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The general (2,2) gauged sigma model with three-form flux

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Abstract: We find the conditions under which a Riemannian manifold equipped with a closed three-form and a vector field define an on-shell $\mathcal{N} = (2,2)$ supersymmetric gauged sigma model. The conditions are that the manifold admits a twisted generalized Kähler structure, that the vector field preserves this structure, and that a so-called generalized moment map exists for it. By a theorem in generalized complex geometry, these conditions imply that the quotient is again a twisted generalized Kähler manifold; this is in perfect agreement with expectations from the renormalization group flow. This method can produce new $\mathcal{N} = (2,2)$ models with NS flux, extending the usual Kähler quotient construction based on Kähler gauged sigma models.

Keywords: Superstring Vacua, Extended Supersymmetry, Differential and Algebraic Geometry, Flux compactifications.

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1. Introduction

The geometry of flux backgrounds in string theory has recently become clearer. For example, the geometry underlying the most general (2,2) nonlinear sigma model has been long described as "bihermitian" [1]. It describes what happens when the complex structures $I_\pm$ felt by the left- and right-movers are allowed to be different, and, crucially, when a Neveu-Schwarz (NS) three-form H is introduced. For $I_+ = I_-$ and $H = 0$ the manifold will be Kähler as familiar; but in general, for $H \neq 0$, the manifold will not be Kähler with respect to either complex structure. This leads to some loss of computational power.

Twenty years after its definition, however, bihermitian geometry has been reinterpreted as generalized Kähler geometry [2].[1] Among the applications of the new approach are new constructions of (2,2) sigma models and an expression for the off-shell action (extending the usual $\int d^4\theta K$ for the Kähler case) in the case in which I_+ and I_- do not commute [3]. It has also been applied to topological sigma models (for example [4 − 6]) and to $\mathcal{N} = 2$ NS vacua in supergravity [7]. The broader field of generalized *complex* geometry [8, 2]

[1] What we call generalized Kähler geometry in this paper is called twisted generalized Kähler geometry in [2].

has also led to a similar classification of Ramond-Ramond $\mathcal{N}=1$ vacua [9] and to related developments concerning brane calibrations [10, 11].

Most of these developments are formal; it would be nice to have tools to produce explicit examples of flux compactifications. In the case without fluxes one such a tool is the gauged linear sigma model [12], which leads to the so-called Kähler quotient construction — a particular case of hamiltonian reduction [13].

It is the aim of this paper to generalize this useful tool to the case with H-flux. Specifically, we work out the conditions for $(2,2)$ supersymmetry of the general gauged sigma model.

As we have mentioned, the most general (2,2) ungauged sigma model was written in [1]. However, unless the two complex structures commute, the off-shell formulation of this model requires the introduction of complicated semi-chiral multiplets. For this reason, the (2,2) gauged sigma model was analyzed in [14] only in the case $[I_+, I_-] = 0$, and the general case was left alone.

While at that time this omission was justifiably perceived as that of a pathological case, new developments have changed the situation somewhat. As an example, in the more general setting of Ramond-Ramond (RR) vacua, cases with non-commuting complex structures are relevant for example for supergravity duals to superconformal theories (as recently demonstrated for example in [16] for the beta-deformation in [17] and pointed out on general grounds in [18] and [11]). One can expect the generalized Kähler case to provide a stepping stone towards finding such solutions [19], much as the process of adding branes on the tip of a conical Calabi-Yau gives Sasaki-Einstein gravity duals.

We perform our analysis on shell, even though the problem of finding the general off-shell action in the non-commuting case has now been solved in [3], as mentioned above. The off-shell approach to gauging $(2,2)$ sigma-models has been explored in the very recent paper [20]. We first perform the physical computation in section 2. Although the result is formally identical to the commuting case, the computation is lengthy, and details are left to the appendix A. We then proceed to interpret the result geometrically in section 3. In this latter task our work is facilitated by a series of papers that appeared last year [21 − 26] which analyzed the conditions for generalized complex and generalized Kähler reduction. The motivation for those papers was mathematically clear: given that generalized complex geometry has symplectic geometry as an important special case, and that hamiltonian reduction is an important result in symplectic geometry, it is natural to wonder if a generalized complex (or Kähler) structure can be reduced too.

We refer in particular to [22], which contains a theorem whose hypotheses are exactly the same as the conditions we find for the (2,2) gauged sigma model. We find, then, perfect agreement between that mathematical theorem and physical expectations.

Another advantage of having such mathematical literature available (one which in fact constituted a major motivation for this work) is the possibility to tap into the reservoir of explicit examples those papers contain. It turns out, unfortunately, many of them are unsuitable for physical consideration, for reasons we discuss; we do, however, provide illustrations for several of the general features. In particular, we provide an example with flux which is a one-parameter deformation of the standard bihermitian structure on $S^3 \times \mathbb{R}$;

from the physical viewpoint, it is a $(2,2)$ deformation of the near-horizon geometry of the NS five-brane in flat space. We also sketch an example of reduction without flux but with non-commuting complex structures.

One possible direction in which it would be interesting to extend the present paper is the addition of a potential term to the sigma model. Similarly to [12], this should produce more interesting examples, presumably not yet considered in the mathematical literature.

2. The (2,2) gauged sigma model

In this section, we describe the gauged sigma model in terms of (1,1) superfields. In the following section we will show how this is connected to the existence of a moment map. We will start, however, with a subsection showing how already the *ungauged sigma model* implies the existence of two moment maps. We anticipate that the condition for the existence of the *gauged* sigma model will be stronger, in that these two moment maps are to be equal.

2.1 The supersymmetric ungauged sigma model and two moment maps

We start from the ungauged (1,1) sigma model whose target is a Riemannian manifold M equipped with a closed three-form H:

$$-\int_\Sigma g_{mn} D_+\phi^m D_-\phi^n \, d^2x d^2\theta + \int_B H_{mnp}\partial_t\phi^m D_+\phi^n D_-\phi^p \, d^3x d^2\theta \tag{2.1}$$

where the three-manifold B is such that $\partial B = \Sigma$, and t is a local coordinate in B normal to Σ, such that $\frac{\partial}{\partial t}|_\Sigma$ is an outward normal. If one imposes an extended supersymmetry

$$\delta\phi^m = \epsilon^+ D_+\phi^n I_{+\,n}^{m} + \epsilon^- D_-\phi^n I_{-\,n}^{m} \; . \tag{2.2}$$

one gets [1] that $I_\pm$ are complex structures, that the fundamental forms $\omega^\pm_{mn} \equiv g_{mp} I^p_{\pm\,n}$ are antisymmetric, and that

$$\nabla_\pm \omega_\pm = 0 \, , \qquad (\Rightarrow d\omega_\pm = \pm \iota_{I_\pm} H) \tag{2.3}$$

where $\nabla_\pm$ are covariant derivatives with connection $\Gamma^{\rm LC} \pm \frac{1}{2} g^{-1} H$. (This computation is a particular case of the one we perform later on for the gauged model, so we do not review it here.) This geometry is called *bihermitian geometry* because the forms $\omega_\pm$ make the geometry hermitian in two ways; it has been studied by mathematicians (see for example [27]) and then shown to be equivalent to generalized Kähler geometry (to be reviewed in section 3) in [2].

If one further imposes invariance of the model under a one-parameter family of diffeomorphisms generated by a vector field ξ, one also gets [14]

$$L_\xi H = L_\xi I_+ = L_\xi \omega_+ = 0 \; . \tag{2.4}$$

In particular it follows that (locally)

$$\iota_\xi H = d\alpha_\xi \tag{2.5}$$

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a for α_ξ some one-form.

Introduce now, temporarily, the operator ∂ and the (p,q) form decomposition for I_+. Following [15, 14], rewrite $L_\xi I_+ = 0$ as

$$[\partial, \iota_{\xi^{0,1}}] = 0 \; ; \tag{2.6}$$

taking the $(1,0)$ part of (2.3) and using (2.6), one gets $\partial(\iota_{\xi^{0,1}}\omega + i\alpha^{1,0}) = 0$, which locally implies

$$\iota_{\xi^{0,1}}\omega + i\alpha^{1,0} = \partial f \tag{2.7}$$

for some function f. Summing this with its complex conjugate, and using that on a function $d^c = \iota_{I_+} d$, one gets

$$\iota_\xi \omega + \iota_{I_+}(\alpha + d(\mathrm{Im} f)) = d\mathrm{Re} f \; . \tag{2.8}$$

We can now notice that α is only defined up to exact forms anyway (see (2.5)), and reabsorb $d\mathrm{Im} f$; we will then define $\mathrm{Re} f \equiv \mu$ and call it *generalized moment map*. We ask the reader to accept the name for the time being: it will be explained in the next section.

This discussion, however, could be repeated *verbatim* for the $-$ sector, leading, at this stage, to *two* moment maps $\mu_\pm$. We will see shortly that gauging the invariance under ξ requires these two to be equal.

2.2 Gauging: review of the bosonic case

On our way to gauging (2.1), we now review quickly the bosonic case, following [28, 29]. The bosonic action reads

$$S = -\frac{1}{2}\int_\Sigma g_{mn} d\phi^m \wedge * d\phi^n + \int_B H \; ; \tag{2.9}$$

again we call ξ the vector under which the model is invariant, which means that ξ is an isometry ($L_\xi g = 0$) and that $L_\xi H = 0$. As always, although the action contains an integral over the three-dimensional manifold B, the equations of motion are two-dimensional, because a field variation $L_{\delta\phi}$ acts as follows:

$$L_{\delta\phi}\int_B H = \int_B \{d, \iota_{\delta\phi}\} H = \int_B d(\iota_{\delta\phi} H) = \int_\Sigma \iota_{\delta\phi} H \tag{2.10}$$

where we have used the "magic Cartan formula"

$$L_v = d\iota_v + \iota_v d \tag{2.11}$$

valid for any vector field v.

Gauging the model means that we want to promote invariance under ξ to an invariance

$$\delta\phi = \lambda \xi(\phi) \, , \tag{2.12}$$

with λ a function on Σ. This is accomplished by introducing a vector field A which transforms as

$$\delta A = -d\lambda \; . \tag{2.13}$$

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To write the gauged action, it is convenient to introduce the covariant derivative $d^A = d\phi + A\xi$, so that $d^A(f(\phi)) = df(\phi) + A\xi^m \frac{\partial f(\phi)}{\partial\phi^m}$. It is covariant in the sense that

$$\delta(d^A\phi) = \lambda d^A(\xi) \ .$$

One is familiar with various particular cases of this covariant derivatives, notably the case in which the components of ξ are linear in ϕ as in most gauge theories, and the case in which they are constant, which appears for example in many gauged supergravity theories.

One is tempted to change all the derivatives d in the action (2.9) into covariant d^A's; this, however, would spoil the Stokes argument in (2.10). One way around this is to introduce a compensating three-dimensional integral. The other way, which we will use in this paper, is to write the gauged action as

$$S = -\int_\Sigma \left(\frac{1}{2}g_{mn}d^A\phi^m \wedge *d^A\phi^n + \alpha\wedge A + \frac{1}{e^2}dA\wedge *dA\right) + \int_B H \tag{2.14}$$

with e being the gauge coupling and α defined by (2.5) (from now on we will drop the subscript ξ on α). When varying the last term in parentheses in (2.14) with respect to ϕ, one now gets

$$\int_\Sigma \{\iota_{\delta\phi}, d\}\alpha A = \int_\Sigma (\iota_{\delta\phi}\iota_\xi HA - \iota_{\delta\phi}\alpha dA) \ ;$$

the first term completes the variation of H in (2.10) so that one gets

$$\delta\phi^m \, H_{mnp} d^A\phi^n \wedge d^A\phi^p$$

from their combination. For the action to be invariant under the local transformations (2.12), (2.13), it actually also turns out [28] that the condition

$$\iota_\xi\alpha = 0 \tag{2.15}$$

has to be satisfied. We will find it later in the supersymmetric case.

It is interesting to consider what happens if one follows the renormalization group flow in this model. As in [12], the gauge coupling e diverges in the infrared limit, and hence the kinetic term is negligible. This makes the gauge field A an auxiliary field, and one can integrate it out. If one does that, one gets a sigma model with target $M' = M/\mathrm{U}(1)$, where the $\mathrm{U}(1)$ action is generated by ξ. The metric g' and the NS three-form H' on the manifold M' are given by

$$g'_{mn} = g_{mn} - \xi^2\tilde{\xi}_m\tilde{\xi}_n + \xi^{-2}\alpha_m\alpha_n \,, \qquad \tilde{H}' = H + d(\tilde{\xi}\wedge\alpha) \tag{2.16}$$

where

$$\tilde{\xi}_m = \frac{1}{\xi^2}g_{mn}\xi^n$$

is the one-form dual to ξ (so that $\iota_\xi\tilde{\xi} = 1$) and $\xi^2 = g_{mn}\xi^m\xi^n$. For $\alpha = 0$, the form of the metric in (2.16) would be the metric obtained on the quotient by the Kaluza-Klein procedure. The extra piece is not inconsistent, however, since the metric still satisfies

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$\xi^m g'_{mn} = 0$ thanks to (2.15), hence it is well-defined for tangent vectors on the quotient manifold (which are defined as equivalence classes of tangent vectors on M under the equivalence relation $v \sim v + c\xi, c \in \mathbb{R}$). To understand better the geometric meaning of (2.16), let us rewrite g' as

$$g'_{mn} = g_{pq} Q^p_{+m} Q^q_{+n} = g_{pq} Q^p_{-m} Q^q_{-n}\,, \quad \text{where} \quad Q^m_{\pm n} \equiv \delta^m{}_n - \xi^m \left(\tilde{\xi}_n \pm \frac{\alpha}{\xi^2} \right) . \tag{2.17}$$

The matrices $Q_\pm$ are projectors, i.e. $Q_\pm Q_\pm = Q_\pm$. Their kernel is one-dimensional and consists of vectors proportional to ξ. The image consists of vectors v satisfying the condition $g_{mn} v^m \xi^n \pm \alpha(v) = 0$. It is easy to see that in any equivalence class of tangent vectors under the relation $v \sim v + c\xi$ there is a unique representative which belongs to the image of Q_+ (or Q_-). Thus the formula for the metric g' can be described in words as follows. Consider two vectors $v'_{1,2}$ on the quotient manifold M', whose scalar product we want to compute. We can think of them as tangent vectors on M defined up to shifts by ξ. In each of the two equivalence classes we find the (unique) representatives v_1 and v_2 which satisfy $\langle v_i, \xi\rangle + \alpha(v_i) = 0$ (we chose to work with Q_+ for definiteness). We define the scalar product of v'_1 and v'_2 to be the scalar product of v_1 and v_2 with respect to the metric g. (In the KK procedure, one would take representatives $v_{1,2}$ which are orthogonal to ξ). This is what is expressed by (2.17).

We will see below that the projectors $Q_\pm$ in (2.17) play a similar role for all tensors in the gauged supersymmetric model.

Let us also notice that H' in (2.16) is a basic form, and hence it also lives on the quotient M'. By definition, a basic form is annihilated both by ι_ξ and by L_ξ. For the first of these, one has to use (2.5), (2.15) and the identity $\iota_\xi d(\tilde{\xi}) = 0$ (which can be seen most easily by taking coordinates adapted to the fibration, in which $\xi = \partial_\psi$ for an angular coordinate ψ). $L_\xi H' = 0$ then follows easily from H' being closed and from (2.11).

2.3 The gauged supersymmetric action

We will now introduce the supersymmetric model, much along the lines of the bosonic gauged model described in the previous subsection. In analogy to the introduction of A there, we want to introduce an $\mathcal{N} = 1$ vector multiplet Γ_α to gauge (2.1). Since later we want to require that the action be $\mathcal{N} = 2$-supersymmetric, we also introduce an extra $\mathcal{N} = 1$ scalar multiplet S, which together with Γ_α should form an $\mathcal{N} = 2$ vector multiplet.

All in all the action we introduce reads

$$\begin{aligned} &- \int_\Sigma g_{mn} D^\Gamma_+ \phi^m D^\Gamma_- \phi^n \; d^2x d^2\theta + \int_B H_{mnp} \partial_t \phi^m D_+ \phi^n D_- \phi^p d^3x d^2\theta \\ &+ \int_\Sigma \left(\frac{1}{e^2} (W_\alpha W^\alpha + D_\alpha S\, D^\alpha S) - S\,\mu(\phi) - \alpha_m (D_+ \phi^m \Gamma_- + D_- \phi^m \Gamma_+) \right) d^2x d^2\theta \end{aligned} \tag{2.18}$$

where the covariant derivative

$$D^\Gamma_\alpha = D_\alpha + \Gamma_\alpha \xi^m \frac{\partial}{\partial \phi^m} \;;$$

for example $D^\Gamma_\alpha \phi^m = D_\alpha \phi^m + \Gamma_\alpha \xi^m$.

So far this is a (1,1) model. We now want to see what are the consequences of imposing a second supersymmetry in both left-moving and right-moving sectors, similarly to the result in [1] quoted around equation (2.3).

2.4 The second supersymmetry transformation

The second supersymmetry transformation for all the fields is easy to guess. Let us start from the $\mathcal{N}=2$ vector, made up by Γ_α and S. If we had an $\mathcal{N}=2$ superfield V, we could expand its dependence on θ^2_α as

$$V = v - i\theta_2^\alpha \Gamma_\alpha + i\theta_2^+\theta_2^- S$$

where v, Γ_α, S are functions of θ_1 only, and the factors are for later convenience. The gauge transformation for such a superfield consists of shifting $V \to V + \mathrm{Re}\Lambda$, with Λ a chiral $\mathcal{N}=2$ superfield. In particular, we can use this gauge freedom to send V to an "$\mathcal{N}=1$ Wess-Zumino gauge" in which $v=0$, by choosing Λ so that $\Lambda_{|\,\theta_2^\alpha=0} = v/2$. This gives

$$V \to -i\theta_2^\alpha\Gamma_\alpha + \theta_2^+\theta_2^-(iS - D^2 v) \tag{2.19}$$

In $\mathcal{N}=2$ terms, the second supersymmetry transformation is easy to express as $\delta_2 V = \epsilon^\alpha D_{2\,\alpha} V_{|\,\theta_2^\alpha=0}$. This gives the transformation laws

$$\delta_2(v, \Gamma_\alpha, S) = -\epsilon_2^\beta\,(i\Gamma_\beta,\ (\partial_{\alpha\beta}v - \epsilon_{\alpha\beta}S),\ -i\partial_{\alpha\beta}\Gamma^\alpha) \tag{2.20}$$

Now, by starting from $(0, \Gamma_\alpha, S)$ and composing (2.20) and (2.19), one gets

$$\delta\Gamma_\alpha = \epsilon_\alpha S\,, \qquad \delta S = \epsilon^\alpha W_\alpha \tag{2.21}$$

where we have defined

$$W_\alpha = D^\beta D_\alpha \Gamma_\beta\ . \tag{2.22}$$

As for the ϕ^m, we will generalize in the simplest way the supersymmetry transformations in (2.2):

$$\delta\phi^m = \epsilon^+ D_+^\Gamma \phi^n I^m_{+\,n} + \epsilon^- D_-^\Gamma \phi^n I^m_{-\,n}\ . \tag{2.23}$$

This is in fact the only possible expression for the second supersymmetry transformation which is gauge-invariant and compatible with dimensional analysis.

It is not very difficult to check that the putative second supersymmetry transformation in (2.23) commutes with the one implicit in the superfield notation; it follows in a standard way from the fact that D and Q commute. Much less trivial is the fact that the second supersymmetry transformation obeys by itself the supersymmetry algebra. But first we will require that it leaves our action invariant.

2.5 Invariance of the action

First of all, one can check that supersymmetry variations of the $(D_\alpha S)^2$ and $(W_\alpha)^2$ terms in (2.18) annihilate each other. To check this, one needs to use the identity $D_\alpha W_\beta = D_\beta W_\alpha$ which follows from $D_\alpha W^\alpha = 0$, which in turn follows from (A.1).

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The other terms contain ϕ and are much more involved. The result is that for the action to be supersymmetric the following has to be satisfied:

$$(\omega^{\pm})^t = -\omega^{\pm}\,, \qquad\qquad \nabla_{\pm}\omega_{\pm} = 0\;; \tag{2.24}$$

$$L_\xi g = L_\xi \omega = 0\,, \qquad\qquad \omega_{\pm}\xi \mp I^t_{\pm}\alpha = d\mu\,, \qquad\qquad \iota_\xi\alpha = 0 \tag{2.25}$$

where we recall that $\omega^{\pm}_{mn} \equiv g_{mp}I^{p}_{\pm\,n}$. We have also used $(\omega\xi)_m = \omega_{mn}\xi^n$ for $-(\iota_\xi\omega)_m$. Notice that the conditions in the first line, (2.24), were already present for the ungauged model, as mentioned in section 2.1; the second line, (2.25), contains the conditions specifically arising upon gauging.

Actually, by looking at the second equation in (2.25), we recognize that it was almost implied (locally) by the invariance of the ungauged model; the difference is that, at that point, one could only derive the existence of two separate moment maps. What we are finding here is that to write down a supersymmetric gauged model those two moment maps have to be equal. This can be traced back to the fact that in the gauged action (2.18) there is only room for one function $\mu(\phi)$, and not for two.

2.6 Supersymmetry algebra

We now require that the transformations defined in section 2.4 satisfy the right supersymmetry algebra on-shell.

In fact, they only do so *up to gauge transformations*. Namely, in appendix A we show that (2.20), (2.21) and (2.23) satisfy

$$\begin{aligned}
[\delta_1,\delta_2]S &= -2i(\epsilon_1^+\epsilon_2^+\partial_{\#} + \epsilon_1^-\epsilon_2^-\partial_{=})S\,,\\
[\delta_1,\delta_2]\Gamma_\alpha &= -2i(\epsilon_1^+\epsilon_2^+\partial_{\#} + \epsilon_1^-\epsilon_2^-\partial_{=})\Gamma_\alpha + \Lambda\,,\\
[\delta_1,\delta_2]\phi^m &= -2i(\epsilon_1^+\epsilon_2^+\partial_{\#} + \epsilon_1^-\epsilon_2^-\partial_{=})\phi^m - \Lambda\,\xi^m\,,
\end{aligned} \tag{2.26}$$

with a gauge transformation Λ that happens to be

$$\Lambda = 2\epsilon_1^+\epsilon_2^+\,D_+\Gamma_+ + \left(\epsilon_+^1\epsilon_-^2 + \epsilon_+^2\epsilon_-^1\right)(D_+\Gamma_- + D_-\Gamma_+) + 2\epsilon_1^-\epsilon_2^-\,D_-\Gamma_- \tag{2.27}$$

if the following conditions are satisfied:

$$(I_{\pm})^2 = -1\,, \qquad \mathrm{Nij}(I_{\pm}) = 0\;; \qquad\qquad L_\xi I_{\pm} = 0\,, \tag{2.28}$$

Nij being the Nijenhuis tensor. This time, the only condition really specific to the gauged model is the last one, whereas the first two (that $I_{\pm}$ are complex structures) already arise for the ungauged model.

2.7 The flow to the infrared

As in the bosonic case, the renormalization group flow makes the kinetic term for $\Gamma_{\pm}$ and S in (2.18) negligible and $\Gamma_{\pm}$ and S non-dynamic. Integrating out $\Gamma_{\pm}$ then gives

$$\Gamma_{\pm} = -(\xi^n g_{mn} \pm \alpha_m)D_{\pm}\phi^m \tag{2.29}$$

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which will be useful later. At the same time S becomes a Lagrange-multiplier superfield which constrains the bosonic fields to the hypersurface $\mu(\phi) = 0$. Hence the infrared limit is an ungauged sigma-model whose target M' is the quotient of the hypersurface $\mu = 0$:

$$M' = \{\mu = 0\}/\mathrm{U}(1) \tag{2.30}$$

as familiar from [12]. The formulæ for g' and H' in (2.5) are still valid, with a similar interpretation; the only extra step is that the vectors v' in the discussion after (2.17) have to be tangent to $\{\mu = 0\}$.

It is instructive to see what happens to the second supersymmetry transformation in the infrared, when we integrate out the fields Γ and S as in (2.29). One gets the usual transformations in (2.2), with complex structures

$$I'^{m}_{\pm\ n} = I^{m}_{\pm\ p} Q^{p}_{\pm n} \tag{2.31}$$

with $Q_\pm$ given in (2.17). The geometric meaning of these formulas is as follows. Consider a tangent vector v' on the quotient M' given by (2.30). Again since $Q_\pm \xi = 0$, this can be thought of as a vector on $\{\mu = 0\}$ defined modulo $v' \to v' + \lambda\xi$. There is a unique representative in the equivalence class such that $g_{mn}\xi^m v^n \pm \alpha(v) = 0$. Now one can act on this representative with $I_\pm$ to get a new vector on M; one can verify that this vector is tangent to $\{\mu = 0\}$ by computing $\partial_m \mu\, I'^{m}_{\pm\ n} = 0$. This is done by noticing that $Q_\pm$ in (2.17) can be rewritten, using the second equation in (2.25), as

$$Q^{m}_{\pm n} = \delta^m{}_n - \frac{\xi^m}{\xi^2} I^{p}_{\pm n} \partial_p \mu$$

and by noticing that $\partial_m \mu\, I^{m}_{\pm n} \xi^n = \xi^2$ (using this time both the second and third equation in (2.25)). One can now project the vector back to the quotient M'; hence we have obtained a linear map from TM' to itself, and this map is the complex structure on M'.

3. Geometrical interpretation

We are now ready to reinterpret the conditions we got for the existence of a gauged sigma model in terms of generalized geometry. The ungauged analogue of this is the equivalence between bihermitian geometry and generalized Kähler geometry proved in [2]. Therefore we first give a lightning review of this correspondence. For more details, see [2].

3.1 Bihermitian and generalized Kähler geometries

An *almost generalized complex structure* $\mathcal{J}$ on a manifold is an endomorphism of $T \oplus T^*$, squaring to -1, and hermitian with respect to the metric

$$\mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on $T \oplus T^*$. It follows that it has eigenvalues $\pm i$; call them $L_{\mathcal{J}}$ and $\bar{L}_{\mathcal{J}}$. The so-called *twisted Courant bracket* is the derived bracket [30] on $T \oplus T^*$ with respect to the differential $d + H\wedge$, where H is a closed three-form. Now, if $L_{\mathcal{J}}$ is closed with respect to the twisted Courant bracket, one says that $\mathcal{J}$ is *twisted integrable.*

If we have two commuting generalized complex structures $\mathcal{J}_{1,2}$ such that their product $M \equiv \mathcal{I}\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $T \oplus T^*$, we say that the manifold is *generalized Kähler.*

The reason this geometry is relevant for us is that such a pair $\mathcal{J}_{1,2}$ can be shown to have the form

$$\mathcal{J}_{1,2} = \frac{1}{2}\begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+ \pm I_-)^t \end{pmatrix} \tag{3.1}$$

for some bihermitian structure defined by complex structures I^m_{+n} and two-forms $\omega_\pm$; in particular, twisted integrability of $\mathcal{J}_{1,2}$ is equivalent to the integrability of $I_\pm$ and $d\omega_\pm = \pm\iota_{I_\pm}H$ (which we had in (2.3)).

Alternatively, locally one can replace twisted integrability with respect to H with ordinary integrability, and replace $\mathcal{J}_{1,2}$ with their so-called b-transform:

$$\mathcal{J}_{1,2} \to \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{J}_{1,2} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} .$$

The Kähler case is recovered in these formulæ by taking $H = 0$, $I_+ = I_-$ and $\omega_+ = \omega_-$.

Finally, let us mention a construction that we will need in section 4. To any generalized complex structure $\mathcal{J}$ one can associate locally an inhomogeneous differential form Φ (called *pure spinor*) with certain special properties . There are two features of this correspondence that we will need later.

The first is that twisted integrability for $\mathcal{J}$ translates into the existence of a one-form η and a vector field v such that $(d + H\wedge)\Phi = (\eta \wedge +\iota_v)\Phi$.

The second one concerns the *type* of a pure spinor Φ. This is defined as the smallest degree of a homogeneous component of Φ. It can be shown [2] that the type of a pure spinor Φ is equal to the number of i-eigenvectors of the corresponding $\mathcal{J}$ of the form $(v, 0)^t$ — that is, the dimension of the intersection of the i-eigenspace of $\mathcal{J}$ with T.

For more details, again see [2, 8].

3.2 Generalized moment map

After having reviewed how the conditions from the ungauged sigma model can be cast in the language of generalized Kähler geometry, we will now look at the conditions coming from the *gauged* (2,2) sigma model.

First we recall what an ordinary moment map is. If a vector preserves a symplectic form ω ($L_\xi\omega = 0$), one has by (2.11) that $d\iota_\xi\omega = 0$. Then locally one has

$$\iota_\xi\omega = -d\mu \tag{3.2}$$

for some function μ. This function is called the moment map.

Consider now the second equation in (2.25):

$$\omega_\pm\xi \mp I^t_\pm\alpha = d\mu \ . \tag{3.3}$$

If one takes sum and difference of these equations, one gets

$$\begin{pmatrix} 0 \\ d\mu \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I_+ - I_- & -\left(\omega_+^{-1} + \omega_-^{-1}\right) \\ \omega_+ + \omega_- & -(I_+ - I_-)^t \end{pmatrix} \begin{pmatrix} \xi \\ \alpha_\xi \end{pmatrix} = \mathcal{J} \begin{pmatrix} \xi \\ \alpha_\xi \end{pmatrix} . \tag{3.4}$$

(In the notation of [2], $\mathcal{J} = \mathcal{J}_2$.) In other words, $\mathcal{J}(\xi + \alpha) = d\mu$. This is equivalent to $\mathcal{J}(\xi + \alpha - id\mu) = i(\xi + \alpha - id\mu)$, or in other words

$$\xi + \alpha - id\mu \in L_{\mathcal{J}} . \tag{3.5}$$

This is exactly the definition of a *generalized moment map* in [22]. An action which admits a generalized moment map is called a generalized hamiltonian action.

This name is well motivated: in the Kähler case, $I_+ = I_-$ and $\mathcal{J}$ in (3.4) becomes

$$\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

so that $\alpha = 0$ and $d\mu = \omega\xi$, just as in (3.2).

We would like to emphasize that the existence of the generalized moment map puts a strong additional constraint on the vector field ξ and the corresponding one-form α, even assuming that ξ preserves all the tensors involved. This is in contrast with the Kähler case, where the moment map always exists locally, though there may be global obstructions coming from the nontrivial topology of the target manifold. In the case when $H \neq 0$, if we wanted to gauge while preserving only $\mathcal{N} = (2,1)$ supersymmetry, the situation would be similar: locally we can always solve equation (2.7) for f, while globally we may find an obstruction living in the Dolbeault cohomology group $H^1_{\bar\partial}(M)$. But in the $\mathcal{N} = (2,2)$ case we find an extra strong constraint coming from the requirement that right-moving and left-moving moment maps be identical. We have shown above that this physical constraint corresponds to the requirement that the action by ξ be generalized hamiltonian in the sense of [22].

Another condition in (2.25) was that $\iota_\xi \alpha = 0$. From (3.5), since $L_{\mathcal{J}}$ is isotropic (that is, the metric $\mathcal{I} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is zero when restricted to $L_{\mathcal{J}}$), we know that $\iota_\xi(\alpha - id\mu) = 0$. Taking the real part, we obtain the desired relation, which therefore is not independent. If we had considered a quotient by a group H more complicated than U(1), this relation would have been non-trivial (it would have corresponded to the condition that the moment map be *equivariant* in [22]).

As for the remaining conditions in (2.25), (2.28), together they say

$$L_\xi \mathcal{J}_{1,2} = 0 . \tag{3.6}$$

Hence we have now reinterpreted all the conditions for on-shell (2,2) supersymmetry of the gauged sigma model in terms of generalized Kähler geometry. To summarize:

- the manifold has to be generalized Kähler (just like in the ungauged case);
- ξ has to preserve the generalized Kähler structure (eq. (3.6));

- a generalized moment map ξ (with one-form α) has to exist for the action of ξ.

Looking at [22], we find that these are precisely the hypotheses for their Proposition 12. Hence the quotient by ξ inherits a twisted generalized Kähler structure. Above we have determined the bihermitian structure on the quotient using physical methods (integrating out the fields Γ and S). One possible way to read our result is as a physical proof of the theorem in [22].

4. Examples

We give here a few very simple examples, taking inspiration from some of the mathematical papers on generalized Kähler reduction.

1. Perhaps the simplest twisted generalized Kähler structure is the one living on the Hopf surface which is topologically $S^3 \times S^1$ [31, 2]. This can be described by $\mathbb{C}^2 - \{0\}$ with complex coordinates z_1, z_2, quotiented by $z_i \to 2z_i$. The two complex structures $I_\pm$ give respectively i, i and $i, -i$ on ∂_{z_1}, ∂_{z_2}, and hence they commute. The three-form flux H is the volume form of S^3. If we let $z_1 = r \sin\lambda\, e^{i\phi_1}, z_2 = r\cos\lambda\, e^{i\phi_2}$, where $\lambda \in \left[0, \frac{\pi}{2}\right]$ and both ϕ_1 and ϕ_2 have period 2π, then the metric is

$$ds^2 = \frac{\sum_{i=1}^2 dz_i d\bar{z}_i}{r^2} = \frac{dr^2}{r^2} + d\lambda^2 + \sin^2\lambda\, d\phi_1^2 + \cos^2\lambda\, d\phi_2^2.$$

The three-form H is

$$H = \sin 2\lambda\, d\lambda d\phi_1 d\phi_2.$$

If we do not quotient by $z_i \to 2z_i$, we get the product metric on $S^3 \times \mathbb{R}$. In the context of string theory, it describes the near-horizon geometry of the Neveu-Schwarz fivebrane.

This geometry has $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}$ isometry group, where $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $S^3 \simeq \mathrm{SU}(2)$ by right and left translations, while $\mathbb{R}$ acts by $r \to re^t$. One could try to reduce this model either along one of the left-invariant vector fields on S^3 or along $r\frac{\partial}{\partial r}$. It turns out that this is not possible: even though these vector fields preserve all the tensors concerned, their action is not generalized hamiltonian — that is, no μ exists so that (3.4) is satisfied. One can see this in the following way. From (3.3) one can derive

$$[I_+, I_-]g^{-1}\alpha = (2 + \{I_+, I_-\})\xi\,, \tag{4.1}$$

an equation that we will need again in the appendix. If the two complex structures commute, the left hand side is zero; also, the right hand side then becomes proportional to $P\xi \equiv (1/2)(1 + I_+I_-)\xi$. Now, I_+I_- is an almost product structure [1], and P is a projector defined by it; so we have found that ξ is along one of the two subspaces defined by the almost product structure. This is a general result. Its physical interpretation is that, when the model can be written without using semi-chiral multiplets, the gauging may involve either only the chiral or only the twisted chiral multiplets.

Coming back to $S^3 \times \mathbb{R}$, from the explicit form of $I_\pm$ given above one sees that ξ should either involve only z_1 or only z_2. Of course the two possibilities are equivalent, so let us pick the first one and write $\xi = iz_1\partial_{z_1} - i\bar{z}_1\partial_{\bar{z}_1}$. The reduction along this vector field has been considered by S. Hu [24]. Let us describe this example in some detail; below we modify it to produce a new family of generalized Kähler structures on $S^3 \times \mathbb{R}$ and $S^3 \times S^1$. First we need to choose a one-form α which solves the equation $i_\xi H = d\alpha$. The choice which leads to a hamiltonian action turns out to be

$$\alpha = \cos^2 \lambda d\phi_2.$$

Note that this form is smooth everywhere on S^3. The corresponding moment map turns out to be

$$\mu = -\log r + const.$$

Thus the zero-level of the moment map is a submanifold given by $r = const$, which is a three-sphere. The quotient of this submanifold by the vector field

$$\xi = \frac{\partial}{\partial \phi_1}$$

can be parametrized by λ and ϕ_2 and can be identified with a disc. The reduced metric turns out to be

$$d\lambda^2 + \tan^2 \lambda d\phi_2^2.$$

This is precisely the metric which corresponds to the $\mathcal{N} = 2$ minimal model $\mathrm{SU}(2)/\mathrm{U}(1)$ [32]. This is hardly surprising: the first step in the generalized Kähler reduction in this case amounts to fixing r to be constant, thereby reducing the theory to the $\mathcal{N} = (1,1)$ SU(2) WZW model, while the second step consists of gauging the adjoint action of the maximal torus of SU(2) and integrating out the gauge $\mathcal{N} = (1,1)$ supermultiplet, which gives the supercoset $\mathrm{SU}(2)/\mathrm{U}(1)$.

We can modify the above construction to produce a one-parameter family of generalized Kähler structures on $S^3 \times \mathbb{R}$ and $S^3 \times S^1$. Consider $S^3 \times \mathbb{R} \times \mathbb{R}^2$, where we regard $\mathbb{R}^2$ as a flat Kähler manifold with a complex coordinate $z_3 = x_3 + iy_3$ and a metric $ds^2 = |dz_3|^2$. Let us quotient this generalized Kähler manifold by a vector field

$$iz_1 \frac{\partial}{\partial z^1} - i\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \zeta \frac{\partial}{\partial y_3},$$

where ζ is an arbitrary real number. (More generally, we could consider a product $S^3 \times \mathbb{R} \times Y$, where Y is a Kähler manifold with a U(1) symmetry). The moment map is now

$$\mu = -\log r - \zeta x_3 + const.$$

The equation $\mu = 0$ allows to express x_3 in terms of r, so the quotient of the submanifold $\mu = 0$ can be naturally identified with $S^3 \times \mathbb{R}$ parametrized by $r, \lambda, \phi_1, \phi_2$. The reduced metric is

$$\frac{dr^2}{r^2}\left(1 + \frac{1}{\zeta^2}\right) + d\lambda^2 + \frac{\zeta^2 \sin^2 \lambda d\phi_1^2 + \cos^2 \lambda (1+\zeta^2) d\phi_2^2}{\sin^2 \lambda + \zeta^2}.$$

In the limit $\zeta \to \infty$ it reduces to the standard metric on $S^3 \times \mathbb{R}$. The three-form H' on the reduced manifold is

$$H' = \sin 2\lambda \, d\lambda d\phi_1 d\phi_2 + d(\tilde{\xi} \wedge \alpha) = \sin 2\lambda \, d\lambda d\phi_1 d\phi_2 + d\left(\frac{\sin^2 \lambda \cos^2 \lambda \, d\phi_1 d\phi_2}{\sin^2 \lambda + \zeta^2}\right).$$

Note that the reduced metric is invariant with respect to $r \to re^t$, so we can make periodic identification of $\log r$ and produce a one-parameter deformation of the standard generalized Kähler structure on $S^3 \times S^1$. We can compute the corresponding forms $\omega'_\pm$ following the same geometric procedure as for the metric g': we restrict $\omega_\pm$ to the hypersurface $\mu = 0$ and define the value of $\omega'_\pm$ on the vectors $v'_{1,2}$ tangent to the quotient to be the value of $\omega_\pm$ on the specially chosen representatives of $v'_{1,2}$ (those which lie in the image of $Q_\pm$). In this way we obtain:

$$\omega'_+ = -\frac{1+\zeta^2}{\sin^2\lambda + \zeta^2}\left[\sin\lambda\cos\lambda\left(\frac{\zeta^2}{1+\zeta^2}d\lambda d\phi_1 - d\lambda d\phi_2\right) + \sin^2\lambda\frac{dr d\phi_1}{r} + \cos^2\lambda\frac{dr d\phi_2}{r}\right] .$$

One can also compute the complex structure I'_+; by finding the $(1,0)$ forms with respect to this complex structure and integrating them, one obtains complex coordinates $z'_{1+} = r^{1+\zeta^{-2}}\sin\lambda\, e^{i\phi_1}$, $z'_{2+} = r\cos\lambda\, e^{i\phi_2} = z_2$. The form ω'_- and the complex structure I'_- are obtained from ω'_+ and I'_+ by changing $\phi_2 \to -\phi_2$.

2. We now want to give an example without NS flux, but with non-commuting complex structures. For this, we turn to [23, 21]. These authors apply to $\mathbb{C}^k$ a procedure devised in [2] to deform an ordinary Kähler structure into a generalized Kähler one. To describe the idea we will need the pure spinors Φ introduced in section 3.1. The pure spinors for the initial Kähler case corresponding to the generalized complex structures (3.1) read $\Phi_1 = \Omega$, $\Phi_2 = e^{i\omega}$, where Ω is the $(k,0)$ form (it would in general only exist locally, but we are considering $\mathbb{C}^k$) and ω is the Kähler form. Now the deformation is described by

$$\Phi_1 \to \exp\left[\beta^{ij}\left(\partial_i - i\omega_{i\bar{k}}d\bar{z}^{\bar{k}}\right)\left(\partial_j + i\omega_{j\bar{l}}d\bar{z}^{\bar{l}}\right)\right]\Phi_1\,,$$

where β is a holomorphic Poisson bivector. It so happens that the same operator acting on Φ_2 leaves it invariant.

Choosing different bivectors β and reducing along different vector fields ξ produces many examples of generalized Kähler structures on a certain class of toric manifolds (not on all, because of the condition that the bivector β be holomorphic Poisson). Unfortunately, it appears that for all these examples the generalized Kähler structure before reduction is defined not on all of $\mathbb{C}^k$, but on some open set obtained by excluding a lower-dimensional submanifold (defined by a real equation). For instance, the examples in [23] start with $\mathbb{C}^3$ and reduce it by the action $z^i \to e^{i\psi}z^i$, to yield a generalized Kähler structure on $\mathbb{CP}^2$. In order for the generalized Kähler structure to be invariant, one has to take the Poisson bivector to have degree two in the z^i, so that Φ_1 is homogeneous of degree 3. The Poisson bivector being non-constant causes the norm of Φ_1 go to zero on a certain locus and the generalized Kähler is not well-defined there.

Mathematically this is harmless, since one can usually arrange so that the hypersurface $\mu = \text{const}$ does not intersect with the troublesome locus. Physically, however, the model one starts with has to be defined on the whole of the manifold in order for the gauged sigma-model to make sense.

A way to circumvent this problem is as follows. Let us start from $\mathbb{C}^4$ and take the (integrated) action of ξ to be $(z^1, z^2, z^3, z^4) \to (e^{i\psi}z^1, e^{i\psi}z^2, e^{-i\psi}z^3, e^{-i\psi}z^4)$. Then the pure spinors we start with are both of degree zero with respect to the action of ξ. This means that we can take the bivector β to be constant. This does not lead to the problem described above, and one can safely perform the reduction, getting a bihermitian structure on the conifold.

Unfortunately, by taking β to be constant we have given up the NS flux as well. Also, it should be emphasized that we have not changed the flat metric on $\mathbb{C}^4$: the model we start with is still the usual free sigma model, only with a very particular choice of a (2,2) supersymmetry algebra. What one produces after reduction is a pair of non-commuting, complex structures on the conifold, covariantly constant with respect to the Levi-Civita connection given by the reduced metric. This metric has holonomy U(3); the Calabi-Yau metric on the conifold is more complicated and is found by following the renormalization group further down in the infrared.

So this bihermitian structure is not very interesting *per se*; it is, however, an example in which the two complex structures $I_\pm$ do not commute. A way to see it is the following. If $I_\pm$ commute, they are simultaneously diagonalizable; by looking at (3.1), we can produce k eigenvectors of either $\mathcal{J}_1$ or $\mathcal{J}_2$ that are purely in T. In other words, the sum of the types (see end of section 3.1) of the two pure spinors Φ_1 and Φ_2 is k.

For example, in the Kähler case, $\Phi_1 = \Omega$ is a k-form and has degree zero; $\Phi_1 = e^{i\omega} = 1 + i\omega + \cdots$ has inside the differential form 1, which has degree zero, so the type of Φ_2 is zero. The sum of the two is k, and indeed in this case $I_+ = I_-$. After the deformation by the bivector β, however, the type of Φ_- is lowered; the sum of the two types can no longer be zero, and by the reasoning above this means $[I_+, I_-] \neq 0$.

There are other constructions of generalized Kähler manifolds, but it is not obvious whether they admit a hamiltonian action. Notably, the construction by Hitchin [33], that closely parallels the physical construction in [3], appears to be fairly general; and the even-dimensional semi-simple groups are bihermitian, as pointed out in [2]. It would be interesting to consider their reduction.

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A. Details about supersymmetry

We raise and lower indices with the tensor $\epsilon^{\alpha\beta}$, defined so that $\epsilon^{+-} = 1$. The derivatives $D_\pm$ satisfy $(D_+)^2 = i\partial_{++}$, $(D_-)^2 = i\partial_{=}$. Some useful equalities are

$$D_\alpha D_\beta = i\partial_{\alpha\beta} - \epsilon_{\alpha\beta} D^2 \ ; \quad D^2 D_\alpha = -D_\alpha D^2 = -i\partial_{\alpha\beta} D^\beta \ ; \quad D^\alpha D_\beta D_\alpha = 0 \tag{A.1}$$

where $D^2 = D_+ D_-$.

Let us now look at the details of the computations in sections 2.5 and 2.6. The methods are standard, if a little complicated; here we list some of the steps in which the computation departs from the one for the ungauged model.

For the variation of the action, it is useful to notice that

$$\frac{\delta}{\delta\phi^m}\left(\int g_{np}D_+^\Gamma\phi^n D_-^\Gamma\phi^p\right) = -g_{mn}[D_+^\Gamma, D_-^\Gamma]\phi^n - 2\Gamma_{mnp}D_+^\Gamma\phi^n D_-^\Gamma\phi^p$$
$$+(\Gamma_+ D_-^\Gamma\phi^n - \Gamma_- D_+^\Gamma\phi^n)(L_\xi g)_{mn}\ .$$

(In computing this, one needs $\xi^m\partial_m(D_+\phi^n) = D_+\xi^n$.) This can then be specialized to the variation under supersymmetry. Another slight modification is given by the integration by parts. In the case of linear gaugings, for example, all the covariant derivatives are given in the appropriate representation, so that any scalar is acted on by a straight derivative; hence one can integrate covariant derivatives by parts. In the present case, however, a function may still be transforming non-trivially under the vector ξ. One can, however, integrate by parts the straight derivative, and add and subtract the connection piece, so that for example

$$\int A_{[mn]}D_-^\Gamma D_+^\Gamma\phi^m D_+^\Gamma\phi^n = \int\Big(-\frac{1}{2}A_{mn,p}D_-^\Gamma\phi^p D_+^\Gamma\phi^m D_+^\Gamma\phi^n + \Gamma_- D_+^\Gamma\phi^m D_+^\Gamma\phi^n(L_\xi A)_{mn}\Big).$$

Using all this, the total variation is

$$\begin{aligned}
\delta S = \int \epsilon^+\Big[&2D_-^\Gamma D_+^\Gamma\phi^m D_+^\Gamma\phi^n\Big(-\omega^+_{(mn)}\Big) + D_-^\Gamma\phi^p D_+^\Gamma\phi^m D_+^\Gamma\phi^n\Big(\nabla_p^+\omega^+_{mn}\Big)\\
&+\big(SD_+^\Gamma\phi^m + D_+^\Gamma\phi^n I_+{}^m{}_n(D_+\Gamma_- + D_-\Gamma_+)\big)\Big(g_{mp}\,\xi^p + \alpha_m - I_+{}^p{}_m\partial_p\,\mu\Big)\\
&+S\Gamma_+\Big(\alpha_m\xi^m\Big) + \Gamma_+ D_+^\Gamma\phi^m D_-^\Gamma\phi^n\Big(I_+{}^p{}_m(L_\xi g)_{pn}\Big) + \Gamma_+ D_-^\Gamma\phi^m D_-^\Gamma\phi^n\Big(I_+{}^p{}_m(L_\xi g)_{pn}\\
&+(L_\xi\omega)_{mn}\Big)\Big]\\
+\epsilon^-\Big[&2D_+^\Gamma D_-^\Gamma\phi^m D_-^\Gamma\phi^n\Big(\omega^-_{(mn)}\Big) + D_+^\Gamma\phi^p D_-^\Gamma\phi^m D_-^\Gamma\phi^n\Big(-\nabla_p^-\omega^-_{mn}\Big)\\
&+(SD_-^\Gamma\phi^m - D_-^\Gamma\phi^n I_-{}^m{}_n(D_+\Gamma_- + D_-\Gamma_+))\Big(g_{mp}\,\xi^p - \alpha_m - I_-{}^p{}_m\partial_p\,\mu\Big)\\
&+S\Gamma_-\Big(-\alpha_m\xi^m\Big) + \Gamma_+ D_-^\Gamma\phi^m D_+^\Gamma\phi^n\Big(-I_-{}^p{}_m(L_\xi g)_{pn}\Big)\\
&+\Gamma_- D_+^\Gamma\phi^m D_+^\Gamma\phi^n\Big(I_-{}^p{}_m(L_\xi g)_{pn} + (L_\xi\omega)_{mn}\Big)\Big]\ .
\end{aligned}$$

Let us now look at the commutator of two second supersymmetry transformations. The one on S is uneventful. For the one on Γ_+, one only needs to add and subtract a term $\epsilon_1^+\epsilon_2^+D_+^2\Gamma_+$; one piece goes towards building the right supersymmetry algebra, the other goes to the gauge transformation Λ given in (2.27). The most complicated commutator is obviously the one evaluated on ϕ^m. One gets:

$$
\begin{aligned}
[\delta_1,\delta_2]\phi^m = \; & 2\epsilon_1^+\epsilon_2^+\Big[(I_+^2)^m{}_n\, i\partial_{\mathbin{+\!\!+}}\phi^n + (I_+)^m{}_n(D_+\Gamma_+)\xi^n + D_+^\Gamma\phi^n D_+^\Gamma\phi^p\Big(\mathrm{Nij}(I_+)^m{}_{np}\Big) \\
& \qquad -\Gamma_+ D_+^\Gamma\phi^n\Big((L_\xi I_+)^m{}_n\Big)\Big] \\
& +2\epsilon_1^-\epsilon_2^-\Big[(I_-^2)^m{}_n\, i\partial_{=}\phi^n + (I_-)^m{}_n(D_-\Gamma_-)\xi^n + D_-^\Gamma\phi^n D_-^\Gamma\phi^p\Big(\mathrm{Nij}(I_-)^m{}_{np}\Big) \\
& \qquad -\Gamma_- D_-^\Gamma\phi^n\Big((L_\xi I_-)^m{}_n\Big)\Big] \\
& +(\epsilon_1^+\epsilon_2^- + \epsilon_1^-\epsilon_2^+)\Big[S(I_+-I_-)^m{}_n\,\xi^n + \frac{1}{2}[I_+,I_-]^m{}_n[D_+^\Gamma,D_-^\Gamma]\phi^n \\
& +D_+^\Gamma\phi^n D_-^\Gamma\phi^p\Big(I^m_{+\;q}I^q_{-p,n} + I^m_{+\;n,q}I^q_{-p} - I^m_{-\;q}I^q_{+n,p} - I^m_{-\;p,q}I^q_{+n}\Big) \\
& +\frac{1}{2}\{I_+,I_-\}^m{}_n\xi^n(D_+\Gamma_- + D_-\Gamma_+) + \Gamma_+D_-^\Gamma\phi^n\Big(-(L_\xi I_-)^m{}_n\Big) \\
& +\Gamma_-D_+^\Gamma\phi^n\Big(-(L_\xi I_+)^m{}_n\Big)
\end{aligned} \tag{A.2}
$$

We can now use some of the conditions coming from the action to massage the $(\epsilon_1^+\epsilon_2^- + \epsilon_1^-\epsilon_2^+)$ term in this result. First of all, one can show

$$
[I_+,I_-]^m{}_q\left(\Gamma + \frac{1}{2}g^{-1}H\right)^q{}_{np} = I^m_{+\;q}I^q_{-p,n} + I^m_{+\;n,q}I^q_{-p} - I^m_{-\;q}I^q_{+n,p} - I^m_{-\;p,q}I^q_{+n}\,, \tag{A.3}
$$

which is already useful in the ungauged case [1]. To derive this identity, note that since $I_\pm$ are covariantly constant with respect to the connections $\Gamma_\pm$, we can express ordinary derivatives of $I_\pm$ in terms of $I_\pm$ and $\Gamma_\pm$:

$$
I^m_{\pm\;n,p} = \Gamma^q_{\pm pn}I^n_{\pm q} - \Gamma^m_{\pm pq}I^q_{\pm n}.
$$

Substituting this into the expression on the r.h.s. of (A.3), using $\Gamma^m_{-np} = \Gamma^m_{+pn}$, and collecting similar terms, we get the expression on the l.h.s. of (A.3).

Using that $L_\xi g = L_\xi\omega_\pm = 0$, we also get that $L_\xi I_\pm = 0$. Finally, from (3.3) we can derive

$$
[I_+,I_-]g^{-1}\alpha = (2 + \{I_+,I_-\})\xi\,, \qquad [I_+,I_-]g^{-1}d\mu = -2(I_+-I_-)\xi\;. \tag{A.4}
$$

The first of these two has already been used in section 4. So the $(\epsilon_1^+\epsilon_2^- + \epsilon_1^-\epsilon_2^+)$ term in (A.2) now reads

$$
\begin{aligned}
& [I_+,I_-]^m{}_n\left(\frac{1}{2}[D_+^\Gamma,D_-^\Gamma]\phi^n + \Gamma^n_{+pq}D_+^\Gamma\phi^p D_-^\Gamma\phi^q - \frac{1}{2}Sg^{np}\partial_p\mu + \frac{1}{2}g^{np}\alpha_p(D_+\Gamma_- + D_-\Gamma_+)\right) \\
& -(D_+\Gamma_- + D_-\Gamma_+)\xi^m\;.
\end{aligned}
$$

The first line is now proportional to the equation of motion for ϕ; the second is a piece of the gauge transformation Λ we claimed in (2.27) — the other pieces having already been obtained in (A.2). This completes the computation.

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