Asymptotics of quantum channels: application to matrix product states

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This work derives an analytical formula for the asymptotic state — the quantum state resulting from an infinite number of applications of a general quantum channel on some initial state. For channels admitting multiple fixed or rotating points, conserved quantities — the *left* fixed/rotating points of the channel — determine the dependence of the asymptotic state on the initial state. The analytical formula stems from results regarding conserved quantities, including the fact that, for any channel admitting a full-rank fixed point, conserved quantities commute with that channel's Kraus operators up to a phase. In the same way that asymptotic states depend on initial states, the thermodynamic limit of (noninjective) matrix product states (MPS) depends on the MPS boundary conditions. Expectation values of local observables in such MPS are calculated in the thermodynamic limit, showing that effects due to the interaction of "twisted" boundary conditions with decaying bond degrees of freedom can persist. It is shown that one can eliminate all decaying bond degrees of freedom and calculate the same expectation values using mixtures of MPS with reduced bond dimension and multiple boundary conditions.

I. QUANTUM CHANNELS

Quantum channels, or completely-positive trace-preserving maps, are the most general maps between quantum systems. They enjoy a diverse range of applications, primarily in the quantum information community [1], but also in studies of matrix product states [2, 3], entanglement renormalization [4, 5], computability theory [6], and even biological inference processes [7]. The canonical form of a quantum channel $\mathcal{A}$ and its adjoint $\mathcal{A}^{\ddagger}$ (a generalization of the Heisenberg picture defined under the Frobenius norm) is [8–10]

$$\mathcal{A}\left(\rho\right)=\sum_{\ell}A^{\ell}\rho A^{\ell\dagger}\qquad\text{and}\qquad\mathcal{A}^{\ddagger}\left(O\right)=\sum_{\ell}A^{\ell\dagger}OA^{\ell}\,,\tag{1}$$

where $\mathcal{A}$ acts on states ρ and $\mathcal{A}^{\ddagger}$ on operators O. The matrices A^{ℓ} are called the Kraus operators of $\mathcal{A}\equiv\left\{A^{\ell}\right\}$, eq. (1) is the Kraus form of $\mathcal{A}$, and the only requirement for the channel to be trace preserving is (for I identity)

$$\sum_{\ell}A^{\ell\dagger}A^{\ell}=I\,.\tag{2}$$

Quantum channels can be represented as matrices acting on a vectorized density matrix, i.e., the $D\times D$ matrix ρ written as a D^2-dimensional vector. Vectorization essentially "flips" the bra part in each of the outer products making up ρ and $\mathcal{A}$ is written as a $D^2\times D^2$ matrix of the form $\hat{A}=\sum_{\ell}A^{\ell}\otimes A^{\ell\star}$ acting on the vectorized ρ strictly from the left. This *matrix or Liouville representation* of $\mathcal{A}$ [11] is equivalent to the Kraus representation (1), and I slightly abuse notation by ignoring hats and not distinguishing the two.

In the matrix representation, channels can be studied in terms of their eigenvalues and eigenmatrices. The eigenvalues of all channels are contained in the unit disk, and this work focuses on the eigenvalues/matrices Ψ on the periphery of that disk, i.e.,

$$\mathcal{A}\left(\Psi\right)=e^{i\Delta}\Psi\qquad\text{for some real }\Delta\,.\tag{3}$$

Such eigenmatrices are called the channel's (right) *rotating points*, and those with $\Delta=0$ are called *fixed points*. The Ψ's do not have to be physical states themselves, but they are a matrix basis for such states. Since $\mathcal{A}$ may not be diagonalizable, the eigenmatrices J of its adjoint — left rotating points — may be different from Ψ:

$$\mathcal{A}^{\ddagger}\left(J\right)=e^{-i\Delta}J\,.\tag{4}$$

Left rotating points will be called *conserved quantities* because their expectation value is either constant or oscillates with successive powers of $\mathcal{A}$, but does not decay:

$$\text{Tr}\{J\mathcal{A}^{n}(\rho)\}=\text{Tr}\{\mathcal{A}^{\ddagger n}(J)\rho\}=e^{-in\Delta}\text{Tr}\{J\rho\}\,.\tag{5}$$

The general block structure of Ψ's is already well-known [12–16], and here the focus is on the structure of the J's. It is important to note that there are as many conserved quantities as there are rotating points (more technically, the Jordan normal form of $\mathcal{A}$ contains only trivial Jordan blocks for all eigenvalues on the periphery of the unit disk; see, e.g., Prop 6.2 in Ref. [17]).

In the limit of many applications of $\mathcal{A}$, all eigenmatrices with eigenvalues not on the periphery of the unit disk will become irrelevant and all that will be left of the channel is the projection onto the subspace spanned by the rotating points. The collective effect of many applications of $\mathcal{A}$ is quantified by the channel's *asymptotic* *projection* $\mathcal{P}_{\mathcal{A}}$,

$$\mathcal{P}_{\mathcal{A}}(\rho) \equiv \lim_{n\rightarrow\infty} \mathcal{A}^{\alpha n}(\rho)\,, \tag{6}$$

which projects onto the eigenspace of the peripheral spectrum of the channel. The extra parameter α allows one to take the limit in such a way as to remove the eigenvalues $e^{i\varDelta}$ arising from application of $\mathcal{A}$ on ρ. For any $\varDelta = \frac{2\pi}{N}n$ (for some positive integers n, N), rotating points of $\mathcal{A}$ are fixed points of $\mathcal{A}^N$, so one simply takes $\alpha = N$ to get rid of the extra phases. Other $\varDelta$ which are not rational multiples of 2π can similarly be removed to arbitrary accuracy [12, 17, 18] by remembering that irrational numbers are limits of sequences of rationals. The above limit is a direct generalization of the large time limit of Markovian/Lindbladian channels $\mathcal{A}_t = e^{t\mathcal{L}}$ for some Lindbladian $\mathcal{L}$. However, in that case, $\lim_{t\rightarrow\infty} e^{t\mathcal{L}}$ can produce residual unitary evolution which cannot be removed by clever manipulation of the limit.

The asymptotic projection is expressible in terms of (superoperator) projections onto the eigenspaces of the rotating points,

$$\mathcal{P}_{\mathcal{A}}(\rho) = \sum_{\varDelta,\mu} \varPsi_{\varDelta\mu} \mathrm{Tr}\left\{J^{\varDelta\mu\dagger}\rho\right\}\,, \tag{7}$$

where the rotating points are indexed by their eigenvalue $e^{i\varDelta}$ and μ counts any degeneracies for each $\varDelta$. In that sense, conserved quantities are as important as fixed points despite being less well-understood. Conveniently, the rotating points and their corresponding conserved quantities can be made biorthogonal, $\mathrm{Tr}\{J^{\varDelta\mu\dagger}\varPsi_{\varTheta\nu}\} = \delta_{\varDelta\varTheta}\delta_{\mu\nu}$. The $\varPsi$'s thus determine the basis elements of a generalized Bloch vector [19, 20] of the asymptotic state $\mathcal{P}_{\mathcal{A}}(\rho)$ while the J's determine the coefficients of said Bloch vector. The biorthogonality condition easily implies that $\mathcal{P}_{\mathcal{A}}$ is really a projection — $\mathcal{P}_{\mathcal{A}}^2 = \mathcal{P}_{\mathcal{A}}$.

If a channel has a unique fixed point $\varPsi$ and no rotating points, then the unique conserved quantity is the identity (due to the necessity of trace preservation) and $\mathcal{P}_{\mathcal{A}}(\rho) = \varPsi\mathrm{Tr}\{\rho\} = \varPsi$. Channels with more non-trivial $\mathcal{P}_{\mathcal{A}}$ are therefore those with multiple fixed or rotating points. As a simple example of such a channel, consider $\mathcal{A} = \{A\}$ acting on 2×2 matrices with one Kraus operator $A = \mathrm{diag}\{1, e^{i\theta}\}$. Such a channel sports two fixed points, the identity and the Pauli matrix Z, and two rotating points $\sigma_\pm$ with eigenvalues $\varDelta = \pm\theta$. In fact, since there is only one Kraus operator, such a channel is actually unitary. For a non-unitary example, set $\theta = \pi$ (so $A = Z$) and add the Pauli matrix X as another Kraus operator [normalizing both A's by $\frac{1}{\sqrt{2}}$ to satisfy trace preservation (2)]. This channel has the identity as the unique fixed point and Y as the only rotating point with $\varDelta = \pi$. Since both Kraus operators are Hermitian, the left and right fixed points are the same; we will see examples when they are not later. Other examples of $\mathcal{P}_{\mathcal{A}}$ come from recovery maps in quantum error-correction, which take a state which has undergone an error and project it back into the protected subspace of the quantum code [21].

	FP unique?	∃ full-rank FP?	∃ rot. point?
ergodic [22–24]	Yes		
faithful [here]		Yes	
irreducible [17, 25]	Yes	Yes	
mixing [24]	Yes		No
primitive [17, 26]	Yes	Yes	No

Table I. Some types of channels; FP=fixed point. A blank entry means there is no requirement for that definition. For semigroups, mixing is also known as relaxing [27] and faithful is also known as minimal [28]. Primitive is equivalent to strongly irreducible [26].

II. STRUCTURE OF CONSERVED QUANTITIES

A. Faithful channels

The first part focuses on channels that do not contain a decaying subspace. This means that no populations $|\psi\rangle\langle\psi|$ decay completely to zero under many applications of the channel: $\langle\psi|\mathcal{P}_{\mathcal{E}}(|\psi\rangle\langle\psi|)|\psi\rangle \neq 0$ for all states $|\psi\rangle$, a channel $\mathcal{E}$, and its asymptotic projection $\mathcal{P}_{\mathcal{E}}$. Equivalently, the channel has to have one fixed point ρ which is of full rank ($\langle\psi|\rho|\psi\rangle > 0$ for all $|\psi\rangle$). The structural differences between such channels and channels which do admit decay warrant a special definition:

Definition. A channel $\mathcal{E} \equiv \{E_\ell\}$ is *faithful* if it admits a full-rank (i.e., faithful) fixed point ρ. In other words,

$$\exists\,\rho > 0 \;\text{ such that }\; \mathcal{E}(\rho) = \rho\,. \tag{8}$$

Here, I always use $\mathcal{E}$ to denote faithful channels and later show how $\mathcal{E}$ can be extended to channels $\mathcal{A}$ which act on a larger Hilbert space and admit a decaying subspace. In this sense, $\mathcal{E}$ is the faithful channel of $\mathcal{A}$. Note that the number of fixed points is independent of this condition, and Table I relates this definition to others.

The first result is regarding the relationship between the conserved quantities J and the Kraus of operators of $\mathcal{E}$. It is a generalization of a theorem for fixed points of faithful channels [12, 29–31], which states that a conserved quantity J with eigenvalue $\varDelta = 0$ commutes with all of the Kraus operators. It is shown that conserved quantities with $\varDelta \neq 0$ commute up to a phase. For the aforementioned example $\mathcal{E} = \{E\}$ with $E = \mathrm{diag}\{1, e^{i\theta}\}$, the conserved quantity σ_+ satisfies $\sigma_+ E = e^{-i\theta} E\sigma_+$. This turns out to be true for all faithful channels and reduces to known results for ergodic channels ([27], Thm. 9). It can be proven using Thms. 4.1-4.2 and Corollary 4.3 in Ref. [32]; a more direct proof is in the appendix.

Proposition 1. *Let $\mathcal{E}=\{E_\ell\}$ be a faithful channel. Let J be a conserved quantity of $\mathcal{E}$, i.e., $\mathcal{E}^{\ddagger}(J)=e^{-i\Delta}J$ for some real Δ. Then, for all ℓ,*

$$JE_\ell = e^{-i\Delta}E_\ell J\,. \tag{9}$$

Assuming $\mathcal{E}^{\ddagger}(J_1)=e^{-i\Delta_1}$ and $\mathcal{E}^{\ddagger}(J_2)=e^{-i\Delta_2}$, Eq. (9) easily implies that $\mathcal{E}^{\ddagger}(J_1J_2)=e^{-i(\Delta_1+\Delta_2)}J_1J_2$. Combined with the fact that there must be $\leq D^2$ conserved quantities, this implies that there are some constraints on Δ such that there remain finitely many eigenvalues. This brings us to the second result about the eigenvalues of a specific subset of conserved quantities.

Each conserved quantity $J=J_{\mathsf{dgn}}+J_{\mathsf{nil}}$ can be decomposed into a diagonalizable part J_{dgn} and a nilponent part J_{nil} [33] ($J_{\mathsf{nil}}^N=0$ for $N\leq D$, the dimension of the Hilbert space). While Δ can be an irrational multiple of 2π for strictly nilpotent J, it turns out that $e^{i\Delta}$ are Nth roots of unity for all diagonalizable J with $N\leq D$. In other words, given any conserved quantity J, J^D is either a zero, the identity, or a projection. This extends similar results ([17], Thm. 6.6; [2], Prop. 3.3; [34], Corr. 3) to faithful channels. It is not, however, as thorough a characterization of the peripheral spectrum as Ref. [18], Thm. 9.

Proposition 2. *Let $\mathcal{E}=\{E_\ell\}$ be a faithful channel. Let J_{dgn} be such that $\mathcal{E}^{\ddagger}(J_{\mathsf{dgn}})=e^{-i\Delta}J_{\mathsf{dgn}}$ for some real Δ and assume J_{dgn} is diagonalizable. Then, there exists an integer n such that*

$$\Delta=\frac{2\pi}{N}n \quad \textit{for some} \quad N\leq D\,. \tag{10}$$

Let us assume a unitary conserved quantity, $J^\dagger J=JJ^\dagger=I$, and show that the above two propositions extend known results ([17], Prop. 6.7) from irreducible to faithful channels. Proposition 2 readily implies that $\mathcal{E}$ is covariant (more specifically, invariant or symmetric) under J,

$$J\mathcal{E}(\rho)J^\dagger=\mathcal{E}(J\rho J^\dagger) \quad \forall\rho\,, \tag{11}$$

so conserved quantities are symmetries of the channel. Proposition 2 implies that $J^{N\leq D}=I$, so the set $\{J^n\}_{n=0}^{N-1}$ forms the symmetry group $\mathbb{Z}_N$. Note that the symmetry group is never infinite for finite dimension D. Generalizing this, the set of unitary conserved quantities thus forms a finite group under which $\mathcal{E}$ is covariant. This is a one-way Noether-type theorem linking conserved quantities to symmetries (see Ref. [35] or Ref. [36], Ch. 2.6, for the semigroup analogue). This cannot be extended to a two-way theorem because symmetries of a channel are not always conserved quantities. A simple counterexample is the channel $\mathcal{E}=\{X/\sqrt{2},Z/\sqrt{2}\}$, for which the Hadamard operation H taking $X\leftrightarrow Z$ is a symmetry, but is not conserved [$\mathcal{E}^{\ddagger}(H)=0$].

B. General channels

Now let us extend faithful channels to channels which do not contain a full-rank fixed point. While Props. 1-2 break down for general channels, the extension below implies that, for every general channel, there is a corresponding faithful channel for which they hold.

Any faithful channel $\mathcal{E}=\{E_\ell\}$ can be extended to a channel $\mathcal{A}=\{A^\ell\}$ which contains a decaying subspace (also, transient subspace [37]). Specifically, the Kraus operators of $\mathcal{A}$ are

$$A^\ell=\begin{pmatrix}E_\ell & A^\ell_{\text{◳}}\\ 0 & A^\ell_{\text{◲}}\end{pmatrix}\equiv\begin{pmatrix}A^\ell_{\text{◰}} & A^\ell_{\text{◳}}\\ 0 & A^\ell_{\text{◲}}\end{pmatrix}. \tag{12}$$

The dimensions of the square matrices E_ℓ and $A^\ell_{\text{◲}}$ can differ, and the bounds of ℓ can change by padding the same E with two different pairs of matrices in ◳ ("upper right") and ◲ ("lower right") to make two different A's. The zero matrix in ◱ is necessary to make sure that ◰ is the largest invariant subspace; thus, all rotating points of $\mathcal{A}$ are the same as those of $\mathcal{E}$. In addition, $\mathcal{A}$ needs to be a legitimate channel, i.e., satisfy eq. (2). Writing out the A^ℓ's in blocks [as in eq. (12)] yields the conditions

$$\sum_\ell A^{\ell\dagger}_{\text{◰}}A^\ell_{\text{◰}}=P \tag{13a}$$

$$\sum_\ell A^{\ell\dagger}_{\text{◰}}A^\ell_{\text{◳}}=0 \tag{13b}$$

$$\sum_\ell (A^\ell_{\text{◳}})^\dagger A^\ell_{\text{◳}}+A^{\ell\dagger}_{\text{◲}}A^\ell_{\text{◲}}=Q\,, \tag{13c}$$

where Q is the projection on ◲ and $P=I-Q$ is the projection onto ◰ (with $\mathrm{Tr}\{P\}\equiv D$). For each faithful channel $\mathcal{E}$, there are an infinite number of possible extensions $\mathcal{A}$. Conversely, an arbitrary channel $\mathcal{A}$ either is a faithful channel or contains one. The remaining two completely positive maps associated with this decomposition of $\mathcal{A}$, $\{A^\ell_{\text{◳}}\}$ and $\{A^\ell_{\text{◲}}\}$, are both trace-decreasing.

Now let us develop the required notation. Just like P and Q split the Hilbert space into two parts, they can be used to split the space of operators on a Hilbert space into four "corners" {◰, ◳, ◱, ◲} [28]. Each of the four corners corresponds to its own superoperator projection. For example,

$$\mathcal{P}_{\text{◳}}(O)\equiv POQ\equiv O_{\text{◳}} \tag{14}$$

for any operator O. The other three projections are defined accordingly. One can graphically determine which corner a product of operators belongs to by multiplying their blocks as matrices (e.g., $A_{\text{◱}}B_{\text{◳}}\in$ ◲). Moreover, the four-corners projections add graphically ($\mathcal{P}_{\text{◰}}+\mathcal{P}_{\text{◲}}\equiv\mathcal{P}_{\text{▚}}$) and are Hermitian ($\mathcal{P}^{\ddagger}_{\boxplus}=\mathcal{P}_{\boxplus}$). Analogous to studying operators in terms of their matrix elements, one can study superoperators in terms of their four-corners decomposition. For example,

$$\mathcal{P}_{▘}\mathcal{A}\mathcal{P}_{▗}(\rho) = P\mathcal{A}\left(Q\rho Q\right)P = \sum_{\ell} A^{\ell}_{▝}\rho_{▗}(A^{\ell}_{▝})^{\dagger} \tag{15}$$

is the map $\{A^{\ell}_{▝}\}$ which transfers $\rho_{▗}$ from ▗ to ▘. "Diagonal" elements are denoted as $\mathcal{A}_{⊞} \equiv \mathcal{P}_{⊞}\mathcal{A}\mathcal{P}_{⊞}$ for convenience, so the faithful channel $\mathcal{E} \equiv \mathcal{P}_{▘}\mathcal{A}\mathcal{P}_{▘}$ and similarly $\{A^{\ell}_{▗}\} \equiv \mathcal{P}_{▗}\mathcal{A}\mathcal{P}_{▗}$.

With conditions (12) and (13), $\mathcal{A}$ contains a decaying subspace of dimension $\mathrm{Tr}\{Q\}$ and the same rotating points as $\mathcal{E}$. But what about the conserved quantities? Those are not the same because, by trace preservation, they need to make sure that all state populations (and sometimes some coherences) in ▗ are transferred to ▘. For example, the identity is (always) a conserved quantity of $\mathcal{A}$, but the analogous conserved quantity of $\mathcal{E}$ is P. Denoting the conserved quantities of $\mathcal{E}$ as $J_{▘}$, it will now be shown how to extend them to form J, the conserved quantities of $\mathcal{A}$. Having defined this notation, it is easy to write out the conserved quantities of the extended channel $\mathcal{A}$.

Proposition 3. *The conserved quantities of $\mathcal{A}$ corresponding to eigenvalues $e^{i\Delta}$ are*

$$J = J_{▘} + J_{▗} = J_{▘} - (\mathcal{A}^{\ddagger}_{▗} - e^{-i\Delta})^{-1}\mathcal{A}^{\ddagger}(J_{▘})\,, \tag{16}$$

where $J_{▘}$ are conserved quantities of $\mathcal{A}_{▘} = \mathcal{E}$.

An important corollary of the above proposition is that $J_{▞} = 0$. After plugging in this formula for J into $\mathcal{P}_{\mathcal{A}}$ (7), this means that the asymptotic projection has only two pieces:

$$\mathcal{P}_{\mathcal{A}} = \mathcal{P}_{▘}\mathcal{P}_{\mathcal{A}}\mathcal{P}_{▚} \equiv \mathcal{P}_{\mathcal{E}} + \mathcal{P}_{\mathcal{A}}\mathcal{P}_{▗}\,, \tag{17}$$

where the *faithful projection* (for semigroups, minimal projection [28])

$$\mathcal{P}_{\mathcal{E}}(\cdot) \equiv \mathcal{P}_{\mathcal{A}}\mathcal{P}_{▘}(\cdot) = \sum_{\Delta,\mu}\Psi_{\Delta\mu}\mathrm{Tr}\{J^{\Delta\mu\dagger}_{▘}\cdot\} \tag{18}$$

is the asymptotic projection of the faithful channel $\mathcal{E}$. The piece $\mathcal{P}_{\mathcal{E}}$ is responsible for preserving parts of an initial state ρ which is in ▘ while the piece $\mathcal{P}_{\mathcal{A}}\mathcal{P}_{▗}$ is a channel mapping states from ▗ onto the subspaces spanned by the rotating points of $\mathcal{A}$, all located in ▘. The key result here is that the rotation induced by Δ, besides inducing phases on the rotating points, also contributes to the decay of information from ▗ into ▘. Namely, the inverse of the piece $(\mathcal{A}^{\ddagger} - e^{-i\Delta})_{▗}$ modulates the decoherence induced during the decay in a way that depends on how close the eigenvalues of $\mathcal{A}_{▗}$ are to the phases $e^{i\Delta}$

$$\mathcal{P}_{\mathcal{A}}\mathcal{P}_{▗}(\rho) = -\sum_{\Delta,\mu}\Psi_{\Delta\mu}\mathrm{Tr}\left\{J^{\Delta\mu\dagger}_{▘}\left[\mathcal{A}(\mathcal{A} - e^{i\Delta})^{-1}_{▗}\right](\rho_{▗})\right\}\,, \tag{19}$$

where the superoperator in square brackets acts on $\rho_{▗}$. The $\Delta = 0$ case reduces to known results ([12], Lemma 5.8; [38], Prop. 7),

$$\mathcal{P}_{\mathcal{A}}\mathcal{P}_{▗} = \mathcal{P}_{\mathcal{E}}\mathcal{A}\left(\mathcal{I} - \mathcal{A}\right)^{-1}_{▗}\,, \tag{20}$$

where $(\mathcal{I} - \mathcal{A})^{-1}_{▗}$ (with $\mathcal{I}$ the superoperator identity) can be thought of as the quantum version of the fundamental matrix from classical Markov chains [39]. These formulas also reduce to the Lindbladian result ([28], Prop. 3) if we let $\mathcal{A} = e^{\mathcal{L}} \to \mathcal{I} + \mathcal{L}$ for some Lindbladian $\mathcal{L}$ and $e^{-i\Delta} \to 1 - i\Delta$. In the Lindblad case, some dependence on Δ can be canceled by properly tuning $\mathcal{L}_{▗}$ ([36], Sec. 3.2.3).

III. APPLICATION: INFORMATION PRESERVING STRUCTURES

This section lists some uses of the above result and includes an algorithm that outputs a properly organized $\mathcal{P}_{\mathcal{A}}$ given a channel $\mathcal{A}$.

A. Asymptotic probabilities

Expounding on the above, eq. (19) allows us to find the asymptotic [38] (also, reachability [37]) probabilities of a given initial state ρ to reach a particular subspace of ▘. The new result here is determination of the *coherences* reached by ρ, assuming knowledge of the left ($J^{\Delta\mu}_{▘}$) and right ($\Psi_{\Delta\mu}$) rotating points of $\mathcal{E}$. To show this, recall that the $\Psi_{\Delta\mu}$'s can be made orthonormal, $\mathrm{Tr}\{\Psi^{\dagger}_{\Delta\mu}\Psi_{\Theta\nu}\} = \delta_{\Delta\Theta}\delta_{\mu\nu}$. (Loosely speaking, this is because the Ψ's are a matrix basis used to write all asymptotic density matrices and so must be well-behaved; for more rigor, see Sec. III C.) To determine the coefficient in front of the basis element $\Psi_{\Delta\mu}$ in the asymptotic state $\rho_{\infty} = \mathcal{P}_{\mathcal{A}}(\rho)$, instead of applying $\mathcal{A}$ a sufficiently large number of times to determine $\mathcal{P}_{\mathcal{A}}$, simply calculate

$$\mathrm{Tr}\{\Psi^{\dagger}_{\Delta\mu}\rho_{\infty}\} = \mathrm{Tr}\left\{J^{\Delta\mu\dagger}_{▘}\left[\mathcal{I} - \mathcal{A}(\mathcal{A} - e^{i\Delta})^{-1}_{▗}\right](\rho)\right\}\,. \tag{21}$$

B. Error-correction of a decoherence-free subspace

Let us assume that now all of ▘ consists of rotating or fixed points, so $\mathcal{A}_{▘} = \mathcal{E}$ is a unitary channel. An example of this case is $\mathcal{A}_{▘} = \{E\}$, where $E = \mathrm{diag}\{1, e^{i\theta}\}$ is the Kraus operator that mentioned before. The necessary and sufficient condition on the A's for this to hold is

$$A^{\ell}_{▘} = a_{\ell}U \tag{22}$$

for some unitary U, real a_{ℓ}, and such that $\sum_{\ell}|a_{\ell}|^2 = 1$ to satisfy the condition (13a). Since there is no decay in

▘, that portion forms a *decoherence-free subspace* (DFS) [40] and $\mathcal{P}_{\mathcal{E}} = \mathcal{P}_{\text{▘}}$. The form of $A_{\text{▘}}$ also implies that $\mathcal{P}_{\text{▘}}\mathcal{A}\mathcal{P}_{\text{▞}} = 0$ and the statement of Prop. 1 implies that the rotating points reduce to being outer products of eigenstates of U.

The form (12) of A with the above restriction on $A_{\text{▘}}$ generalizes the previous DFS condition from eq. (11) of Ref. [41] (see also Refs. [42, 43] for different formulations). The difference is that now $A_{\text{▝}}$ does not have to be zero, so information from ▗ flows into the DFS ▘. For example, in quantum error-correction, ▘ is the logical subspace, ▗ is the orthogonal error subspace, and the piece $\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{▗}}$ plays the role of a "recovery channel" which attempts to recover the leaked information after an error [21]. It turns out one can remove the inverse term from $\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{▗}}$, putting the piece in Kraus form. Setting $A_{\text{▗}} = 0$ and $A_{\text{▘}} = P$ (unitary evolution within DFS is trivial) eliminates $\mathcal{A}_{\text{▗}}$ and reduces $\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{▗}}$ to the transfer map (15),

$$\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{▗}} = \mathcal{P}_{\text{▘}}\mathcal{A}\mathcal{P}_{\text{▗}}\,, \tag{23}$$

with Kraus operators $A_{\text{▝}}$. Condition (13b) on $A_{\text{▝}}$ reduces to $\sum_\ell A^\ell_{\text{▝}} = 0$, which is automatically satisfied by the set of operators $\{\pm A^\ell_{\text{▝}}/\sqrt{2}\}$. However, the channel created by those operators is the same as $\{A^\ell_{\text{▝}}\}$, so $\mathcal{P}_{\mathcal{A}}$ embeds an arbitrary recovery channel from the error subspace ▗ to code subspace ▘.

C. How to find $\mathcal{P}_{\mathcal{A}}$

In a more complicated case than a DFS, ▘ is factorized into a DFS and an auxiliary subspace, forming a *noiseless subsystem (NS)* [44]. Evolution on the DFS is still unitary while the auxiliary subspace contains one fixed and no rotating points. The Kraus operators for $\mathcal{E} = \mathcal{A}_{\text{▘}}$ are then $A^\ell_{\text{▘}} = U \otimes B^\ell$, where U acts on the DFS and B^ℓ are Kraus operators on the auxiliary space. This reduces to the DFS case (22) if the dimension of the auxiliary space is one. In the most general case, the rotating and fixed points of $\mathcal{E}$ can be block-diagonalized into a direct sum of blocks, with each block being an NS [12–16]. In that case, the Kraus operators can be written as

$$A^\ell_{\text{▘}} = \bigoplus_{\varkappa} U_\varkappa \otimes B^{\ell,\varkappa}\,, \tag{24}$$

where $U_\varkappa$ is unitary and the Kraus map $\{B^{\ell,\varkappa}\}_\ell$ for each $\varkappa$ is primitive (see Table I). This blocks-of-factors structure or *shape* of $A^\ell_{\text{▘}}$ is the most general form of an information-preserving structure [12] and has deep connections to the theory of matrix algebras [17]. The key to organizing the rotating points and conserved quantities is converting to a *canonical basis* — a basis which respects the above block structure. In such a basis (utilizing the block index $\varkappa$), rotating points are of the form $\Psi^\varkappa_{\Delta\mu} = e^\varkappa_\mu \otimes \varrho^\varkappa$ (where μ is now used to label the matrix units $e^\varkappa_\mu$ of the space of $U_\varkappa$ and $\varrho^\varkappa$ is the unique fixed point of $\{B^{\ell,\varkappa}\}_\ell$) while their dual conserved quantities are $J^{\varkappa\Delta\mu}_{\text{▘}} = e^\varkappa_\mu \otimes P^\varkappa$ (where $P^\varkappa$ is the identity on the auxiliary subspace). Thus, conserved quantities in each block are related to rotating points via a division by (i.e., inversion of all nonzero eigenvalues of) the auxiliary fixed point, $J^{\varkappa\Delta\mu}_{\text{▘}} = \Psi^\varkappa_{\Delta\mu}(\varrho^\varkappa)^{-1}$ [32, 45]. It is well-known among experts that $\{J^{\varkappa\Delta\mu}_{\text{▘}}\}$ form a *matrix algebra* — a vector space (where the vectors are matrices) that is closed under multiplication and the conjugate transpose operation. It is important to keep in mind that all of this extra structure in ▘ does not put any constraints on the remaining parts $\{A_{\text{▝}}, A_{\text{▗}}\}$ of $\mathcal{A}$, the extension of $\mathcal{E}$; this is why it was avoided until now. Moreover, $\{J^{\varkappa\Delta\mu}\}$ do *not* have to form a matrix algebra.

There exist several algorithms to determine the shape (24) of $\mathcal{A}$ [12, 30, 46–50]. A straightforward way [12] to find the form (24) for a general channel $\mathcal{A}$ is to diagonalize $\mathcal{A}$ and apply standard matrix algebra techniques [46, 48] to find a canonical basis for the algebra of conserved quantities in ▘. Using Prop. 3, I slightly extend the algorithm from Ref. [12] to one that finds and organizes not just the conserved quantities restricted to ▘, but the full conserved quantities as well. Once again, the main new inclusion is the determination of conserved quantities whose eigenvalue is modulus one (as opposed to exactly one).

Algorithm. *Finding and organizing* $\mathcal{P}_{\mathcal{A}}$

Find the rotating points Ψ *and conserved quantities* J *by diagonalizing* $\mathcal{A}$

Construct $\mathcal{P}_{\mathcal{A}}$ *and* P*, the projection onto* range$\{\mathcal{P}_{\mathcal{A}}(I)\}$

Find the projected conserved quantities $J_{\text{▘}} \equiv PJP$

Decompose the algebra spanned by $J_{\text{▘}}$ *into canonical form using, e.g., Refs. [46, 48]*

Determine a canonical basis $\Psi^\varkappa_{\Delta\mu}$ *for the rotating points and* $J^{\varkappa\Delta\mu}_{\text{▘}}$ *for the conserved quantities*

Extend $J^{\varkappa\Delta\mu}_{\text{▘}}$ *to* $J^{\varkappa\Delta\mu}$ *via Prop. 3.*

Note that ▘ is the range of $\mathcal{P}_{\mathcal{A}}(I)$, i.e., $\mathcal{P}_{\mathcal{A}}(I) \propto P$, because I is dual to the maximally mixed fixed point $\frac{1}{\mathrm{Tr}\{P\}}P$ and is the only conserved quantity with nonzero trace.

IV. APPLICATION: MATRIX PRODUCT STATES

For those who skimmed Secs. II-III, those parts focused on the distinction between a channel $\mathcal{A}$ and its corresponding faithful channel $\mathcal{E} \equiv \mathcal{P}_{\text{▘}}\mathcal{A}\mathcal{P}_{\text{▘}}$ — $\mathcal{A}$ restricted to the largest invariant subspace ▘ (equivalently, the range of $\mathcal{A}$'s maximal-rank fixed point). The block ▗ thus forms a decaying subspace, but the asymptotic projection $\mathcal{P}_{\mathcal{A}}$ (7) of $\mathcal{A} = \{A^\ell\}$ nevertheless retains information from states in ▗ by transferring it into ▘ through the operators $A^\ell_{\text{▝}}$. Here, this decomposition is applied to matrix product states (MPS) in order to obtain an unambiguous thermodynamic limit for any MPS that is translationally invariant in the bulk, but has non-trivial boundary effects. Then, I show how one can absorb any dependence of said limit on the decaying parts ▗ of the bond degrees of freedom into the boundary conditions. This allows one to shorten the bond dimension and use the transfer matrix $\mathcal{A}_{\text{▘}} = \mathcal{E}$ instead of the full $\mathcal{A}$.

A. What are MPS?

Our playground is now a one-dimensional lattice consisting of $2M+1$ spins. Each spin is d-dimensional and indexed by the physical index ℓ. An MPS $|\Phi\rangle$ that is translationally-invariant in the bulk of the lattice can be written as

$$|\Phi^{\{B\}}_{\mathcal{A}}\rangle \propto \sum_{\ell_{-M},\cdots,\ell_M=0}^{L-1} \mathrm{Tr}\{BA^{\ell_{-M}}\cdots A^{\ell_M}\}|\ell_{-M}\cdots\ell_M\rangle\,, \tag{25}$$

where A^ℓ is an L-dimensional vector of $N\times N$ matrices (for some *bond dimension* N) and B is an $N\times N$ matrix quantifying the boundary conditions. The bond dimension determines the degree of entanglement of the spins, with $N=1$ corresponding to a separable state. Physically meaningful boundaries are either $B=I$ (the identity) for translationally invariant MPS's or $B=|r\rangle\langle l|$ for some states $|r\rangle, |l\rangle$ quantifying the effect of the boundary on the right and left ends of the chain.

By performing similarity transformations on the A's, all MPSs can be put into a canonical form [3, 51], in which the A's satisfy eq. (2) and therefore form a Kraus map $\mathcal{A}\equiv\{A^\ell\}_{\ell=0}^{L-1}$. This map is usually called a *transfer channel* (also, double tensor [52]), and it appears when one of the lattice sites from eq. (25) is traced out.

Continuing to trace out more sites while also taking the thermodynamic limit of the MPS ($M\rightarrow\infty$), one can obtain the normalization of the state:

$$\lim_{M\rightarrow\infty}\langle\Phi^{\{B\}}_{\mathcal{A}}|\Phi^{\{B\}}_{\mathcal{A}}\rangle = \lim_{M\rightarrow\infty}\mathrm{T}{\scriptstyle\mathrm{R}}\{\mathcal{A}^{\alpha(2M+1)}\mathcal{B}\} = \mathrm{T}{\scriptstyle\mathrm{R}}\{\mathcal{P}_{\mathcal{A}}\mathcal{B}\}\,, \tag{26}$$

where $\mathcal{B}\equiv B\otimes B^\star$, the trace is over superoperator space, and α is the parameter that eliminates phases stemming from rotating points. The addition of α, which physically is equivalent to blocking sites of the MPS and taking the limit of blocks, allows one to define an unambiguous and non-pathological thermodynamic limit for general boundary conditions.

As an example, for periodic boundary conditions $B=I$ and faithful channels $\mathcal{E}$ containing rotating points $\mathit{\Psi}_{\Delta=\frac{2\pi}{N}n}$ and conserved quantities $J^{\Delta=\frac{2\pi}{N}n}$ satisfying $\mathrm{Tr}\{J^{\Delta\dagger}\mathit{\Psi}_{\Delta'}\}=\delta_{\Delta,\Delta'}$, the normalization is

$$\lim_{M\rightarrow\infty}\langle\Phi^{\{I\}}_{\mathcal{E}}|\Phi^{\{I\}}_{\mathcal{E}}\rangle = \lim_{M\rightarrow\infty}\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}\alpha(2M+1)n}\,. \tag{27}$$

Picking $\alpha=1$ yields zero whenever $2M+1$ and N are noncommensurate {e.g., [53], Eq. (130)}. In contrast, setting $\alpha=N$ gives N (as was also noticed recently in Ref. [51]).

A similar equation occurs if one wants to evaluate observables in the thermodynamic limit (see below). In this way, the transfer channel and boundaries determine the properties of the MPS in the thermodynamic limit. One can also use $\mathcal{B}$ to get rid of any undesired components of $\mathcal{P}_{\mathcal{A}}$ [54]. Note that $|\Psi^{\{B\}}_{\mathcal{P}_{\mathcal{A}}}\rangle$ is also the fixed-point MPS that $|\Phi^{\{B\}}_{\mathcal{A}}\rangle$ flows to under RG transformations [55–57], and

$$\lim_{M\rightarrow\infty}\langle\Phi^{\{B\}}_{\mathcal{A}}|\Phi^{\{B\}}_{\mathcal{A}}\rangle = \langle\Psi^{\{B\}}_{\mathcal{P}_{\mathcal{A}}}|\Psi^{\{B\}}_{\mathcal{P}_{\mathcal{A}}}\rangle\,, \tag{28}$$

so simplifying $\mathcal{P}_{\mathcal{A}}$ also yields insight into the structure of RG fixed points.

B. Boundary effects in the thermodynamic limit

The exact connection between quantum channels and MPSs has been well-studied for the case when the MPS is injective — when its corresponding transfer channel has only one fixed point and no rotating points. Since the results from the previous section are exactly about cases where there are arbitrary numbers of fixed and rotating points, here we will quantify the connection between asymptotics of quantum channels and the thermodynamic limit of non-injective MPS. The approach is somewhat reverse of what has been done before (see Sec. 3.2.2 of [3]): instead of first considering a general MPS, I consider a general channel $\mathcal{A}$ and simplify its corresponding MPS in the thermodynamic limit by applying the above results about $\mathcal{A}$'s structure. Since applying identical transformations U to each site is the same as changing basis for the Kraus operators of $\mathcal{A}$,

$$A^\ell\rightarrow\sum_{\ell'}U_{\ell\ell'}A^{\ell'}\,, \tag{29}$$

more technically this is a study of sets of MPS related by local unitaries.

Let us apply the four-corners decomposition onto the MPS in order to determine which blocks are relevant in the thermodynamic limit. Assume that each $A^\ell = A^\ell_{\text{▜}}$ now has a decaying subspace ▗ and that powers of $\mathcal{A}$ eventually transfer the state completely into ▘. Recall that $A^\ell_{\text{▘}}\equiv E_\ell$ and so the channel which determines the right fixed points is $\mathcal{A}_{\text{▘}}\equiv\{E_\ell\}=\mathcal{E}$. After some algebra, the coefficient $\mathrm{Tr}\{B(A_{\ell_{-M}}\cdots A_{\ell_M})\}$ in the MPS

(25) becomes equal to

$$\mathrm{Tr}\left\{B_{\text{◰}}(E_{\ell_{-M}}\cdots E_{\ell_M})\right\}+\mathrm{Tr}\left\{B_{\text{◳}}(A_{\text{◲}}^{\ell_{-M}}\cdots A_{\text{◲}}^{\ell_M})\right\}$$
$$+\sum_{m=-M}^{M}\mathrm{Tr}\left\{B_{\text{◱}}(E_{\ell_{-M}}\cdots E_{\ell_{m-1}})A_{\text{◳}}^{\ell_m}(A_{\text{◲}}^{\ell_{m+1}}\cdots A_{\text{◲}}^{\ell_M})\right\}. \tag{30}$$

The first term corresponds to the usual MPS $|\Phi_{\mathcal{E}}^{\{B\}}\rangle$ whose transfer matrix $\mathcal{E}$ is faithful. The second term vanishes in the thermodynamic limit because its corresponding transfer matrix does not have any fixed points. When $B_{\text{◱}}\neq 0$, the third term is present and has the form of a translationally-invariant domain wall excitation. Therefore, the decaying subspace ◲ corresponds to extra degrees of freedom on each site which house such an excitation. This excitation is never present for periodic boundary conditions ($B=I$), allowing one to straightforwardly derive a standard irreducible form for MPS with such boundary conditions in which the first and second terms are decomposed into smaller irreducible blocks [3, 51].

Let us continue to focus on "twisted" boundaries $B_{\text{◱}}\neq 0$. The main result is that, in the thermodynamic limit, contributions from extra degrees of freedom corresponding to ◲ can equivalently be described by considering only $\mathcal{A}_{\text{◰}}=\mathcal{E}$, but given a *mixture* of MPS having different boundary conditions. Culminating with Eq. (37), it will be shown that, in the thermodynamic limit, expectation values of local observables with an MPS $|\Phi_{\mathcal{A}}^{\{B\}}\rangle$ can be equivalently calculated from expectation values with the MPS $\{|\Phi_{\mathcal{E}}^{\{B_k\}}\rangle\}_{k=0}^{K}$, where $K>1$ and B_k are distinct boundary conditions dependent on $\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{◲}}$ and B.

Let us evaluate the expectation value of an observable O on a site in the thermodynamic limit. The number of lattice sites between the site which supports O and both boundaries is infinite and α is used to remove any phases occurring due to rotating points [see eq. (6)]. This allows one to simplify a previous form of such a limit, eq. (133) of Ref. [53], and remove any convergence issues arising from such phases. After some algebra,

$$\lim_{M\to\infty}\langle\Phi_{\mathcal{A}}^{\{B\}}|O|\Phi_{\mathcal{A}}^{\{B\}}\rangle=\mathrm{TR}\{\mathcal{P}_{\mathcal{A}}\mathcal{O}\mathcal{P}_{\mathcal{A}}\mathcal{B}\}\equiv\mathrm{TR}\{\mathcal{O}_{\mathcal{A}}\mathcal{B}\}, \tag{31}$$

where the corresponding superoperator is

$$\mathcal{O}\equiv\sum_{k,\ell=0}^{d-1}\langle\ell|O|k\rangle A_k\otimes A_\ell^\star\,. \tag{32}$$

To finish the calculation, decompose $\mathcal{P}_{\mathcal{A}}$ using eq. (17) and $\mathcal{O}$ using the block form of A^ℓ (12), yielding $\mathcal{O}\mathcal{P}_{\text{◰}}=\mathcal{P}_{\text{◰}}\mathcal{O}\mathcal{P}_{\text{◰}}$ and correspondingly

$$\lim_{M\to\infty}\langle\Phi_{\mathcal{A}}^{\{B\}}|O|\Phi_{\mathcal{A}}^{\{B\}}\rangle=\mathrm{TR}\{\mathcal{O}_{\mathcal{E}}(\mathcal{B}_{\text{◰}}+\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{◲}}\mathcal{B}\mathcal{P}_{\text{◰}})\}, \tag{33}$$

where $\mathcal{O}_{\mathcal{E}}\equiv\mathcal{P}_{\mathcal{E}}\mathcal{O}\mathcal{P}_{\mathcal{E}}$ and $\mathcal{B}_{\text{◰}}\equiv\mathcal{P}_{\text{◰}}\mathcal{B}\mathcal{P}_{\text{◰}}$. The $\mathcal{B}_{\text{◰}}$ term is the standard contribution of boundary effects located in ◰ and corresponds to the first term in the form of the MPS (30). By contrast, the only piece of $\mathcal{B}$ contributing to the second term in Eq. (33) is $\mathcal{P}_{\text{◲}}\mathcal{B}\mathcal{P}_{\text{◰}}=B_{\text{◱}}\otimes B_{\text{◱}}^\star$, corresponding to the *third* term in the form of the MPS (30). As a sanity check, taking periodic boundary conditions ($B=I=I_{\text{▚}}$) yields $\mathcal{P}_{\text{◲}}\mathcal{B}\mathcal{P}_{\text{◰}}=0$ and so only the first term in Eq. (33) remains. In general, the domain-wall-like excitations from the third term in Eq. (30) combined with "twisted" boundary conditions $B_{\text{◱}}\neq 0$ *can* contribute to the thermodynamic limit of the MPS.

C. Absorbing boundary effects

One can interpret the contribution of ◲ in a different way by thinking of both terms from Eq. (33) as coming from the effective boundary on ◰,

$$\overline{\mathcal{B}}\equiv\mathcal{B}_{\text{◰}}+\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{◲}}\mathcal{B}\mathcal{P}_{\text{◰}}=\overline{\mathcal{B}}_{\text{◰}}\,. \tag{34}$$

Since $\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{◲}}$ is a channel from ◲ to a subspace of ◰, one can decompose it in terms of some Kraus operators $F^k=F^k_{\text{◱}}$: $\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\text{◲}}=\sum_{k=1}^{K}F^k\otimes F^{k\star}$. (These Kraus operators are of course related to the rotating points $R_{\Delta\mu}$ and the ◲ pieces of conserved quantities $L_{\text{◲}}^{\Delta\mu}$ from the previous section.) The rank K is bounded by $\min\{\dim\text{◰},\dim\text{◲}\}$, so it is independent of the system size M. This shows that the effects of ◲ can just as well be simulated by a *superposition* of effective boundary conditions $B_{\text{◰}}$ with those from the set $\{F^k_{\text{◱}}B_{\text{◱}}\}_{k=1}^{K}$,

$$\overline{\mathcal{B}}=\sum_{k=0}^{K}\mathcal{B}_k\equiv\sum_{k=0}^{K}B_k\otimes B_k^\star\,, \tag{35}$$

where $B_0=B_{\text{◰}}$ and $B_{k>0}=F^k_{\text{◱}}B_{\text{◱}}$. Plugging in the above form for $\overline{\mathcal{B}}$ into Eq. (33),

$$\lim_{M\to\infty}\langle\Phi_{\mathcal{A}}^{\{B\}}|O|\Phi_{\mathcal{A}}^{\{B\}}\rangle=\mathrm{TR}\{\mathcal{O}_{\mathcal{E}}\overline{\mathcal{B}}\}=\sum_{k=0}^{K}\mathrm{TR}\{\mathcal{O}_{\mathcal{E}}\mathcal{B}_k\}\,. \tag{36}$$

Working backwards, each term in the sum over k corresponds to the thermodynamic limit of the MPS $|\Psi_{\mathcal{E}}^{\{B_k\}}\rangle$:

$$\lim_{M\to\infty}\langle\Phi_{\mathcal{A}}^{\{B\}}|O|\Phi_{\mathcal{A}}^{\{B\}}\rangle=\sum_{k=0}^{K}\lim_{M\to\infty}\langle\Phi_{\mathcal{E}}^{\{B_k\}}|O|\Phi_{\mathcal{E}}^{\{B_k\}}\rangle\,. \tag{37}$$

Therefore, when calculating expectation values of local observables, one can drop ◲ as long as one includes a *mixture* of MPS with different boundary conditions.

The same occurs with two observables $O^{(1)}$ and $O^{(2)}$ (with corresponding superoperators $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$) separated by some number of sites W,

$$\lim_{M\to\infty}\langle\Phi_{\mathcal{A}}^{\{B\}}|O^{(1)}O^{(2)}|\Phi_{\mathcal{A}}^{\{B\}}\rangle=\mathrm{Tr}\left\{\mathcal{P}_{\mathcal{A}}\mathcal{O}^{(1)}\mathcal{A}^W\mathcal{O}^{(2)}\overline{\mathcal{B}}\right\}, \tag{38}$$

and take the $W \to \infty$ limit by blocking sites in order to get rid of any phases from rotating points. This yields

$$\lim_{M,W\to\infty} \langle \Phi_{\mathcal{A}}^{\{B\}} | O^{(1)} O^{(2)} | \Phi_{\mathcal{A}}^{\{B\}} \rangle = \mathrm{Tr}\left\{ \mathcal{O}_{\mathcal{E}}^{(1)} \mathcal{O}_{\mathcal{E}}^{(2)} \overline{\mathcal{B}} \right\}, \tag{39}$$

where $\mathcal{O}_{\mathcal{E}}^{(i)} = \mathcal{P}_{\mathcal{E}} \mathcal{O}^{(i)} \mathcal{P}_{\mathcal{E}}$. Similarly, consider an observable touching the left boundary:

$$\lim_{M\to\infty} \langle \Phi_{\mathcal{A}}^{\{B\}} | O^{(L)} | \Phi_{\mathcal{A}}^{\{B\}} \rangle = \mathrm{Tr}\{ \mathcal{O}^{(L)} \mathcal{P}_{\mathcal{A}} \mathcal{B} \} = \mathrm{Tr}\{ \mathcal{O}_{\mathcal{E}}^{(L)} \overline{\mathcal{B}} \}. \tag{40}$$

Somewhat surprisingly, considering an observable touching the right boundary produces something completely different:

$$\lim_{M\to\infty} \langle \Phi_{\mathcal{A}}^{\{B\}} | O^{(R)} | \Phi_{\mathcal{A}}^{\{B\}} \rangle = \mathrm{Tr}\{ \mathcal{P}_{\mathcal{A}} \mathcal{O}^{(R)} \mathcal{B} \}. \tag{41}$$

Notice how $\mathcal{P}_{\mathcal{A}}$ now comes before the observable [cf. the first equality of Eq. (40)], which results in a series of new terms stemming from combinations of $A^{\ell}_{\boxplus}$ and $A^{\ell}_{\boxminus}$ with B. Why is there an asymmetry between the two boundaries? This has to do with the fact that we had initially assumed an asymmetric form for our MPS, $A^{\ell} = A^{\ell}_{\blacksquare}$. The domain wall-type excitations represented by the third term in Eq. (30) are such that there is always a $A_{\boxminus}$ at the right-most site M.

Since one has to block sites in order to have a valid thermodynamic limit, one might imagine that effects of periodicities in the MPS (i.e., effects of the rotating points) are eliminated. This is not the case due to the presence of the eigenvalues $e^{i\Delta}$ in the piece $\mathcal{P}_{\mathcal{A}} \mathcal{P}_{\boxminus}$ (19). This piece in turn affects the boundary conditions $\{B_i\}_i$ required to make sure Eq. (37) is satisfied. Thus, MPS with rotating points retain some of their properties even in a thermodynamic limit which blocks sites.

V. CONCLUSION

An important property of quantum channels $\mathcal{A}$ is their asymptotics, i.e., their behavior in the limit of infinite applications, akin to the infinite-time limit of Lindbladians [28]. An infinite product of $\mathcal{A}$ produces the channel's asymptotic projection $\mathcal{P}_{\mathcal{A}}$ — a projection on all of the non-decaying eigenspaces of the channel (i.e., whose eigenvalues have unit modulus). The superoperator $\mathcal{P}_{\mathcal{A}}$ can be constructed out of the channel's left and right rotating points, or as they are called here, conserved quantities J and steady-state basis elements Ψ. There has been a lot of overlapping work quantifying such asymptotics, but there have remained a few gaps in the literature when it came to considering conserved quantities with eigenvalue other than one. The aim of the first half of this work is to close those gaps in simple and standalone fashion.

I start off with two results about channels admitting a full-rank fixed point, which I call faithful. The first is that any J commute with a faithful channel's Kraus operators up to a phase. The second is that the eigenvalue of any diagonalizable J of a faithful channel is an Nth root of unity, where N is bounded by the dimension of the channel's Hilbert space. A third result deals with determining the dependence of the asymptotic state on initial state and on properties of $\mathcal{A}$. An analytical formula is derived that quantifies the dependence of the final state on initial states located in $\mathcal{A}$'s decaying eigenspaces (i.e., whose eigenvalues are less than one in modulus).

The aim of the second half of this work is to apply the third result above to matrix product states (MPS), where asymptotics come into play in the thermodynamic limit or in the limit of infinite renormalization transformations. In the same way that asymptotic states depend on initial states, the thermodynamic limit of MPS (whose transfer matrices admit more than one fixed point) depends on the boundary conditions. In such situations, the effects of any decaying bond degrees of freedom can be absorbed in the boundary conditions. Quantitatively, it is shown that the thermodynamic expectation value of a local operator O with an MPS having transfer matrix $\mathcal{A}$ and boundary condition B is equivalent to a sum of expectations values with MPS having $\mathcal{A}$ restricted to its largest invariant subspace and several different boundary conditions $\{B_i\}$ (37). Since similar two-dimensional MPS (often called "PEPS" [58]) and multiscale entanglement renormalization ansatz (MERA [4, 5]) states also correspond to a transfer channel, such techniques may further generalized to study PEPS dependence on boundaries and dependence of MERA hierarchies on their "caps".

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Appendix: Proofs

Proposition 1. *Let $\mathcal{E} = \{E_\ell\}$ be a faithful channel. Let J be a conserved quantity of $\mathcal{E}$, i.e., $\mathcal{E}^{\ddagger}(J) = e^{-i\Delta} J$ for some real Δ. Then, for all ℓ,*

$$J E_\ell = e^{-i\Delta} E_\ell J. \tag{A.1}$$

Proof. I extend an often-used [12, 30] application of the dissipation function [59] from fixed points to rotating points. An analogous extension for Lindbladians is in Ref. [28]. Let

$$X_\ell \equiv J E_\ell - e^{-i\Delta} E_\ell J \tag{A.2}$$

for each Kraus operator E_ℓ. Then, after some algebra,

$$\sum_\ell X_\ell^\dagger X_\ell = \mathcal{E}^\ddagger \left(J^\dagger J\right) - J^\dagger J \,. \tag{A.3}$$

Now multiply both sides by a full-rank fixed point ρ_{ss} and take the trace. Moving $\mathcal{E}^\ddagger$ under the Hilbert-Schmidt inner product so that it acts on ρ_{ss} yields

$$\mathrm{Tr}\left\{\mathcal{E}\left(\rho_{\mathsf{ss}}\right) J^\dagger J\right\} - \mathrm{Tr}\left\{\rho_{\mathsf{ss}} J^\dagger J\right\} = 0 \tag{A.4}$$

for the right-hand side, meaning that

$$\sum_\ell \mathrm{Tr}\left\{\rho_{\mathsf{ss}} X_\ell^\dagger X_\ell\right\} = 0\,.$$

Since $X_\ell^\dagger X_\ell \geq 0$ and $\rho_{\mathsf{ss}} > 0$, the only way for the above to hold is for $X_\ell = 0$, which implies the statement. □

Proposition 2. *Let $\mathcal{E} = \{E_\ell\}$ be a faithful channel. Let J_{dgn} be such that $\mathcal{E}^\ddagger(J_{\mathsf{dgn}}) = e^{-i\varDelta} J_{\mathsf{dgn}}$ for some real $\varDelta$ and assume J_{dgn} is diagonalizable. Then, there exists an integer n such that*

$$\varDelta = \frac{2\pi}{N} n \quad \textit{for some} \quad N \leq D\,. \tag{A.5}$$

Proof. I utilize tools from [2, 17, 27], which proved similar results to smaller families of quantum channels. First, there must exist an $N \geq 1$ such that J_{dgn}^N is a projection ($J_{\mathsf{dgn}}^{2N} = J_{\mathsf{dgn}}^N$). To show this, assume by contradiction that all powers of J_{dgn} are distinct. Then, there is an infinite sequence of conserved quantities J_{dgn}^N with eigenvalues $e^{-iN\varDelta}$ due to eq. (9). But the Hilbert space is D-dimensional, so there are at most D^2 fixed/rotating points. Moreover, $e^{-iN\varDelta} = 1$; otherwise, J_{dgn}^N would have a different eigenvalue than J_{dgn}^{2N}. Therefore, the exists an $N \leq D^2$ such that $\varDelta = \frac{2\pi}{N} n$ for $n \in \{0, 1, \cdots N-1\}$.

Now I show that $N \leq D$. Since J_{dgn}^N is a projection, J_{dgn} has eigenvalues which are Nth roots of unity. Since J_{dgn} is diagonalizable, one can introduce projections $\varPi_k$ on the eigenspace of each root of unity, write

$$J_{\mathsf{dgn}} = \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}n} \varPi_n \qquad \text{and define} \qquad \varPi \equiv \sum_{n=0}^{N-1} \varPi_n\,. \tag{A.6}$$

Let $R \equiv \mathrm{Tr}\{\varPi\} \leq D$ and note that $J_{\mathsf{dgn}} = \varPi J_{\mathsf{dgn}} \varPi$ is invertible on the range of $\varPi$. Observing the equation

$$\mathcal{E}^\ddagger(J_{\mathsf{dgn}}) = \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}n} \mathcal{E}^\ddagger(\varPi_n) = e^{-i\frac{2\pi}{N}} J_{\mathsf{dgn}}\,, \tag{A.7}$$

$\mathcal{E}^\ddagger$ is an automorphism on the subspace spanned by the $\varPi_n$'s ([17], Thm. 6.6). Since each $\varPi_n$ must be contained in the range of at least one E_ℓ, there exists a linear superposition of Kraus operators $E \in \mathrm{span}\{E_\ell\}$ such that $\varPi E \varPi$ is invertible on the range of $\varPi$. Now plug in this E into eq. (9), take the determinant on the range of $\varPi$ (i.e., treating both $\varPi J_{\mathsf{dgn}} \varPi$ and $\varPi E \varPi$ as $R \times R$ matrices), and simplify:

$$\det\{J_{\mathsf{dgn}}\} \det\{\varPi E \varPi\} = e^{-iR\varDelta} \det\{\varPi E \varPi\} \det\{J_{\mathsf{dgn}}\}. \tag{A.8}$$

Since both determinants are nonzero, divide them out to obtain $e^{-iR\varDelta} = 1$ and, since $R \leq D$, obtain the statement. □

Proposition 3. *The conserved quantities of $\mathcal{A}$ corresponding to eigenvalues $e^{i\varDelta}$ are*

$$J = J_{\text{▘}} + J_{\text{▗}} = J_{\text{▘}} - (\mathcal{A}^\ddagger_{\text{▗}} - e^{-i\varDelta})^{-1} \mathcal{A}^\ddagger(J_{\text{▘}})\,, \tag{A.9}$$

where $J_{\text{▘}}$ are conserved quantities of $\mathcal{A}_{\text{▘}} = \mathcal{E}$.

Proof. I generalize previous results ([12], Lemma 5.8; [38], Prop. 7) to the case of rotating points. Start by writing the eigenvalue equation

$$J e^{-i\varDelta} = \mathcal{A}^\ddagger\left(J\right) \tag{A.10}$$

in terms of the four-corners decomposition of J and

$$\mathcal{A}^\ddagger = \begin{bmatrix} \mathcal{E}^\ddagger_{\text{▘}} & 0 & 0 \\ \mathcal{P}_{\text{▞}} \mathcal{A}^\ddagger \mathcal{P}_{\text{▘}} & \mathcal{A}^\ddagger_{\text{▞}} & 0 \\ \mathcal{P}_{\text{▗}} \mathcal{A}^\ddagger \mathcal{P}_{\text{▘}} & \mathcal{P}_{\text{▗}} \mathcal{A}^\ddagger \mathcal{P}_{\text{▞}} & \mathcal{A}^\ddagger_{\text{▗}} \end{bmatrix}. \tag{A.11}$$

The three zeroes in the above decomposition for $\mathcal{A}^\ddagger$ can be derived by brute-force use of eq. (12). The eigenvalue equation is equivalent to

$$J_{\text{▘}} e^{-i\varDelta} = \mathcal{A}^\dagger_{\text{▘}}(J_{\text{▘}}) \tag{A.12a}$$

$$J_{\text{▞}} e^{-i\varDelta} = \mathcal{P}_{\text{▞}} \mathcal{A}^\ddagger(J_{\text{▘}}) + \mathcal{A}^\ddagger_{\text{▞}}(J_{\text{▞}}) \tag{A.12b}$$

$$J_{\text{▗}} e^{-i\varDelta} = \mathcal{P}_{\text{▗}} \mathcal{A}^\ddagger(J_{\text{▘}}) + \mathcal{P}_{\text{▗}} \mathcal{A}^\ddagger(J_{\text{▞}}) + \mathcal{A}^\ddagger_{\text{▗}}(J_{\text{▗}})\,. \tag{A.12c}$$

First, let's look at the term (A.12b):

$$\mathcal{P}_{\text{▞}} \mathcal{A}^\ddagger(J_{\text{▘}}) = \mathcal{P}_{\text{▝}} \mathcal{A}^\ddagger(J_{\text{▘}}) + \mathcal{P}_{\text{▖}} \mathcal{A}^\ddagger(J_{\text{▘}})\,. \tag{A.13}$$

Using eqs. (9) and (13), one can see that

$$\mathcal{P}_{\text{▝}} \mathcal{A}^\ddagger(J_{\text{▘}}) = \sum_\ell A^{\ell\dagger}_{\text{▘}} J_{\text{▘}} A^\ell_{\text{▝}} = e^{-i\varDelta} J_{\text{▘}} \sum_\ell A^{\ell\dagger}_{\text{▘}} A^\ell_{\text{▝}} = 0\,, \tag{A.14}$$

and similarly for the second term $\mathcal{P}_{\text{▖}} \mathcal{A}^\ddagger(J_{\text{▘}})$. This reduces eq. (A.12b) to

$$J_{\text{▞}} e^{-i\varDelta} = \mathcal{A}^\ddagger_{\text{▞}}(J_{\text{▞}})\,. \tag{A.15}$$

This can in turn be used to show that $J_{\text{▞}} = 0$ [28]. Assume by contradiction that $J_{\text{▞}} \neq 0$. Then, there must exist a corresponding right fixed point $\varPsi$ with $\mathcal{A}_{\text{▞}}(\varPsi) = 0$. But we have already assumed that all right fixed points are in ▘. Therefore, $J_{\text{▞}} = 0$.

The remaining eq. (A.12c) becomes

$$J_{\text{▗}} e^{-i\varDelta} = \mathcal{P}_{\text{▗}} \mathcal{A}^\ddagger(J_{\text{▘}}) + \mathcal{A}^\ddagger_{\text{▗}}(J_{\text{▗}}) \tag{A.16}$$

and can be used to solve for $J_{\text{▗}}$ in terms of $J_{\text{▘}}$ (since $\mathcal{A} - e^{i\varDelta}$ is invertible on ▗), obtaining the statement. □

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