. Letting tildes denote unnormalized operators, we set

$$\widetilde{\mathcal{O}}^{(1/2,1/2)}_{H,\text{free}}(\varphi, z) = z^1 \operatorname{tr}Q(\varphi) + z^2 \operatorname{tr}\tilde{Q}(\varphi), \tag{E.30}$$

so that all operators in the free sector of the 1D theory are simply powers of this operator:

$$\widetilde{\mathcal{O}}^{(j,j)}_{H,\text{free}}(\varphi, z) = [\widetilde{\mathcal{O}}^{(1/2,1/2)}_{H,\text{free}}(\varphi, z)]^{2j}. \tag{E.31}$$

The basic result is

$$\langle (\operatorname{tr} Q)^m(\varphi_1)(\operatorname{tr}\tilde{Q})^m(\varphi_2)\rangle = \langle (\operatorname{tr} Q)^m(\varphi_1)(\operatorname{tr}\tilde{Q})^m(\varphi_2)\rangle_\sigma = m!\left(-\frac{N_c \operatorname{sign}\varphi_{12}}{8\pi r}\right), \tag{E.32}$$

by counting $m!$ equivalent contractions. We compute the two-point functions

$$\langle \widetilde{\mathcal{O}}^{(j,j)}_{H,\text{free}}(\varphi_1, z_1)\widetilde{\mathcal{O}}^{(j,j)}_{H,\text{free}}(\varphi_2, z_2)\rangle = (2j)!\left(\frac{N_c}{8\pi r}\right)^{2j}\langle z_1, z_2\rangle^{2j}(\operatorname{sign}\varphi_{12})^{2j}.$$

In terms of the normalized operators

$$\mathcal{O}^{(j,j)}_{H,\text{free}}(\varphi, z) = \frac{1}{\sqrt{(2j)!}}\left(\frac{8\pi r}{N_c}\right)^j \widetilde{\mathcal{O}}^{(j,j)}_{H,\text{free}}(\varphi, z), \tag{E.33}$$

we then compute the three-point functions

$$\begin{aligned}&\langle \mathcal{O}^{(j_1,j_1)}_{H,\text{free}}(\varphi_1, z_1)\mathcal{O}^{(j_2,j_2)}_{H,\text{free}}(\varphi_2, z_2)\mathcal{O}^{(j_3,j_3)}_{H,\text{free}}(\varphi_3, z_3)\rangle\\ &= \lambda^{\text{free}}_{(j_1,j_1),(j_2,j_2),(j_3,j_3)}\langle z_1, z_2\rangle^{j_{123}}\langle z_2, z_3\rangle^{j_{231}}\langle z_3, z_1\rangle^{j_{312}}(\operatorname{sign}\varphi_{12})^{j_{123}}(\operatorname{sign}\varphi_{23})^{j_{231}}(\operatorname{sign}\varphi_{31})^{j_{312}}\end{aligned} \tag{E.34}$$

for j_1, j_2, j_3 satisfying the triangle inequality, where

$$\lambda^{\text{free}}_{(j_1,j_1),(j_2,j_2),(j_3,j_3)} = \frac{j_{123}!j_{231}!j_{312}!}{\sqrt{(2j_1)!(2j_2)!(2j_3)!}}\binom{2j_1}{j_{123}}\binom{2j_2}{j_{231}}\binom{2j_3}{j_{312}}. \tag{E.35}$$

We claim that the corresponding operators on the Coulomb branch side are given by

$$\mathcal{O}^{(1/2,1/2)}_{C,\text{free}}(\varphi, z) = \sqrt{\frac{2r}{N_c}}(z^1\mathcal{M}^{(-1,\vec{0})}(\varphi) + z^2\mathcal{M}^{(1,\vec{0})}(\varphi)). \tag{E.36}$$

To see this, one can match two-point functions. The shift operators are

$$\mathcal{M}^{(-1,\vec{0})}_N = \frac{1}{r^{1/2}}\sum_{I=1}^{N_c}\frac{\prod_{J\neq I}(\frac{1+B_{IJ}}{2} - i\sigma_{IJ})}{\prod_{J\neq I}(-i\sigma_{IJ} + \frac{B_{IJ}}{2})}e^{\frac{i}{2}\partial_{\sigma_I}+\partial_{B_I}}, \tag{E.37}$$

$$\mathcal{M}^{(1,\vec{0})}_N = -\frac{1}{r^{1/2}}\sum_{I=1}^{N_c}\frac{(\frac{1-B_I}{2} + i\sigma_I)\prod_{J\neq I}(\frac{1-B_{IJ}}{2} + i\sigma_{IJ})}{\prod_{J\neq I}(i\sigma_{IJ} - \frac{B_{IJ}}{2})}e^{-\frac{i}{2}\partial_{\sigma_I}-\partial_{B_I}}, \tag{E.38}$$

from which we obtain

$$\langle \mathcal{M}^{(\mp 1,\vec{0})}(\varphi_1)\mathcal{M}^{(\pm 1,\vec{0})}(\varphi_2)\rangle|_{\varphi_1<\varphi_2} = \frac{Z_{S^3}[\mathcal{I}_\pm]}{Z_{S^3}}, \quad \mathcal{I}_\pm \equiv \pm\frac{1}{r}\sum_{I=1}^{N_c} \frac{(\frac{1}{2}+i\sigma_I)\prod_{J\neq I}(\frac{1}{2}+i\sigma_{IJ})^2}{\prod_{J\neq I}(i\sigma_{IJ})(1+i\sigma_{IJ})}.$$

Since the integrand of Z_{S^3} without insertions is invariant under $\sigma_I \leftrightarrow -\sigma_I$, inserting $\mathcal{I}_\pm$ is equivalent to inserting

$$\frac{\mathcal{I}_\pm(\sigma_1,\ldots,\sigma_{N_c}) + \mathcal{I}_\pm(-\sigma_1,\ldots,-\sigma_{N_c})}{2} = \pm\frac{N_c}{2r}\,. \tag{E.39}$$

It follows that

$$\langle \mathcal{M}^{(-1,\vec{0})}(\varphi_1)\mathcal{M}^{(1,\vec{0})}(\varphi_2)\rangle = -\frac{N_c \operatorname{sign}\varphi_{12}}{2r}\,, \tag{E.40}$$

thus substantiating the stated map.

E.2 Another Mirror Symmetry Example

Let us end with a simpler example where we can check mirror symmetry. We have that SQCD with gauge group $SU(2)$ and three fundamental hypers is dual to $U(1)$ SQED with four charged hypers, because both theories are mirror dual to the $U(1)^4$ necklace quiver gauge theory [76]. Their Coulomb branch is given by $\mathbb{C}^2/\mathbb{Z}_4$: it has three holomorphic generators $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ subject to the chiral ring relation $\mathcal{X}\mathcal{Y} = \mathcal{Z}^4$, whose quantization is $\mathcal{X}\star\mathcal{Y} = (\mathcal{Z}^4)_\star + O(1/r)$. The generators have dimensions $\Delta_\mathcal{Z} = 1$ and $\Delta_\mathcal{X} = \Delta_\mathcal{Y} = 2$. Let us identify $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ in the SQCD theory.

To compute correlation functions, we use that the vacuum wavefunction (5.36) is

$$\Psi_0(\sigma, B) = \delta_{B,0}\frac{[\frac{1}{2\pi}\Gamma(\frac{1}{2}-i\sigma)\Gamma(\frac{1}{2}+i\sigma)]^3}{\frac{1}{2\pi}\Gamma(1-2i\sigma)\Gamma(1+2i\sigma)} = \delta_{B,0}\frac{\sinh(\pi\sigma)}{4\sigma\cosh^2(\pi\sigma)} \tag{E.41}$$

and the gluing measure is

$$\mu(\sigma, B) = (-1)^{3|B|}(4\sigma^2 + B^2). \tag{E.42}$$

Using $|\mathcal{W}| = 2$, this gives the S^3 partition function

$$Z = \frac{1}{2}\int d\sigma\, \mu(\sigma,0)\Psi_0(\sigma,0)^2 = \frac{1}{12\pi}, \tag{E.43}$$

in agreement with the S^3 partition function of the four-node quiver theory and SQED with four flavors (see, e.g., [38]).

The Coulomb branch chiral ring operators are gauge-invariant products of Φ and GNO monopole operators with $b \in \mathbb{Z}$. The smallest-dimension such operator is the GNO monopole $\mathcal{M}^{(1,-1)}$. This operator has $\Delta = 1$, so it should correspond to $\mathcal{Z}$ in the four-node quiver theory. Matching the normalization of the two-point function gives

$$\mathcal{Z} = \frac{1}{4\pi}\mathcal{M}^{(1,-1)}. \tag{E.44}$$

There are three operators with $\Delta = 2$: $\mathcal{M}^{(1,-1)} \star \mathcal{M}^{(1,-1)}$, $\operatorname{tr}\Phi^2$, and the dressed monopole $\Phi\mathcal{M}^{(1,-1)}$. Clearly, $\mathcal{M}^{(1,-1)} \star \mathcal{M}^{(1,-1)} = (4\pi)^2 \mathcal{Z} \star \mathcal{Z}$, so we expect to obtain $\mathcal{X}$ and $\mathcal{Y}$ as linear combinations of $\operatorname{tr}\Phi^2$ and $\Phi\mathcal{M}^{(1,-1)}$. We find that

$$\mathcal{X} = \frac{1}{64\pi^2}\left(\operatorname{tr}\Phi^2 - 4\mathcal{M}^{(1,-1)} \star \mathcal{M}^{(1,-1)} - \frac{1}{2r^2} + 4i\left(\Phi\mathcal{M}^{(1,-1)} - \frac{i}{2r}\mathcal{M}^{(1,-1)}\right)\right), \tag{E.45}$$

$$\mathcal{Y} = \frac{1}{64\pi^2}\left(\operatorname{tr}\Phi^2 - 4\mathcal{M}^{(1,-1)} \star \mathcal{M}^{(1,-1)} - \frac{1}{2r^2} - 4i\left(\Phi\mathcal{M}^{(1,-1)} - \frac{i}{2r}\mathcal{M}^{(1,-1)}\right)\right) \tag{E.46}$$

obey the following relations:

$$[\mathcal{X}, \mathcal{Z}]_\star = \frac{1}{4\pi r}\mathcal{X}, \quad [\mathcal{Y}, \mathcal{Z}]_\star = -\frac{1}{4\pi r}\mathcal{Y}, \quad \mathcal{X} \star \mathcal{Y} = \left(\mathcal{Z} + \frac{1}{8\pi r}\right)^4_\star. \tag{E.47}$$

These are precisely the relations obeyed in the four-node quiver theory. In addition, one can check that $\langle\mathcal{X}\rangle = \langle\mathcal{Y}\rangle = \langle\mathcal{Z}\rangle = 0$, just as in the four-node quiver theory. The last relation in (E.47) shows that the Coulomb branch is indeed $\mathbb{C}^2/\mathbb{Z}_4$.

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