arXiv:1809.07324v1 [quant-ph] 19 Sep 2018

Dissipative self-interference and robustness of continuous error-correction to miscalibration

Victor V. Albert,[1,*] Kyungjoo Noh,[2] and Florentin Reiter[3]

[1] *Institute for Quantum Information and Matter and Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, CA 91125, USA*
[2] *Yale Quantum Institute, Departments of Applied Physics and Physics, Yale University, New Haven, CT 06520, USA*
[3] *Department of Physics, Harvard University, Cambridge, MA 02138, USA*

(Dated: September 21, 2018)

We derive an effective equation of motion within the steady-state subspace of a large family of Markovian open systems (i.e., Lindbladians) due to perturbations of their Hamiltonians and system-bath couplings. Under mild and realistic conditions, competing dissipative processes destructively interfere without the need for fine-tuning and produce *no dissipation* within the steady-state subspace. In quantum error-correction, these effects imply that continuously error-correcting Lindbladians are robust to calibration errors, including miscalibrations consisting of operators undetectable by the code. A similar interference is present in more general systems if one implements a particular Hamiltonian drive, resulting in a coherent cancellation of dissipation. On the opposite extreme, we provide a simple implementation of universal Lindbladian simulation.

Understanding how to reservoir-engineer [1] open quantum systems is important for the success of noisy intermediate-scale quantum (NISQ) [2] technologies. In this context, one often encounters the problem of experimentally controlling time-evolution within a particular subspace of states, e.g., in order to stabilize states [3–7] and phases of matter [8–12], generate gates using Zeno dynamics [13–16], or protect against unwanted errors [17–22]. Resolving this problem revolves around variants of either perturbation theory or adiabatic elimination. In the case of interest here, one applies a perturbation $\mathcal{O}$ to an unperturbed Lindbladian $\mathcal{L}$ [23–27] such that the resulting leading-order time-evolution within the steady-state subspace of $\mathcal{L}$ is governed by an effective Lindbladian $\mathcal{L}_{\mathsf{eff}}$. In general, $\mathcal{L}_{\mathsf{eff}}$ is difficult to put explicitly in Lindblad form since there is a complex interplay between dissipation and coherent evolution inherent in $\mathcal{L}$ *and* arising from $\mathcal{O}$. Cases in which $\mathcal{L}_{\mathsf{eff}}$ (to 1st [28–30] or 2nd [31–42] order in $\mathcal{O}$) can be simplified are highly sought after since they yield physical intuition, are numerically tractable, and provide Hydrogen-atom-like starting points for more complex scenarios. Due to the aforementioned complexity, such cases are scarce relative to the many combinations of steady-state structures [43], perturbation types [44, Sec. 6.1], and features of $\mathcal{L}$ [45, Sec. 2.1].

In this Letter, we derive an $\mathcal{L}_{\mathsf{eff}}$ for arbitrary Hamiltonian and jump-operator perturbations to certain $\mathcal{L}$ admitting decoherence-free subspaces (DFS) [46–48], demonstrating surprising and (to an extent) generic interference effects. Being an extension of an effective operator formalism (EOF) [33] applicable to a variety of Rydberg [49–51], photonic [52, 53], and trapped-ion [22] platforms, our formalism and its predicted interference effects should be observable in and useful to many quantum technologies.

Minimal example.—To gain intuition into the interference effects, consider first a simple three-level system $\{|0\rangle, |1\rangle, |e\rangle\}$ [see Fig. 1(a)] where the excited level $|e\rangle$ resides at an energy $H = \delta|e\rangle\langle e|$ and decays into $|0\rangle$ under jump operator $F = \sqrt{\Gamma}|0\rangle\langle e|$ (with corresponding dissipator $\mathcal{D}[F](\cdot) \equiv F(\cdot)F^\dagger - \frac{1}{2}\{F^\dagger F, (\cdot)\}$). The states $\{|0\rangle, |1\rangle\}$ form a DFS. Now,

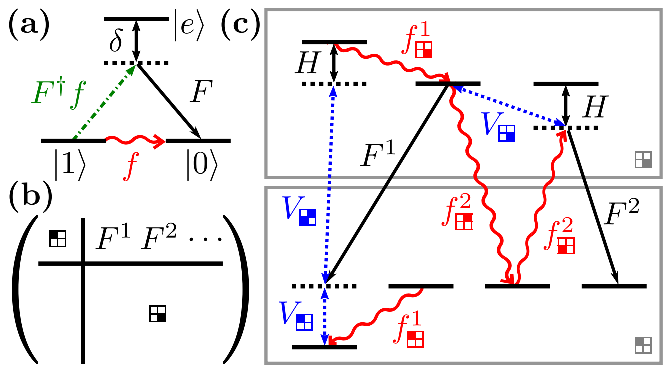

Figure 1. (color online) **(a)** A three-level open system with unperturbed steady states $\{|0\rangle, |1\rangle\}$ (forming a DFS), Hamiltonian $H = \delta|e\rangle\langle e|$, and jump F (black arrows). To leading order, a jump perturbation f (red wavy arrow) induces two processes which destructively interfere with each other [see Eq. (1)]: one is simply f itself while the other occurs via a virtual transition though $|e\rangle$ via $F^\dagger f$ (green dotted-dashed arrow). **(b)** Sketch of the block matrix formed by jumps F^ℓ satisfying the condition (4) necessary for a generalized interference effect. Each jump operator occupies its own block. The levels ⊞ represent the DFS while ⊞ are decaying via the unperturbed Lindbladian $\mathcal{L} = \{H, F^\ell\}$ (5). **(c)** Energy levels of a system satisfying the assumptions of the EJOF. The perturbations include Hamiltonian (V; blue dotted arrows) and jump perturbations $\{f^1, f^2\}$.

assume a small additional decay $|1\rangle \to |0\rangle$ arising from the same coupling to the bath as the $|e\rangle \to |0\rangle$ decay. Under such decay, $F \to F + f$ with perturbation $f \equiv \sqrt{\gamma}|0\rangle\langle 1|$ and $\Gamma \gg \gamma$. Naturally, one would think that the leading-order $O(\gamma)$ dissipation due to f will be $\mathcal{D}[f]$. However, our formalism identifies an additional $O(\gamma)$ effective process that interferes with this dissipation via the virtual transition $|1\rangle \to |e\rangle \to |0\rangle$. While neither the strong (F) nor the weak (f) dissipation alone cause the $|1\rangle \to |e\rangle$ part of that transition, perturbing $F^\dagger F \to (F + f)^\dagger(F + f)$ in $\mathcal{D}[F]$ yields the term $F^\dagger f \propto |e\rangle\langle 1|$ which, when followed by F, produces that transition. That transition is also mediated by the inverse of the non-Hermitian "Kamiltonian" $K = (\delta - \frac{i}{2}\Gamma)|e\rangle\langle e|$ governing evolution of $|e\rangle$.

Leading-order dissipative evolution within the DFS is a superposition of both processes and is governed by $\mathcal{L}_{\text{eff}}$ with effective jump operator

$$F_{\text{eff}} = f + \tfrac{i}{2} F K^{-1} F^{\dagger} f = \sqrt{\gamma} \frac{\delta}{\delta - \frac{i}{2}\Gamma} |0\rangle\langle 1| \,. \tag{1}$$

In the limit of large energy $\delta \gg \Gamma$, the virtual $|1\rangle \to |e\rangle \to |0\rangle$ transition is off-resonant, the second term in F_{eff} goes to zero, and one reduces to the intuitive case ($F_{\text{eff}} = f$). However, when $\Gamma \gg \delta$, destructive interference between the two terms makes the effective dissipation disappear entirely ($F_{\text{eff}} = 0$). Although this cancellation can be understood nonperturbatively using dark-state physics [54], here we show that the perturbative interference holds much more generally than previously thought. Generalizing this three-level example, $F_{\text{eff}} = 0$ at zero energy for $f = \sqrt{\gamma}|0\rangle\langle\psi|$ with any $|\psi\rangle$. Extending to four or more levels, we will see that $F_{\text{eff}} = 0$ for a much larger family of $\{F, f\}$.

Generic cancellation.—It is uncommon in Hamiltonian perturbation theory for a correction to be zero for *any* perturbation (unless a symmetry is present). In this example, we observe such a cancellation not due to symmetry, but to inherent destructive interference between generalizations of the two processes discussed above. Consider an $N + 2$-level system $\{|0\rangle, |1\rangle, |e\rangle, |h\rangle, \cdots\}$ with $\{|0\rangle, |1\rangle\}$ forming a DFS with corresponding projection $I_{▘} = |0\rangle\langle 0| + |1\rangle\langle 1|$. To simplify notation, we partition operators O into four corners [30]: $O_{▘} \equiv I_{▘} O I_{▘}$ acting on the DFS, $O_{▗} \equiv I_{▗} O I_{▗}$ (with $I_{▗} \equiv 1 - I_{▘}$) acting on the N decaying states, the "lowering operator" $O_{▝} \equiv I_{▘} O I_{▗}$ mapping decaying states into the DFS, and the "raising operator" $O_{▖} \equiv I_{▗} O I_{▘}$ taking states out of the DFS. Assume no Hamiltonians ($H = 0$, for now) and an unperturbed jump $F = F_{▝}$, meaning that F maps one directly into the DFS (▘) from the decaying space (▗). This jump can have *any* combination of the $2N$ decay channels from the N excited states into $\{|0\rangle, |1\rangle\}$, with the only restriction that it is surjective,

$$F \left(F^{\dagger} F\right)^{-1} F^{\dagger} = I_{▘} \,. \tag{2}$$

Interestingly, randomly generated jumps do this: all but a measure-zero set of $F = F_{▝}$ consisting of random entries [55] satisfy (2). For now, perturb F with any small f satisfying $f = f^{\ell}_{▜}$, i.e., any f not mapping ▘ into ▗. Applying Eqs. (1,2) yields the effective jump

$$F_{\text{eff}} = f_{▘} - F \left(F^{\dagger} F\right)^{-1} F^{\dagger} f = f_{▘} - f_{▘} = 0 \tag{3}$$

to leading order in *any* jump perturbation $f_{▜}$. (We will later prove that $f_{▗}$ doesn't participate at all.) Therefore, a random jump $F = F_{▝}$ perturbed by any small perturbation not mapping out of the DFS generically produces no leading-order dissipation within the DFS.

This cancellation can be extended to multiple unperturbed jumps F^{ℓ}, granted that (2) holds for each F^{ℓ} and the additional "orthogonality" condition

$$F^{\ell} F^{\ell' \dagger} = \delta_{\ell\ell'} F^{\ell} F^{\ell\dagger} \tag{4}$$

is satisfied. This condition implies that a block matrix consisting of $\{F^{\ell}\}$ will look like Fig. 1(b). Conditions (2,4) imply that $K^{-1} = \sum_{\ell} (-\frac{i}{2} F^{\ell\dagger} F^{\ell})^{-1}$ and $F^{\ell} K^{-1} F^{\ell'\dagger} \propto I_{▘} \delta_{\ell\ell'}$, yielding once again no dissipative evolution ($F^{\ell}_{\text{eff}} = 0$) for *any* $\{f^{\ell}_{▜}\}$. Having described the most interesting effect, we now state our general result—a formalism for tackling perturbations to a large class of Lindbladians.

General result.—Let the unperturbed Lindbladian $\mathcal{L}$ consist of a Hamiltonian H and jump operators F^{ℓ},

$$\mathcal{L}(\cdot) = -i[H, \cdot] + \sum_{\ell} \mathcal{D}[F^{\ell}](\cdot) \,. \tag{5}$$

Consider coherent and dissipative perturbations, respectively,

$$H \to H + V \qquad \text{and} \qquad F^{\ell} \to F^{\ell} + f^{\ell} \,. \tag{6}$$

Since $\mathcal{L}$ governs the evolution of a system coupled to a bath [10, 56, 57], V is a modification of the system Hamiltonian while f^{ℓ} modifies the system-bath coupling. If $\mathcal{L}$ is a desired reservoir engineering operation, then $\{V, f^{\ell}\}$ can be thought of as uncontrollable coherent evolution and miscalibrations in the engineered dissipation, respectively. The resulting superoperator perturbation has terms both 1^{st} and 2^{nd} order in $\{V, f^{\ell}\}$, $\mathcal{O} = \mathcal{O}_1 + \mathcal{O}_2$, and perturbation theory within the steady-state subspace yields the Lindbladian [58, Supplement]

$$\mathcal{L}_{\text{eff}} = \mathcal{P}_{\infty} \mathcal{O} \mathcal{P}_{\infty} - \mathcal{P}_{\infty} \mathcal{O}_1 \mathcal{L}^{-1} \mathcal{O}_1 \mathcal{P}_{\infty} \,, \tag{7}$$

where $\mathcal{L}^{-1}$ is the Drazin pseudoinverse [28, Eq. (D4)] and the asymptotic projection $\mathcal{P}_{\infty} = \mathcal{I} - \mathcal{L}\mathcal{L}^{-1}$ (with $\mathcal{I}$ identity) projects onto all steady states of $\mathcal{L}$ [30, 45]. The above expression is not particularly illuminating as it is not in Lindblad form. However, since $\mathcal{L}_{\text{eff}}$ is a Lindbladian, it must be expressible in terms of some effective Hamiltonian H_{eff}, jump operators F^{ℓ}_{eff}, and/or a completely positive (CP) map $\mathcal{E}_{\text{eff}}$ and its adjoint $\mathcal{E}^{\ddagger}_{\text{eff}}$ [24, Prop. 5], all depending on the unperturbed pieces $\{H, F^{\ell}\}$ and perturbations $\{V, f^{\ell}\}$. Generally, the expressions may not be simple and the dependence not explicit, but we are able to express $\mathcal{L}$ in Lindblad form given the following assumptions. We assume $\mathcal{L}$ admits a unique DFS $I_{▘}$ and that (A) the unperturbed Hamiltonian acts only on the decaying subspace ($H = H_{▗}$) and (B) unperturbed jump operators map decaying states directly into the DFS ($F^{\ell} = F^{\ell}_{▝}$). We assume these hold from now on, noting there are no restrictions on $\{V, f^{\ell}\}$; see Fig. 1(c) for an example. To simplify $\mathcal{L}_{\text{eff}}$, we introduce Kamiltonians

$$K = H - \tfrac{i}{2} \sum_{\ell} F^{\ell\dagger} F^{\ell} \tag{8a}$$

$$K_{\text{eff}} = V_{▚} - \tfrac{i}{2} \sum_{\ell} \left(F^{\ell\dagger} f^{\ell}_{▘} + f^{\ell\dagger}_{▘} F^{\ell}\right) \,. \tag{8b}$$

As we have seen, $K = K_{▗}$ and its corresponding superoperator

$$\mathcal{K}(\cdot) \equiv -i\left(K(\cdot) - (\cdot) K^{\dagger}\right)$$

govern evolution within ▗ [45, Sec. 2.1.3]. As we will see shortly, pieces of the effective Kamiltonian $K_{\text{eff}} = (K_{\text{eff}})_{\text{▖}}$ map one out of and into the DFS. We picked K_{eff} to depend only on $\{V_{\text{▖}}, f_{\text{▖}}\}$ because $\{V_{\text{▘}}, f_{\text{▘}}\}$ participate differently and $\{V_{\text{▗}}, f_{\text{▗}}\}$ do not feature to this order. The resulting simplified $\mathcal{L}_{\text{eff}}$ (7) is as follows [55].

Proposition (EJOF). *Let $\mathcal{L}$ be a Lindbladian with a unique DFS $I_{\text{▘}}$, Hamiltonian $H_{\text{▗}}$, and jump operators $\{F^{\ell}_{\text{▝}}\}$ (5). Perturb $\mathcal{L}$ with a Hamiltonian V and jump perturbations $\{f^{\ell}\}$ (6). The effective Lindbladian (7) within the DFS is*

$$\mathcal{L}_{\text{eff}}(\cdot) = -i[H_{\text{eff}}, (\cdot)] + \sum_{\ell} \mathcal{D}[F^{\ell}_{\text{eff}}](\cdot) + \mathcal{E}_{\text{eff}}(\cdot) - \tfrac{1}{2}\left\{\mathcal{E}^{\ddagger}_{\text{eff}}(I), (\cdot)\right\}, \tag{9}$$

where the effective Hamiltonian, jumps, and CP map are

$$H_{\text{eff}} = \tfrac{1}{2}\left(V_{\text{▘}} - K_{\text{eff}} K^{-1} K_{\text{eff}}\right) + H.c. \tag{10a}$$

$$F^{\ell}_{\text{eff}} = f^{\ell}_{\text{▘}} - F^{\ell} K^{-1} K_{\text{eff}} \tag{10b}$$

$$\mathcal{E}_{\text{eff}}(\cdot) = -\sum_{\ell,\ell'} F^{\ell'} \mathcal{K}^{-1}\left(f^{\ell}_{\text{▖}}(\cdot) f^{\ell\dagger}_{\text{▖}}\right) F^{\ell'\dagger}. \tag{10c}$$

This effective jump-operator formalism (EJOF) reduces to the EOF [33] (see also [36, Lemma 3]) when $f^{\ell} = 0$ (and $V_{\text{▗}} = 0$). Therefore, the EOF, derived via adiabatic elimination, can alternatively be derived using time-independent perturbation theory [55].

The first term (10a) represents the resulting coherent evolution within the DFS. It consists of $V_{\text{▘}}$, a 1st-order effect, and the effective Hamiltonian $K_{\text{eff}} K^{-1} K_{\text{eff}} + H.c.$ reminiscent of Hamiltonian perturbation theory. In the latter, K_{eff} (8b) maps states in the kernel (▘) of K into the range (▗) using both coherent ($V_{\text{▖}}$) and dissipative ($F^{\dagger\ell} f^{\ell}_{\text{▘}}$) terms, returning via $V_{\text{▝}}$ and $f^{\ell\dagger}_{\text{▘}} F$, respectively, with the "energy" denominator determined by K^{-1}. Thus there are cross-terms consisting of leaving via dissipation and returning via a Hamiltonian and visa-versa. Interestingly, the participating dissipative perturbation $f^{\ell}_{\text{▘}}$ cannot map one out of the DFS, instead conspiring with $F^{\dagger\ell}$ to provide the dissipative analogue of $V_{\text{▘}}$. A similar story occurs in the effective jump F^{ℓ}_{eff} (10b) and is the key reason behind the highlighted cancellation. The first part of F^{ℓ}_{eff} comes from the first piece $\mathcal{P}_{\infty} O_2 \mathcal{P}_{\infty} = \sum_{\ell} \mathcal{D}[f^{\ell}_{\text{▘}}]$ in Eq. (7), which is itself a Lindbladian. However, the second piece $-\mathcal{P}_{\infty} O_1 \mathcal{L}^{-1} O_1 \mathcal{P}_{\infty}$, which surprisingly is *not* a Lindbladian, contributes the interference term $F^{\ell} K^{-1} K_{\text{eff}}$. This term consists of leaving the DFS through $(K_{\text{eff}})_{\text{▖}}$ and returning to the DFS via F^{ℓ} while paying an "energy" penalty determined by the eigenvalues of K. The third term (10c) [with $\mathcal{E}^{\ddagger}_{\text{eff}}(I) = \sum_{\ell} f^{\ell\dagger}_{\text{▖}} f^{\ell}_{\text{▖}}$] results from a nonzero $f^{\ell}_{\text{▖}}$, mapping one out of the DFS and recovering via F^{ℓ}, with "energy" denominator determined by the *superoperator* $\mathcal{K}^{-1}(\cdot) \neq K^{-1}(\cdot) K^{-1\dagger}$. This term has no analogue in Hamiltonian 2nd-order perturbation theory because it directly connects ▘ to ▗ via one instance of $f_{\text{▖}}$. If K is diagonalizable, we can easily express $\mathcal{K}^{-1}$ using the eigendecomposition of K. However, this formalism remains valid even for non-diagonalizable K [55].

Coherent cancellation.—In our previous examples, we assumed $H = V = 0$ since any initial coherent evolution spoils the interference effect. We now expand those examples to nonzero Hamiltonians $H \neq 0 \neq V$, showing how to restore the interference spoiled by H with a judicious choice of V. We maintain conditions (2,4) and let $f^{\ell}_{\text{▖}} = 0$, so only $\{H_{\text{eff}}, F^{\ell}_{\text{eff}}\}$ contribute to $\mathcal{L}_{\text{eff}}$ (10). The presence of H in K (8a) means that $F^{\ell} K^{-1} F^{\ell'\dagger}$ is no longer the DFS identity and $F^{\ell}_{\text{eff}} \neq 0$. However, since the return to the DFS in F^{ℓ}_{eff} occurs via dissipation only, $V_{\text{▝}}$ does not contribute to F_{eff}. Exploiting this effect, we pick

$$V = \tfrac{i}{2} \sum_{\ell} \left(F^{\ell\dagger} f^{\ell} - f^{\ell\dagger} F^{\ell}\right) + \tilde{V} \tag{11}$$

to cancel the $F^{\ell\dagger} f^{\ell}$ term in K_{eff}, leaving us with $(K_{\text{eff}})_{\text{▖}} = \tilde{V}_{\text{▖}}$ that is dependent only on the coherent perturbation. Picking $\tilde{V}_{\text{▖}} = K \sum_{\ell} (F^{\ell\dagger} F^{\ell})^{-1} F^{\ell\dagger} f^{\ell} + H.c.$, the K out front cancels the K^{-1} in F^{ℓ}_{eff} and removes $f^{\ell}_{\text{▘}}$ via the same effect as that in Eq. (3). In other words, if $H \neq 0$, one can use a particular coherent perturbation to cancel leading-order effects due to unwanted jump perturbations $f^{\ell}_{\text{▘}}$ [55].

Universal dissipation.—In a quick detour from canceling unwanted dissipation, let us instead use a customizable V to see what possible dissipation within ▘ we can generate (c.f. [35]). We assume to have full control over the perturbations, showing that restricting them to $\{V_{\text{▘}}, f^{\ell}_{\text{▘}}\}$ allows universal dissipation within the DFS. First, by letting $\tilde{V} = 0$ in Eq. (11), we cancel K_{eff}-dependent terms in both $\{H_{\text{eff}}, F^{\ell}_{\text{eff}}\}$ (10a-b) and obtain $\mathcal{L}_{\text{eff}} = \{V_{\text{▘}}, f^{\ell}_{\text{▘}}\}$. Second, letting d be the dimension of the DFS, a general $\mathcal{L}_{\text{eff}}$ has $d^2 - 1$ jump operators $\{f^{\ell}\}$. Therefore, if $\mathcal{L}$ has at least $d^2 - 1$ independent jump operators F^{ℓ}, $\mathcal{L}_{\text{eff}}$ generates any dissipation within ▘.

Continuous error-correction.—In conventional QEC, one starts out in a logical state located in the codespace (▘) and attempts to correct errors caused by an error channel $\mathcal{E}$ by acting with a recovery channel $\mathcal{R}$. Ideally, $\mathcal{E}$ consists of correctable noise, so $\mathcal{R}\mathcal{E} = \mathcal{I}$ [59, 60]; we focus on a similar case here. We consider a continuous QEC, where one has the ability to correct noise via an infinitesimal version of the recovery $\mathcal{L} = \mathcal{R} - \mathcal{I}$ [61], whose jumps are the Kraus operators $\{R^{\ell}\}$ of $\mathcal{R}$ (and we additionally removed the DFS-identity Kraus operator $R^0 \propto I_{\text{▘}}$). Instead of perturbing $\mathcal{L}$ with another Lindbladian representing external noise, we consider perturbations to the jump operators of $\mathcal{L}$, which represent *miscalibration* of the recovery itself. Such noise is important since a recovery map is never perfect in real life. It consists of detectable errors $f^{\ell}_{\text{▖}}$ (which we assume are correctable by $\mathcal{L}$), undetectable errors $f^{\ell}_{\text{▘}}$ (which are not correctable since they act nontrivially *within* the codespace [60]), recovery errors $f^{\ell}_{\text{▝}}$, and correctable errors $f^{\ell}_{\text{▗}}$. The EJOF shows that small imperfections of *all* types do not harm the quantum information.

Since $\mathcal{L} = \mathcal{R} - \mathcal{I}$ comes from a recovery operation, each jump F^{ℓ} is an isometry from a subspace of ▗ (corresponding to a distinct error syndrome) into the codespace. Such F^ℓ automatically satisfy conditions (2,4) and, since $\mathcal{R}$ is a channel from ◲ to ◰, $\sum_\ell F^{\ell\dagger} F^\ell = I_{◲}$ [45, Sec. 2.1.4]. Let us further assume that miscalibrations mapping out of the codespace form a channel, $\mathcal{E}(\cdot) = \sum_\ell f^\ell_{◱}(\cdot) f^{\ell\dagger}_{◱}$, consisting of correctable noise, i.e., $\mathcal{R}\mathcal{E}(\rho) \propto \rho$ for all $\rho \in$ ◰. Application of the EJOF results in the following.

Corollary. *Let $\mathcal{L} = \mathcal{R} - \mathcal{I}$ with corresponding recovery channel $\mathcal{R}(\cdot) = \sum_\ell F^\ell (\cdot) F^\ell$ such that $\{F^\ell = F^\ell_{◳}\}$ satisfy conditions (2,4). Assume small miscalibrations $\{f^\ell\}$ in the recovery, $F^\ell \to F^\ell + f^\ell$, such that the pieces $\{f^\ell_{◱}\}$ form a noise channel correctable by $\mathcal{R}$. To leading order, the miscalibrations $\{f^\ell\}$ do not induce errors within the codespace,*

$$\mathcal{L}_{\text{eff}} = 0\,. \tag{12}$$

To prove the above, we have to show that each line in Eq. (10) is zero. First, the simple structure of $\mathcal{L}$ lets us simplify all Kamiltonian inverses: $K = -\frac{i}{2} I_{◲}$ and $\mathcal{K}(\rho_{◲}) = -\rho_{◲}$ for any $\rho_{◲}$. Plugging this into H_{eff} (10a) and using condition (4) yields $H_{\text{eff}} = 0$. Similarly, and most surprisingly, the interference effect discussed above cancels the undetectable miscalibrations, yielding $F^\ell_{\text{eff}} = 0$ (10b). Lastly, simplifications to K and the condition on $\{f^\ell_{◱}\}$ yield the trivial CP map (10c), $\mathcal{E}_{\text{eff}}(\rho) = -\mathcal{R}\mathcal{K}^{-1}\mathcal{E}(\rho) = \mathcal{R}\mathcal{E}(\rho) \propto \rho$ for all $\rho \in$ ◰.

The above robustness corollary shows that even undetectable miscalibrations $f^\ell_{◰}$ in continuous error-recovery operations do not affect the codespace. Qualitatively, it is a statement that holds for any recovery $\mathcal{R}$ that maps into the codespace after one action and is applied rapidly (allowing us to consider $\mathcal{L} = \mathcal{R} - \mathcal{I}$). For example, the statement holds for continuous recoveries for the three-qubit repetition [62] and binomial [63] codes. For the former, $I_{◰} = |000\rangle\langle 000| + |111\rangle\langle 111|$, and its jumps $F^\ell = I_{◰} X^\ell$ (for qubits $\ell \in \{1, 2, 3\}$ and $\{X, Y, Z\}$ the usual Pauli matrices) satisfy conditions (2,4). Terms $f^\ell \propto X^\ell$ are corrected by the continuous recovery while $f^\ell \propto Z^\ell$ are canceled out due to interference, despite being undetectable by the code. The terms $f^\ell \propto Y^\ell$ cannot be corrected since X^ℓ and Y^ℓ are not simultaneously correctable; picking a code correcting both solves this problem.

We cannot make the same statement about all recovery operations since the assumptions of the EJOF no longer hold. The assumption $F^\ell = F^\ell_{◳}$ amounts to the jumps recovering all states (in their range) back into the codespace *after one action*. Another set of cases is where $F^\ell = F^\ell_{◲}$ does not map all states immediately into the codespace, but instead keeps certain states uncorrected (i.e., in ◲) after one action by, e.g., only correcting errors occurring in a localized region [45, Sec. 3.4]. Such systems include local recoveries for topological codes and the above corollary unfortunately *does not apply* to them. Similarly, we cannot guarantee robustness when there is an inherent Hamiltonian ($H_{◲} \neq 0$). While we can still use V to coherently cancel any undetectable miscalibrations $f^\ell_{◰}$ (so that $F^\ell_{\text{eff}} = 0$) as in the coherent cancellation example above, the presence of $\mathcal{K}^{-1}$ in $\mathcal{E}_{\text{eff}}$ (10c) obstructs us from being able to correct any detectable errors $f^\ell_{◱}$. So $\mathcal{L}_{\text{eff}} = 0$ only when either $f^\ell_{◱} = 0 \neq H$ or visa versa.

Conclusion.—We develop an effective jump-operator formalism to tackle general perturbations to a particular class of Lindbladians relevant in quantum optics and error correction. We explicitly solve for the effective Lindbladian $\mathcal{L}_{\text{eff}}$ governing perturbation-induced evolution within the steady-state subspace of an unperturbed Lindbladian $\mathcal{L}$. Using this formalism, we uncover an interference effect that is a generalized version of the interference observed in dark-state physics. This interference occurs in generic Lindbladians of the type we study and can be applied to show that Lindbladian-based error-correction operations are robust to both detectable *and* undetectable calibration noise. While this interference is destroyed when the unperturbed system has a Hamiltonian piece, it can be reinstated with a certain Hamiltonian perturbation. This formalism also provides a simple way to realize universal Lindbladian simulation.

We thank Mikhail D. Lukin, Jacob P. Covey, Richard Kueng, John Preskill, Liang Jiang, Paola Cappellaro, and Mădălin Guţă for illuminating discussions. We acknowledge financial support from the Walter Burke Institute for Theoretical Physics at Caltech (V.V.A.), the Korea Foundation for Advanced Studies (K.N.), and a Feodor-Lynen fellowship from the Alexander von Humboldt-Foundation (F.R.).

---

* valbert4@gmail.com

[1] J. F. Poyatos, J. I. Cirac, and P. Zoller, *Quantum Reservoir Engineering with Laser Cooled Trapped Ions,* Phys. Rev. Lett. **77**, 4728 (1996).

[2] J. Preskill, *Quantum Computing in the NISQ era and beyond,* Quantum **2**, 79 (2018).

[3] C. Altafini, *Coherent control of open quantum dynamical systems,* Phys. Rev. A **70**, 062321 (2004).

[4] B. Kraus, H. P. Büchler, S. Diehl, A. Kantian, A. Micheli, and P. Zoller, *Preparation of entangled states by quantum Markov processes,* Phys. Rev. A **78**, 042307 (2008).

[5] G. Dirr, U. Helmke, I. Kurniawan, and T. Schulte-Herbrüggen, *Lie-semigroup structures for reachability and control of open quantum systems: kossakowski-lindblad generators form lie wedge to markovian channels,* Rep. Math. Phys. **64**, 93 (2009).

[6] F. Ticozzi and L. Viola, *Analysis and synthesis of attractive quantum Markovian dynamics,* Automatica **45**, 2002 (2009).

[7] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, J. M. Petersen, J. I. Cirac, and E. S. Polzik, *Entanglement generated by dissipation and steady state entanglement of two macroscopic objects.* Phys. Rev. Lett. **107**, 080503 (2011).

[8] S. Diehl, A. Micheli, A. Kantian, B. Kraus, H. P. Büchler, and P. Zoller, *Quantum states and phases in driven open quantum systems with cold atoms,* Nat. Phys. **4**, 878 (2008).

[9] T. Prosen and I. Pižorn, *Quantum Phase Transition in a Far-from-Equilibrium Steady State of an X Y Spin Chain,* Phys. Rev. Lett. **101**, 105701 (2008).

[10] F. Verstraete, M. M. Wolf, and J. I. Cirac, *Quantum computation and quantum-state engineering driven by dissipation,* Nat. Phys. **5**, 633 (2009).

[11] J. J. García-Ripoll, S. Dürr, N. Syassen, D. M. Bauer, M. Let-

tner, G. Rempe, and J. I. Cirac, *Dissipation-induced hard-core boson gas in an optical lattice,* New J. Phys. **11**, 013053 (2009).

[12] J. Dengis, R. Konig, and F. Pastawski, *An optimal dissipative encoder for the toric code,* New J. Phys. **16**, 013023 (2014).

[13] A. Beige, D. Braun, B. Tregenna, and P. L. Knight, *Quantum Computing Using Dissipation to Remain in a Decoherence-Free Subspace,* Phys. Rev. Lett. **85**, 1762 (2000).

[14] P. Facchi and S. Pascazio, *Quantum Zeno Subspaces,* Phys. Rev. Lett. **89**, 080401 (2002).

[15] C. Arenz, D. Burgarth, P. Facchi, V. Giovannetti, H. Nakazato, S. Pascazio, and K. Yuasa, *Universal Control Induced by Noise,* Phys. Rev. A **93**, 062308 (2016).

[16] S. Touzard, A. Grimm, Z. Leghtas, S. Mundhada, P. Reinhold, C. Axline, M. Reagor, K. Chou, J. Blumoff, K. Sliwa, S. Shankar, L. Frunzio, R. J. Schoelkopf, M. Mirrahimi, and M. Devoret, *Coherent Oscillations inside a Quantum Manifold Stabilized by Dissipation,* Phys. Rev. X **8**, 021005 (2018).

[17] N. Boulant, L. Viola, E. M. Fortunato, and D. G. Cory, *Experimental implementation of a concatenated quantum error-correcting code.* Phys. Rev. Lett. **94**, 130501 (2005).

[18] J. T. Barreiro, M. Müller, P. Schindler, D. Nigg, T. Monz, M. Chwalla, M. Hennrich, C. F. Roos, P. Zoller, and R. Blatt, *An open-system quantum simulator with trapped ions,* Nature **470**, 486 (2011).

[19] V. Paulisch, H. J. Kimble, and A. González-Tudela, *Universal quantum computation in waveguide QED using decoherence free subspaces,* New J. Phys. **18**, 043041 (2016).

[20] M. Mirrahimi, Z. Leghtas, V. V. Albert, S. Touzard, R. J. Schoelkopf, L. Jiang, and M. H. Devoret, *Dynamically protected cat-qubits: a new paradigm for universal quantum computation,* New J. Phys. **16**, 045014 (2014).

[21] Z. Leghtas, S. Touzard, I. M. Pop, A. Kou, B. Vlastakis, A. Petrenko, K. M. Sliwa, A. Narla, S. Shankar, M. J. Hatridge, M. Reagor, L. Frunzio, R. J. Schoelkopf, M. Mirrahimi, and M. H. Devoret, *Confining the state of light to a quantum manifold by engineered two-photon loss,* Science (80-.). **347**, 853 (2015).

[22] F. Reiter, A. S. Sørensen, P. Zoller, and C. A. Muschik, *Dissipative quantum error correction and application to quantum sensing with trapped ions,* Nat. Commun. **8**, 1822 (2017).

[23] A. A. Belavin, B. Ya. Zel'dovich, A. M. Perelomov, and V. S. Popov, *Relaxation of Quantum Systems with Equidistant Spectra,* Sov. Phys. JETP-USSR **29**, 145 (1969).

[24] G. Lindblad, *On the generators of quantum dynamical semigroups,* Commun. Math. Phys. **48**, 119 (1976).

[25] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *Completely positive dynamical semigroups of N-level systems,* J. Math. Phys. **17**, 821 (1976).

[26] T. Banks, L. Susskind, and M. E. Peskin, *Difficulties for the evolution of pure states into mixed states,* Nucl. Phys. B **244**, 125 (1984).

[27] D. Chruściński and S. Pascazio, *A Brief History of the GKLS Equation,* Open Syst. Inf. Dyn. **24**, 1740001 (2017).

[28] P. Zanardi and L. Campos Venuti, *Coherent Quantum Dynamics in Steady-State Manifolds of Strongly Dissipative Systems,* Phys. Rev. Lett. **113**, 240406 (2014).

[29] P. Zanardi and L. Campos Venuti, *Geometry, robustness, and emerging unitarity in dissipation-projected dynamics,* Phys. Rev. A **91**, 052324 (2015).

[30] V. V. Albert, B. Bradlyn, M. Fraas, and L. Jiang, *Geometry and Response of Lindbladians,* Phys. Rev. X **6**, 041031 (2016).

[31] D. J. Atkins, H. M. Wiseman, and P. Warszawski, *Approximate master equations for atom optics,* Phys. Rev. A **67**, 023802 (2003).

[32] M. Mirrahimi and P. Rouchon, *Singular Perturbations and Lindblad-Kossakowski Differential Equations,* IEEE Trans. Automat. Contr. **54**, 1325 (2009).

[33] F. Reiter and A. S. Sørensen, *Effective operator formalism for open quantum systems,* Phys. Rev. A **85**, 032111 (2012).

[34] E. M. Kessler, *Generalized Schrieffer-Wolff formalism for dissipative systems,* Phys. Rev. A **86**, 012126 (2012).

[35] P. Zanardi, J. Marshall, and L. Campos Venuti, *Dissipative universal Lindbladian simulation,* Phys. Rev. A **93**, 022312 (2016).

[36] R. Azouit, A. Sarlette, and P. Rouchon, in *2016 IEEE 55th Conf. Decis. Control* (IEEE, 2016) pp. 4559–4565.

[37] R. Azouit, F. Chittaro, A. Sarlette, and P. Rouchon, *Structure-preserving adiabatic elimination for open bipartite quantum systems,* IFAC-PapersOnLine **50**, 13026 (2017).

[38] R. Azouit, F. Chittaro, A. Sarlette, and P. Rouchon, *Towards generic adiabatic elimination for bipartite open quantum systems,* Quantum Sci. Technol. **2**, 044011 (2017).

[39] V. Popkov, S. Essink, C. Presilla, and G. Schütz, *Effective quantum Zeno dynamics in dissipative quantum systems,* (2018), arXiv:1806.10422.

[40] S. Das, V. E. Elfving, F. Reiter, and A. S. Sørensen, *Photon scattering from a system of multilevel quantum emitters. I. Formalism,* Phys. Rev. A **97**, 043837 (2018).

[41] S. Das, V. E. Elfving, F. Reiter, and A. S. Sørensen, *Photon scattering from a system of multilevel quantum emitters. II. Application to emitters coupled to a one-dimensional waveguide,* Phys. Rev. A **97**, 043838 (2018).

[42] P. Forni, A. Sarlette, T. Capelle, E. Flurin, S. Deléglise, and P. Rouchon, *Adiabatic elimination for multi-partite open quantum systems with non-trivial zero-order dynamics,* (2018), arXiv:1803.07810.

[43] R. Blume-Kohout, H. K. Ng, D. Poulin, and L. Viola, *Information-preserving structures: A general framework for quantum zero-error information,* Phys. Rev. A **82**, 062306 (2010).

[44] B. Baumgartner and H. Narnhofer, *Analysis of quantum semigroups with GKS–Lindblad generators II. General,* J. Phys. A Math. Theor. **41**, 32 (2008).

[45] V. V. Albert, *Lindbladians with multiple steady states: theory and applications*, Ph.D. thesis, Yale University (2017).

[46] L.-M. Duan and G.-C. Guo, *Preserving Coherence in Quantum Computation by Pairing Quantum Bits,* Phys. Rev. Lett. **79**, 1953 (1997).

[47] P. Zanardi and M. Rasetti, *Noiseless Quantum Codes,* Phys. Rev. Lett. **79**, 3306 (1997).

[48] D. A. Lidar, I. L. Chuang, and K. B. Whaley, *Decoherence-Free Subspaces for Quantum Computation,* Phys. Rev. Lett. **81**, 2594 (1998).

[49] H. Schempp, G. Günter, S. Wüster, M. Weidemüller, and S. Whitlock, *Correlated Exciton Transport in Rydberg-Dressed-Atom Spin Chains,* Phys. Rev. Lett. **115**, 093002 (2015).

[50] D. W. Schönleber, A. Eisfeld, M. Genkin, S. Whitlock, and S. Wüster, *Quantum Simulation of Energy Transport with Embedded Rydberg Aggregates,* Phys. Rev. Lett. **114**, 123005 (2015).

[51] E. Davis, G. Bentsen, and M. Schleier-Smith, *Approaching the Heisenberg Limit without Single-Particle Detection,* Phys. Rev. Lett. **116**, 053601 (2016).

[52] R. Sweke, I. Sinayskiy, and F. Petruccione, *Dissipative preparation of large W states in optical cavities,* Phys. Rev. A **87**, 042323 (2013).

[53] T. Ramos, B. Vermersch, P. Hauke, H. Pichler, and P. Zoller, *Non-Markovian dynamics in chiral quantum networks with*

spins and photons, Phys. Rev. A **93**, 062104 (2016).
[54] The jump $F + f$ annihilates the dark state $|\psi\rangle = \sqrt{\Gamma}|1\rangle - \sqrt{\gamma}|e\rangle$, which forms a DFS along with $|0\rangle$. For $\Gamma \gg \gamma$, $|\psi\rangle \propto |1\rangle$.
[55] See Supplemental Material for a proof of the EJOF and an ancillary MATHEMATICA file for randomly generated examples.
[56] H. J. Carmichael, *Statistical Methods in Quantum Optics 2: Non-classical fields* (Springer-Verlag, Berlin/Heidelberg, 2008).
[57] C. Gardiner and P. Zoller, *Quantum Noise* (Springer, Berlin/Heidelberg, 2000).
[58] K. Macieszczak, M. Guţă, I. Lesanovsky, and J. P. Garrahan, *Towards a Theory of Metastability in Open Quantum Dynamics,* Phys. Rev. Lett. **116**, 240404 (2016).
[59] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2011).
[60] J. Preskill, Lecture notes on Quantum Computation, (2004).
[61] F. Pastawski, L. Clemente, and J. I. Cirac, *Quantum memories based on engineered dissipation,* Phys. Rev. A **83**, 012304 (2011).
[62] M. Ippoliti, L. Mazza, M. Rizzi, and V. Giovannetti, *Perturbative approach to continuous-time quantum error correction,* Phys. Rev. A **91**, 042322 (2015).
[63] J.-M. Lihm, K. Noh, and U. R. Fischer, *Implementation-independent sufficient condition of the Knill-Laflamme type for the autonomous protection of logical qudits by strong engineered dissipation,* Phys. Rev. A **98**, 012317 (2018).

APPENDIX: PROOF OF THE EJOF

Proposition. *Let $\mathcal{L}$ be a Lindbladian with a unique DFS $I_{\text{▘}}$, Hamiltonian $H_{\text{▗}}$, and jump operators $\{F^{\ell}_{\text{▝}}\}$. Perturb $\mathcal{L}$ with Hamiltonian V and jump perturbations $\{f^{\ell}\}$. The effective Lindbladian (S5) within the DFS is*

$$\mathcal{L}_{\text{eff}}(\cdot) = -i[H_{\text{eff}}, (\cdot)] + \sum_{\ell} \mathcal{D}[F^{\ell}_{\text{eff}}](\cdot) + \mathcal{E}_{\text{eff}}(\cdot) - \tfrac{1}{2}\left\{\mathcal{E}^{\ddagger}_{\text{eff}}(I), (\cdot)\right\}, \tag{S1}$$

where the effective Hamiltonian, jumps, and CP map are

$$H_{\text{eff}} = \tfrac{1}{2}\left(V_{\text{▘}} - K_{\text{eff}} K^{-1} K_{\text{eff}}\right) + H.c. \tag{S2a}$$

$$F^{\ell}_{\text{eff}} = f^{\ell}_{\text{▘}} - F^{\ell} K^{-1} K_{\text{eff}} \tag{S2b}$$

$$\mathcal{E}_{\text{eff}}(\cdot) = -\sum_{\ell,\ell'} F^{\ell'} \mathcal{K}^{-1}\left(f^{\ell}_{\text{▖}}(\cdot) f^{\ell\dagger}_{\text{▖}}\right) F^{\ell'\dagger}. \tag{S2c}$$

A similar proof of the EOF [33] using open-system perturbation theory was performed in Ref. [45], Sec. 4.3.5. The adjoint of a superoperator $\mathcal{E}(\cdot) = \sum_i A_i (\cdot) B_i^{\dagger}$ is $\mathcal{E}^{\ddagger}(\cdot) \equiv \sum_i A_i^{\dagger} (\cdot) B_i$. The perturbation $\mathcal{O}$ to $\mathcal{L}$ consists of contributions from V and f^{ℓ} [44, Sec. 6.1]. Let us conveniently split $\mathcal{O}$ into various superoperators responsible for different processes. First, define the generalized commutator $[A, B]^{\star} \equiv AB - BA^{\dagger}$ and Kamiltonians $K = H - \frac{i}{2}\sum_{\ell} F^{\ell\dagger} F^{\ell}$ and $K_{\text{eff}} \equiv V_{\text{▞}} - \frac{i}{2}\sum_{\ell}\left(f^{\ell\dagger}_{\text{▘}} F + F^{\ell\dagger} f^{\ell}_{\text{▝}}\right)$. Then, construct the superoperators

$$\mathcal{V}(\cdot) = -i\left[V_{\text{▚}} - \frac{i}{2}\sum_{\ell}(f^{\ell\dagger}_{\text{▝}} F + F^{\ell\dagger} f^{\ell}_{\text{▝}}), (\cdot)\right]^{\star} \tag{S3a}$$

$$\mathcal{K}_{\text{eff}}(\cdot) = -i[K_{\text{eff}}, (\cdot)]^{\star} \tag{S3b}$$

$$\mathcal{F}(\cdot) = \sum_{\ell}\left(F^{\ell}(\cdot) f^{\ell\dagger} + f^{\ell}(\cdot) F^{\ell\dagger}\right). \tag{S3c}$$

Split $\mathcal{O} = \mathcal{O}_1 + \mathcal{O}_2$ with $\mathcal{O}_1$ containing one instance of either f^{ℓ} or V in each term and $\mathcal{O}_2$ containing two:

$$\mathcal{O}_1 = \mathcal{V} + \mathcal{K}_{\text{eff}} + \mathcal{F} \tag{S4a}$$

$$\mathcal{O}_2 = \sum_{\ell} \mathcal{D}[f^{\ell}]. \tag{S4b}$$

Second-order perturbation theory within the DFS yields the effective Lindbladian [58, Supplement]

$$\mathcal{L}_{\text{eff}} = \mathcal{P}_{\infty}\mathcal{O}\mathcal{P}_{\infty} - \mathcal{P}_{\infty}\mathcal{O}_1\mathcal{K}^{-1}\mathcal{O}_1\mathcal{P}_{\infty} \equiv \mathcal{T}_1 + \mathcal{T}_2. \tag{S5}$$

We have simplified $\mathcal{L}^{-1}$ to $\mathcal{K}^{-1}$ in the second term $\mathcal{T}_2$ due to the assumption that there is no additional dissipation within ▗, $F^{\ell}_{\text{▗}} = 0$ [45, Sec. 2.1.3]. As opposed to Hamiltonian perturbation theory, here the asymptotic projection $\mathcal{P}_{\infty}$ [30, 45] corresponds to a quantum channel arising from the infinite-time limit of evolution due to $\mathcal{L}$, $\mathcal{P}_{\infty} = \lim_{t\to\infty} e^{t\mathcal{L}}$. This channel is trace-preserving, so it is not merely acting on the DFS since it has to map states initially in ▗ into the DFS. We use an analytical formula for it [30, Prop. 3], which for this particular DFS case is

$$\mathcal{P}_{\infty}(\cdot) = \mathcal{P}_{\text{▘}}(\cdot) - \mathcal{P}_{\text{▘}}\mathcal{L}\mathcal{L}^{-1}_{\text{▗}}(\cdot) = \mathcal{P}_{\text{▘}}(\cdot) - \sum_{\ell} F^{\ell}\mathcal{K}^{-1}_{\text{▗}}(\cdot) F^{\ell\dagger}. \tag{S6}$$

Above, the four-corners projection superoperators are $\mathcal{P}_{\text{▘}}(\cdot) = I_{\text{▘}}(\cdot) I_{\text{▘}}$ and $\mathcal{P}_{\text{▗}}(\cdot) = I_{\text{▗}}(\cdot) I_{\text{▗}}$, and $\mathcal{A}_{\boxplus} \equiv \mathcal{P}_{\boxplus}\mathcal{A}\mathcal{P}_{\boxplus}$ given any square combination $\boxplus$. Above, we have substituted $\mathcal{L}^{-1}$ for $\mathcal{K}^{-1}$ and used $\mathcal{P}_{\text{▘}}\mathcal{L}\mathcal{P}_{\text{▗}}(\cdot) = \sum_{\ell} F^{\ell}(\cdot) F^{\ell\dagger}$ [45, Eq. (2.8)]. We use this block notation to derive the EJOF, introducing the remaining four-corners projectors $\mathcal{P}_{\text{▝}}(\cdot) = I_{\text{▘}}(\cdot) I_{\text{▗}}$ and $\mathcal{P}_{\text{▖}}(\cdot) = I_{\text{▗}}(\cdot) I_{\text{▘}}$, noting that they are orthogonal and can add (e.g., ▞ ≡ ▝ + ▖). Most importantly, note that

$$\mathcal{P}_{\infty} = \mathcal{P}_{\text{▘}}\mathcal{P}_{\infty} = \mathcal{P}_{\infty}\mathcal{P}_{\text{▚}} = \mathcal{P}_{\text{▘}}\mathcal{P}_{\infty}\mathcal{P}_{\text{▚}}, \tag{S7}$$

so $\mathcal{P}_{\infty}$ maps all states into ▘ and destroys knowledge of all coherences ▞ between the DFS and the decaying states.

1. The term $\mathcal{T}_1$

Inserting $1 = \mathcal{P}_{\text{▚}} + \mathcal{P}_{\text{▞}}$ and using Eq. (S7), we have

$$\mathcal{T}_1 = (\mathcal{P}_\infty \mathcal{P}_{\text{▚}})\, \mathcal{O}\, (\mathcal{P}_{\text{▘}} \mathcal{P}_\infty) = \mathcal{P}_\infty\, (\mathcal{O}_{\text{▘}} + \mathcal{P}_{\text{▗}} \mathcal{O} \mathcal{P}_{\text{▘}})\, \mathcal{P}_\infty = \mathcal{O}_{\text{▘}} + \mathcal{P}_\infty \mathcal{P}_{\text{▗}} \mathcal{O} \mathcal{P}_{\text{▘}}\,, \tag{S8}$$

so we only need two superoperator elements, $\mathcal{O}_{\text{▘}}$ and $\mathcal{P}_{\text{▗}}\mathcal{O}\mathcal{P}_{\text{▘}}$, for this term. Note that we have applied $\mathcal{P}_\infty \mathcal{P}_{\text{▘}} = \mathcal{P}_{\text{▘}}$ and replaced the rightmost $\mathcal{P}_\infty$ with $\mathcal{P}_{\text{▘}}$ since the states we are perturbing are in ▘. The former element is a projection of $\mathcal{O}$ onto the DFS while the latter is a leakage term into the decaying space. These elements are listed below for all of the terms $\mathcal{A} \in \{\mathcal{V}, \mathcal{K}_{\text{eff}}, \mathcal{F}, \mathcal{D}[f^\ell]\}$ of $\mathcal{O}$.

$\mathcal{A}$	$\mathcal{A}_{\text{▘}}$	$\mathcal{P}_{\text{▗}}\mathcal{A}\mathcal{P}_{\text{▘}}$
$\mathcal{V}$	$-i[V_{\text{▘}}, (\cdot)]$	0
$\mathcal{K}_{\text{eff}}$	0	0
$\mathcal{F}$	0	0
$\mathcal{D}[f^\ell]$	$\mathcal{D}[f^\ell_{\text{▘}}]\,(\cdot) - \frac{1}{2}\{f^{\ell\dagger}_{\text{▖}} f^\ell_{\text{▖}}, (\cdot)\}$	$f^\ell_{\text{▖}}\,(\cdot)\, f^{\ell\dagger}_{\text{▖}}$

Luckily, $\mathcal{P}_{\text{▗}}\mathcal{A}\mathcal{P}_{\text{▘}} = 0$ for $\mathcal{A} \in \{\mathcal{K}, \mathcal{V}\}$ due to the fact that their constituents act from one side at a time (the no-leak property; see [30, Sec. I.B]). Also, $\mathcal{P}_{\text{▗}}\mathcal{F}\mathcal{P}_{\text{▘}} = 0$ since its constituent $F^\ell = F^\ell_{\text{▝}}$ cannot map one into ▗ by construction. From the above table, we see that $\mathcal{V}$ contributes the first term in H_{eff} (S2a) and $\mathcal{D}[f^\ell]$ contributes the dissipator $\mathcal{D}[f^\ell_{\text{▘}}]$. We cannot yet combine all $f^\ell_{\text{▖}}$ terms because we still need to act on $\mathcal{P}_{\text{▗}}\mathcal{D}[f^\ell]\mathcal{P}_{\text{▘}}$ with $\mathcal{P}_\infty$ (S6):

$$\sum_\ell \mathcal{P}_\infty \mathcal{P}_{\text{▗}} \mathcal{D}[f^\ell] \mathcal{P}_{\text{▘}}\,(\cdot) = -\sum_{\ell,\ell'} F^{\ell'} \mathcal{K}^{-1}_{\text{▗}} \left(f^\ell_{\text{▖}}\,(\cdot)\, f^{\ell\dagger}_{\text{▖}} \right) F^{\ell'\dagger} \equiv \mathcal{E}_{\text{eff}}\,(\cdot)\ . \tag{S9}$$

This provides the first term for the $\mathcal{E}_{\text{eff}}$-dependent part of $\mathcal{L}_{\text{eff}}$ (S1). To complete the derivation of $\mathcal{T}_1$, we need to prove that $\mathcal{E}^\ddagger_{\text{eff}}(I) = \sum_\ell f^{\ell\dagger}_{\text{▖}} f^\ell_{\text{▖}}$. The anticommutator term should be $\mathcal{E}^\ddagger_{\text{eff}}(I_{\text{▘}})$ since $\mathcal{E}_{\text{eff}}$ is a channel from ▘ to itself, but padding with $I_{\text{▗}}$ doesn't make any difference and looks simpler. Note that $H = H_{\text{▚}}$ commutes with $I_{\text{▗}}$ and so $\mathcal{K}^\ddagger(I_{\text{▗}}) = -\sum_{\ell'} F^{\ell'\dagger} F^{\ell'}$. Plugging this into $\mathcal{E}^\ddagger_{\text{eff}}(I)$ cancels the $\mathcal{K}^{-1\ddagger}$, yielding

$$\mathcal{E}^\ddagger_{\text{eff}}(I) = \sum_\ell f^{\ell\dagger}_{\text{▖}} \mathcal{K}^{-1\ddagger} \left(-\sum_{\ell'} F^{\ell'\dagger} F^{\ell'} \right) f^\ell_{\text{▖}} = \sum_\ell f^{\ell\dagger}_{\text{▖}} \mathcal{K}^{-1\ddagger} \mathcal{K}^\ddagger(I_{\text{▗}}) f^\ell_{\text{▖}} = \sum_\ell f^{\ell\dagger}_{\text{▖}} f^\ell_{\text{▖}}\,. \tag{S10}$$

This provides the anticommutator term for the $\mathcal{E}_{\text{eff}}$-dependent part of $\mathcal{L}_{\text{eff}}$ (S1). We are left with the K_{eff}-dependent terms in H_{eff} (S2a) and F_{eff} (S2b), which come from $\mathcal{T}_2$.

2. The term $\mathcal{T}_2$

This term is more difficult since two actions of the perturbation are present. We likewise need to determine which superoperator elements are required for the calculation. Since $\mathcal{K}^{-1}$ does not act on ▘ ($\mathcal{K}^{-1} = \mathcal{K}^{-1}_{\text{▞}}$), the first part of $\mathcal{T}_2$ is $\mathcal{K}^{-1}\mathcal{O}_1\mathcal{P}_\infty = \mathcal{K}^{-1}\,(\mathcal{P}_{\text{▞}}\mathcal{O}_1\mathcal{P}_{\text{▘}} + \mathcal{P}_{\text{▗}}\mathcal{O}_1\mathcal{P}_{\text{▘}})\,\mathcal{P}_\infty$. However, we can see that $\mathcal{P}_{\text{▗}}\mathcal{O}_1\mathcal{P}_{\text{▘}} = 0$ from the previous table, so only $\mathcal{K}^{-1}_{\text{▞}}$ participates. Inserting this into $\mathcal{T}_2$ and using Eq. (S7) yields

$$\mathcal{T}_2 = -\,(\mathcal{P}_{\text{▘}}\mathcal{O}_1\mathcal{P}_{\text{▞}} + \mathcal{P}_\infty\mathcal{P}_{\text{▗}}\mathcal{O}_1\mathcal{P}_{\text{▞}})\,\mathcal{K}^{-1}\,(\mathcal{P}_{\text{▞}}\mathcal{O}_1\mathcal{P}_{\text{▘}})\ . \tag{S11}$$

Therefore, three elements are relevant; they are listed in the table below for all of the terms $\mathcal{A} \in \{\mathcal{V}, \mathcal{K}_{\text{eff}}, \mathcal{F}\}$ of $\mathcal{O}_1$:

$\mathcal{A}$	$\mathcal{P}_{\text{▞}}\mathcal{A}\mathcal{P}_{\text{▘}}$	$\mathcal{P}_{\text{▘}}\mathcal{A}\mathcal{P}_{\text{▞}}$	$\mathcal{P}_{\text{▗}}\mathcal{A}\mathcal{P}_{\text{▞}}$
$\mathcal{V}$	0	0	0
$\mathcal{K}_{\text{eff}}$	$-i[(K_{\text{eff}})_{\text{▖}}, (\cdot)]^\star$	$-i[(K_{\text{eff}})_{\text{▖}}, (\cdot)]^\star$	$-i[(K_{\text{eff}})_{\text{▝}}, (\cdot)]^\star$
$\mathcal{F}$	0	0	$\sum_\ell \left(F^\ell\,(\cdot)\, f^{\ell\dagger}_{\text{▘}} + f^\ell_{\text{▘}}\,(\cdot)\, F^{\ell\dagger} \right)$

The first part $\mathcal{K}^{-1}\,(\mathcal{P}_{\text{▞}}\mathcal{O}_1\mathcal{P}_{\text{▘}})$ in $\mathcal{T}_2$ (S11) is shared by all terms, so we simplify it first by noting that the superoperator inverse $\mathcal{K}^{-1}_{\text{▞}}$ can be written in terms of operator inverses due to the restriction $F^\ell = F^\ell_{\text{▝}}$ [45, Eq. (2.8)],

$$\mathcal{K}^{-1}_{\text{▞}}\,(\cdot) = \mathcal{K}^{-1}_{\text{▝}}\,(\cdot) + \mathcal{K}^{-1}_{\text{▖}}\,(\cdot) = -i\,(\cdot)\,K^{-1\dagger} + iK^{-1}\,(\cdot) = i[K^{-1}, (\cdot)]^\star\ . \tag{S12}$$

Plugging this and the first column of the above table into the first part of $\mathcal{T}_2$ yields

$$\mathcal{K}^{-1}\left(\mathcal{P}_{\text{▞}}\mathcal{O}_1\mathcal{P}_{\text{▘}}\right)(\cdot) = \left[K^{-1}, [(K_{\mathsf{eff}})_{\text{▖}}, (\cdot)]^{\star}\right]^{\star} = K^{-1}K_{\mathsf{eff}}(\cdot) + H.c.\,, \tag{S13}$$

where we remember that the state $(\cdot) \in \text{▘}$ and only $(K_{\mathsf{eff}})_{\text{▖}}$ can map $(\cdot)$ into ▗ (so that K^{-1} acts on the result). In the last equality, we let $(K_{\mathsf{eff}})_{\text{▖}} \to K_{\mathsf{eff}}$ since adding $(K_{\mathsf{eff}})_{\text{▝}}$ does not make any difference, i.e., $K^{-1}(K_{\mathsf{eff}})_{\text{▝}} = 0$. Now let us plug this simplified first part as well as all of the nonzero terms from the table into $\mathcal{T}_2$:

$$\mathcal{T}_2 = -\left(\mathcal{P}_{\text{▘}}\mathcal{K}_{\mathsf{eff}}\mathcal{P}_{\text{▞}} + \mathcal{P}_{\text{▘}}\mathcal{F}\mathcal{P}_{\text{▞}} + \mathcal{P}_{\infty}\mathcal{P}_{\text{▗}}\mathcal{K}_{\mathsf{eff}}\mathcal{P}_{\text{▞}}\right)\left(K^{-1}K_{\mathsf{eff}}(\cdot) + H.c.\right). \tag{S14}$$

We now determine the contribution coming from each of the three terms in the leftmost parentheses. Using the above table and substituting $(K_{\mathsf{eff}})_{\text{▘}} \to K_{\mathsf{eff}}$ in the first line below, the first two terms are simple:

$$-\mathcal{P}_{\text{▘}}\mathcal{K}_{\mathsf{eff}}\mathcal{P}_{\text{▞}}\left(K^{-1}K_{\mathsf{eff}}(\cdot) + H.c.\right) = i\left[K_{\mathsf{eff}}K^{-1}K_{\mathsf{eff}}, (\cdot)\right]^{\star} \tag{S15a}$$

$$-\mathcal{P}_{\text{▘}}\mathcal{F}\mathcal{P}_{\text{▞}}\left(K^{-1}K_{\mathsf{eff}}(\cdot) + H.c.\right) = -\sum_{\ell}\left(F^{\ell}K^{-1}K_{\mathsf{eff}}(\cdot)f_{\text{▘}}^{\ell\dagger} + H.c.\right). \tag{S15b}$$

For the third term, note first this curious formula that we will use to eliminate the inverse coming from $\mathcal{P}_{\infty}\mathcal{P}_{\text{▗}}$:

$$\mathcal{K}^{-1}\left(i[K^{-1}, (\cdot)]^{\star}\right) = \mathcal{K}^{-1}\mathcal{K}\left(K^{-1}(\cdot)K^{-1\dagger}\right) = K^{-1}(\cdot)K^{-1\dagger} \tag{S16}$$

for any operator $(\cdot) \in \text{▗}$. Plugging in Eq. (S6) and applying the above formula yields

$$-\mathcal{P}_{\infty}\mathcal{P}_{\text{▗}}\mathcal{K}_{\mathsf{eff}}\mathcal{P}_{\text{▞}}\left(K^{-1}K_{\mathsf{eff}}(\cdot) + H.c.\right) = \sum_{\ell}F^{\ell}K^{-1}K_{\mathsf{eff}}(\cdot)K_{\mathsf{eff}}^{\dagger}K^{-1\dagger}F^{\ell\dagger}. \tag{S17}$$

3. Combining $\mathcal{T}_1$ and $\mathcal{T}_2$

Plugging the $\mathcal{E}_{\mathsf{eff}}$-dependent terms (S9,S10), all $V_{\text{▘}}$- and $f_{\text{▘}}^{\ell}$-dependent terms in the first table above, and Eqs. (S15,S17) yields the effective Lindbladian

$$\begin{aligned}\mathcal{L}_{\mathsf{eff}}(\cdot) = -i[H_{\mathsf{eff}}, (\cdot)] + \mathcal{E}_{\mathsf{eff}}(\cdot) - \left\{\mathcal{E}_{\mathsf{eff}}^{\ddagger}(I_{\text{▘}}), (\cdot)\right\} + \tfrac{i}{2}\left\{K_{\mathsf{eff}}K^{-1}K_{\mathsf{eff}} - H.c., (\cdot)\right\} \\ + \sum_{\ell}\mathcal{D}[f_{\text{▘}}^{\ell}](\cdot) + F^{\ell}K^{-1}K_{\mathsf{eff}}(\cdot)K_{\mathsf{eff}}^{\dagger}K^{-1\dagger}F^{\ell\dagger} - \left(F^{\ell}K^{-1}K_{\mathsf{eff}}(\cdot)f_{\text{▘}}^{\ell\dagger} + H.c.\right),\end{aligned} \tag{S18}$$

where we have absorbed the Hermitian part of $K_{\mathsf{eff}}K^{-1}K_{\mathsf{eff}}$ into H_{eff}. Remarkably, the last term in the second line and the third line simplify to $\sum_{\ell}\mathcal{D}[F_{\mathsf{eff}}^{\ell}]$. Collecting all of the terms in the second line that act nontrivially from both sides into F_{eff} makes this more clear, leaving only the anticommutator term $\sum_{\ell}F_{\mathsf{eff}}^{\ell\dagger}F_{\mathsf{eff}}^{\ell}$ to be determined from the last term below:

$$\mathcal{L}_{\mathsf{eff}}(\cdot) = -i[H_{\mathsf{eff}}, (\cdot)] + \mathcal{E}_{\mathsf{eff}}(\cdot) - \{\mathcal{E}_{\mathsf{eff}}^{\ddagger}(I_{\text{▘}}), (\cdot)\} + \sum_{\ell}F_{\mathsf{eff}}^{\ell}(\cdot)F_{\mathsf{eff}}^{\ell\dagger} - \tfrac{1}{2}\left\{\sum_{\ell}f_{\text{▘}}^{\ell\dagger}f_{\text{▘}}^{\ell} - i\left(K_{\mathsf{eff}}K^{-1}K_{\mathsf{eff}} - H.c.\right), (\cdot)\right\}. \tag{S19}$$

Let us now write $K_{\mathsf{eff}} = V_{\text{▘}} + \frac{i}{2}G_{\text{▘}} + H.c.$, where $G_{\text{▘}} = \sum_{\ell}F^{\ell\dagger}f_{\text{▘}}^{\ell}$. We abbreviate $V_{\text{▘}}^{\dagger} \equiv (V_{\text{▘}})^{\dagger}$ and similarly for G. Plugging this into K_{eff} and simplifying yields

$$\begin{aligned}-i\left(K_{\mathsf{eff}}K^{-1}K_{\mathsf{eff}} - H.c.\right) = &-iV_{\text{▘}}^{\dagger}\left(K^{-1} - K^{-1\dagger}\right)V_{\text{▘}} + \tfrac{i}{4}G_{\text{▘}}^{\dagger}\left(K^{-1} - K^{-1\dagger}\right)G_{\text{▘}} \\ &- \tfrac{1}{2}G_{\text{▘}}^{\dagger}\left(K^{-1} + K^{-1\dagger}\right)V_{\text{▘}} - \tfrac{1}{2}V_{\text{▘}}^{\dagger}\left(K^{-1} + K^{-1\dagger}\right)G_{\text{▘}}.\end{aligned} \tag{S20}$$

We obtain the same for $\sum_{\ell}F_{\mathsf{eff}}^{\ell\dagger}F_{\mathsf{eff}}^{\ell} - f_{\text{▘}}^{\ell\dagger}f_{\text{▘}}^{\ell}$ to finish the proof. For this, we have to use another curious identity that is proven using the definition of K,

$$\sum_{\ell}K^{-1\dagger}F^{\ell\dagger}F^{\ell}K^{-1} = -i\left(K^{-1} - K^{-1\dagger}\right). \tag{S21}$$

Plugging this in, splitting K_{eff} into $V_{\text{▘}}$ and $G_{\text{▘}}$, and simplifying yields

$$\sum_{\ell}F_{\mathsf{eff}}^{\ell\dagger}F_{\mathsf{eff}}^{\ell} - f_{\text{▘}}^{\ell\dagger}f_{\text{▘}}^{\ell} = -i\left(K_{\mathsf{eff}}K^{-1}K_{\mathsf{eff}} - H.c.\right). \tag{S22}$$