Org.

CALIFORNIA INSTITUTE OF TECHNOLOGY

ELECTRON TUBE AND MICROWAVE LABORATORY

NONLINEAR EQUATIONS DESCRIBING BEAM TYPE CROSSED FIELD TRAVELING WAVE TUBES

by
J. W. Sedin

TECHNICAL REPORT NO. 6

October 1957

A REPORT ON RESEARCH CONDUCTED UNDER
CONTRACT WITH THE OFFICE OF NAVAL RESEARCH

NONLINEAR EQUATIONS DESCRIBING BEAM TYPE CROSSED FIELD TRAVELING WAVE TUBES

by

J. W. Sedin

California Institute of Technology

Pasadena, California

A Technical Report to the Office of Naval Research

Prepared for

Office of Naval Research Contract

Nonr 220(13)

October 1957

ABSTRACT

The equations governing the behavior of beam type crossed-field traveling wave tubes are formulated and presented. The geometry is assumed to be two-dimensional. The electron beam is treated as a number of cylinders of charge and space charge forces are included by calculating the field due to such a cylinder when it is placed between two perfectly conducting planes. The nonlinear equations are re-expressed in terms of normalized variables which make them suitable for machine computation. A brief discussion of the procedure for solving these equations numerically is included.

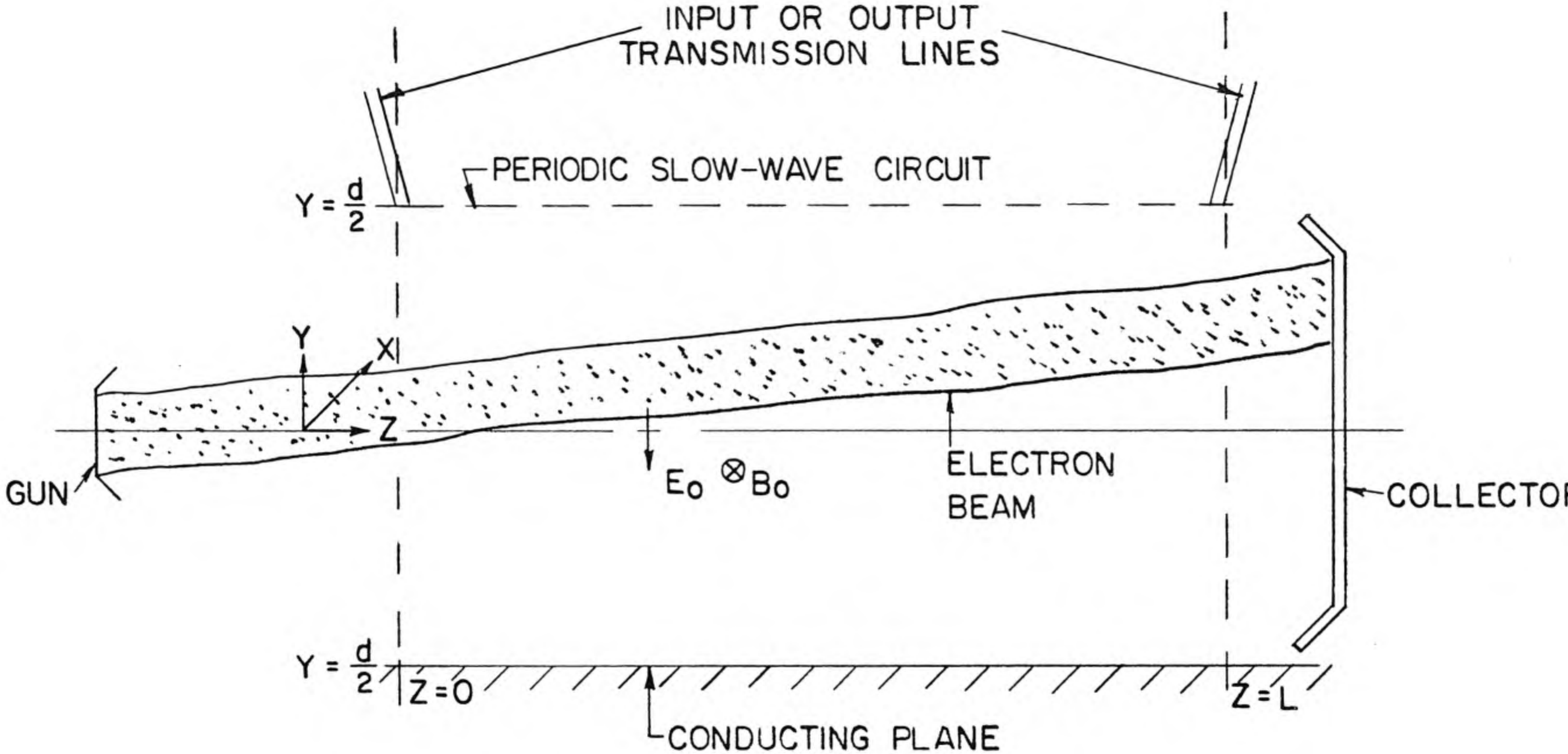
NONLINEAR EQUATIONS DESCRIBING BEAM TYPE
CROSSED FIELD TRAVELING WAVE TUBES

J. W. Sedin

INTRODUCTION

The development in recent years of successful amplifiers and voltage tunable oscillators employing crossed electric and magnetic fields as the beam focusing scheme has resulted in renewed interest in the analysis of this type of device. Most of the work done up to this date has been of the small signal, linearized equations type.[1,2,3,4] The successful operating devices virtually all operate in a nonlinear fashion and cannot be well described by a linear analysis. Feinstein and Kino[5] have done a clever nonlinear analysis of a thin beam crossed field device neglecting space charge effects and assuming that the phase velocity of the circuit wave and the drift velocity of the electron beam are the same and the phase velocity of the circuit wave is constant. The equations derived here will again describe an idealized form of crossed field device. These equations will describe some interesting effects that have been eliminated by the simplifying assumptions of previous work. Among these are space charge interaction effects, nonsynchronism between beam drift velocity and circuit phase velocity, variable beam injection conditions and finite thickness of the electron beam. There is hope that comparison between the results of solving these equations and experimental results may be quantitative. The purpose of this work is to obtain information regarding the effects of changing various parameters on the operating characteristics of crossed field devices.

The system to be analyzed can be represented schematically by the diagram shown in Figure 1. This system consists of a slow-wave circuit located a distance d above a ground plane, the circuit having transitions to external transmission lines at $z = 0$ and $z = L$. Electrons are emitted from some form of gun system to the left of $z = 0$



THE GENERAL FORM OF THE
SYSTEM UNDER CONSIDERATION

FIG. 1

and enter the interaction region $(0 \leq z \leq L; -\frac{d}{2} \leq y \leq \frac{d}{2})$ unmodulated. There is a d.c. magnetic field $+B_o \hat{e}_x$ and a d.c. electric field $-E_o \hat{e}_y$ so that in the absence of space charge effects the electrons have an average drift velocity $u_o = E_o/B_o$ in the z direction. If the electrons travel with a velocity approximately equal to the phase velocity of one of the space harmonics of a wave on the slow-wave circuit, the electron stream will become modulated and energy will be transferred to the circuit. The energy on the circuit is delivered to the external transmission line at $z = 0$ or $z = L$ depending on the direction of power flow on the circuit. The electrons are eventually collected on the circuit itself or on the collector at $z = L$.

This analysis will be carried out using techniques similar to those used by Nordsieck[6], Poulter[7], Tien[8] and Rowe[9] in the large signal analysis of the ordinary traveling wave tube. The slow wave circuit will be replaced by an equivalent transmission line which propagates a wave with the velocity of the desired space harmonic component of a wave on the actual periodic structure. The electron stream will be divided into a number of volume elements. Each of these elements will be replaced by an equivalent "electron" whose motion will be representative of the motion of all the charge in that element. A number of assumptions necessary to make the equations tractable are listed here, and additional assumptions are introduced in the text.

1. All quantities are independent of the x coordinate of the system. The volume elements into which the electron stream is divided are actually cylinders with their axes parallel to the x axis, and these cylinders can move in only the y and z directions.

2. The analysis is nonrelativistic. a.c. magnetic fields are neglected, and the electric field is assumed to be irrotational, $(\nabla \times E = 0)$.

3. The electric field from the charge in a volume element (external to the volume element) is the same as the electric field from an infinite line charge located between two parallel conducting plates and having the same charge per unit length as the cylindrical volume element.

4. Only the fundamental frequency component of voltage is present on the circuit and only one space harmonic component acts on the electron stream. This assumption is good as long as the beam travels at least a

few wavelengths along the circuit because for the general dispersive periodic circuit the harmonics would be out of synchronism, and their integrated effect would be small. In addition, the field of the higher order harmonics are confined more closely to the region of the circuit, and so would have less influence on the beam.

The problem will be treated in three parts: the equations describing propagation on the circuit, the equations describing the motion of the electron beam, and the effects of the space charge fields. The problem will first be treated using a straightforward choice of variables and then using a set of reduced variables suitable for making numerical computations. A brief discussion concerning the procedure for solving these equations numerically is included.

THE CIRCUIT EQUATION

The equation used to describe the circuit in this analysis is the familiar transmission line equation for the voltage as a function of distance and time along a distributed constant transmission line. The purpose of this section is to relate the constants of the transmission line equation to the characteristics of the crossed field device represented schematically by Figure 1.

The equation for the voltage on a distributed constant transmission line driven by a distributed current such as the one shown in Figure 1 is:

$$\frac{\partial^2 V}{\partial t^2} - \frac{1}{LC}\frac{\partial^2 V}{\partial z^2} = \sqrt{\frac{L}{C}}\,\frac{1}{\sqrt{LC}}\,\frac{\partial i}{\partial t} \tag{1}$$

where $V(z,t) \equiv$ the voltage across the line at any point

$i(z,t) \equiv$ the current per unit length flowing into the line at any point

$L \equiv$ the inductance per unit length

$C \equiv$ the capacitance per unit length .

The circuit has been assumed to be lossless; this assumption is merely for convenience, a finite value of R introduces only a slight complication in the numerical work. The constants $1/LC$ and $\sqrt{L/C}$ are chosen to relate the transmission line to the actual slow-wave circuit in much the same manner as is done in the small signal analysis of the ordinary traveling wave tube.[1] $1/LC$ is chosen equal to v_o^2 where v_o is the phase velocity of the desired space harmonic of a free wave on the actual periodic circuit. The homogeneous solution to Equation 1 is then a voltage wave propagating with velocity v_o.

The choice of $\sqrt{L/C}$ is a little more complicated. It is clear that since (1) is a function of z and t only it cannot describe the variations of the fields in the transverse direction. For a circuit propagating a slow wave in the z direction with propagation constant β and assumed to have no variations with x the voltage can be shown to vary as $\sinh \beta y$ in the y direction. Since it is anticipated that the velocity of propagation of waves in the combined circuit and beam system will be near the average drift velocity of the electrons it is assumed that the voltage variation as a function of y, z, and t is

$$V(y,z,t) = V(z,t) \frac{\sinh \beta_e \left(y + \frac{d}{2}\right)}{\sinh \beta_e d} \tag{2}$$

where $\beta_e = \frac{\omega}{u_o} = \frac{\omega B_o}{E_o}$ is the propagation constant of a wave traveling at the electron drift velocity and $V(z,t)$ is a solution to equation (1). The factor multiplying $V(z,t)$ in (2) is the appropriate variation of field with distance away from the circuit so that the characteristic impedance of the transmission line, $Z_o = \sqrt{L/C}$, must be chosen so that $E_z = - \partial V(z,t)/\partial z$ is exactly equal to the field at $y = d/2$ of the desired space harmonic of the slow wave circuit when the equivalent transmission line and the slow wave circuit are carrying the same power. In the analysis of the ordinary traveling wave tube, Z_o is defined in terms of an average field over the region of the beam because it is assumed that the same field acts on all electrons at a particular position in z. The average power carried by a distributed constant transmission line in terms of the voltage on the circuit is $P_{ave} = |V^2| / 2Z_o$. Since $E_z = \partial V(z,t)/\partial z$, for a free wave with propagation constant β the proper choice for Z_o is

$$Z_o = \frac{|V^2|}{2P_{ave}} = \frac{|E_z^2|}{2\beta^2 P_{ave}} \tag{3}$$

where $|E_z|$ is the magnitude of E_z of the desired space harmonic evaluated at the edge of the circuit, $y = d/2$, when the slow wave circuit is carrying P_{ave} with a propagation constant β. Equation (1) can now be written

$$\frac{\partial^2 V}{\partial t^2} - v_o^2 \frac{\partial^2 V}{\partial z^2} = v_o Z_o \frac{\partial i}{\partial t} \quad . \tag{4}$$

The distributed current $i(z,t)$ driving the transmission line is the displacement current flowing into the line due to the motion of charges in the electron stream. The procedure for evaluating $i(z,t)$ in terms of the motion of the charge in the electron beam will be discussed in the section on space charge. The assumptions made in the introduction regarding i will be reiterated. The characteristic impedance Z_o in (4) is in reality a function of frequency for a periodic circuit. In general it decreases for frequencies outside of the band for which the circuit was designed. In addition, the phase velocity of a periodic circuit is also a function of frequency so that net interaction effects are small for frequencies other than those for which the velocity of wave propagation is approximately the same as the electron drift velocity. Because of these facts, although $\partial i/\partial t$ will in general contain harmonics, it is assumed that only the fundamental frequency component of voltage is present on the circuit and only the fundamental frequency component of driving current is effective in (4).

The circuit equation becomes

$$\frac{\partial^2 V}{\partial t^2} - v_o^2 \frac{\partial^2 V}{\partial z^2} = \pm v_o Z_o \frac{\partial i_1}{\partial t} \tag{5}$$

where i_1 has been written for the fundamental component of $i(z,t)$. The $+$ and $-$ signs have been included to allow for both forward and backward wave interaction; $+$ for forward wave interaction, and $-$ for backward wave interaction.

THE EQUATIONS OF MOTION

The general force equation for an electron moving in electric and magnetic fields is

$$m \frac{d^2\bar{r}}{dt^2} = - |e| \bar{E} - |e| \frac{d\bar{r}}{dt} \bar{B} \tag{6}$$

where

e = magnitude of the electronic charge

m = electronic mass

$\bar{r}$ = radius vector to the electron.

Equation (6) can be separated into two equations in the y and z components since it has been assumed that there is no motion in the x direction.

$$\frac{d^2y}{dt^2} = - \eta E_y - \eta \frac{dz}{dt} B_x \tag{7}$$

$$\frac{d^2z}{dt^2} = - \eta E_z + \eta \frac{dy}{dt} B_x \tag{8}$$

$$\eta = \left|\frac{e}{m}\right| .$$

The electric field can now be separated into three parts: the field due to the energy on the circuit, the applied d.c. field, and the field due to the combined effect of all charges present. The fields due to the wave on the circuit can be obtained from the potential function defined in the preceding section.

$$E_y = - \frac{\partial V(y,z,t)}{\partial y} = -\beta_e V(z,t) \frac{\cosh \beta_e (y + \frac{d}{2})}{\sinh \beta_e d} \tag{9}$$

$$E_z = - \frac{\partial V(y,z,t)}{\partial z} = - \frac{\partial V(z,t)}{\partial z} \frac{\sinh \beta_e (y + \frac{d}{2})}{\sinh \beta_e d} . \tag{10}$$

The applied d.c. field is in the negative y direction only, $-E_o \hat{e}_y$. The fields due to space charge will be denoted E_{sy} and E_{sz}. They will be determined explicitly in the next section. The a.c. magnetic fields are neglected so that the only magnetic field to be considered is the applied d.c. magnetic field $+B_o \hat{e}_x$. The equations of motion can now be written

$$\frac{d^2y}{dt^2} = +\eta\ \beta_e\ V(z,t)\ \frac{\cosh \beta_e(y+\frac{d}{2})}{\sinh \beta_e d} -\eta\ E_{sy} +\eta\ E_{oy} -\eta\ \frac{dz}{dt}\ B_{ox} \tag{11}$$

$$\frac{d^2z}{dt^2} = \eta\ \frac{\partial V(z,t)}{\partial z}\ \frac{\sinh \beta_e(y+\frac{d}{2})}{\sinh \beta_e d} - \eta\ E_{sz} +\eta\ \frac{dy}{dt}\ B_{ox} \quad . \tag{12}$$

In the numerical solution of this problem the motion of each representative "electron" must be obtained so (11) and (12) really represent 2N equations where N is the number of "electrons" chosen to represent the beam.

THE EFFECTS OF SPACE CHARGE

The fields due to space charge enter into both the circuit equation and the equations of motion. The fields produced by a line charge located between two infinite, conducting plates will first be determined. The relation between these fields and the fields due to the charge in the beam will then be obtained and the terms due to space charge in the circuit equation (5) and the equations of motion (11) and (12) written out explicitly.

The electric field from an infinite line charge located between two infinite conducting plates can be obtained by writing the expression for the potential of a line charge above a conducting plate, and performing a simple conformal transformation to obtain the case of a line charge between two parallel conducting plates.[10] The fields at y and z due to a line charge q_ℓ at y' and z' are

$$E'_{sy} = +\frac{q_\ell}{2\epsilon d}\left\{\frac{(\sin\frac{\pi y}{d}\cos\frac{\pi y'}{d})\left[\sin\frac{\pi y'}{d}\sin\frac{\pi y}{d}+\cosh\frac{\pi}{d}(z-z')\right] -\sin\frac{\pi y'}{d}\cos\frac{\pi y'}{d}\cos^2\frac{\pi y}{d}}{\left[-\sin\frac{\pi y'}{d}\sin\frac{\pi y}{d}+\cosh\frac{\pi}{d}(z-z')\right]^2 - \cos^2\frac{\pi y'}{d}\cos^2\frac{\pi y}{d}}\right\} \tag{13}$$

$$E'_{sz} = \frac{q_\ell}{2\epsilon d}\left\{\frac{\cos\frac{\pi y'}{d}\cos\frac{\pi y}{d}\sinh\frac{\pi}{d}(z-z')}{\left[-\sin\frac{\pi y'}{d}\sin\frac{\pi y}{d}+\cosh\frac{\pi}{d}(z-z')\right]^2-\cos^2\frac{\pi y'}{d}\cos^2\frac{\pi y}{d}}\right\}. \quad (14)$$

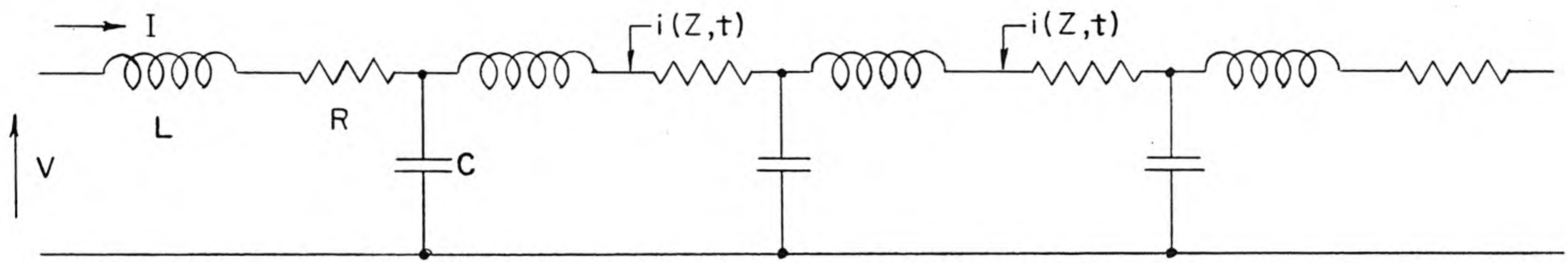
The conducting plates are located at $y = d/2$ and $y = -d/2$ and the line charge is parallel to the x axis.

The fields due to a cylindrical volume of charge $\rho dy'dz'$, located at y', z' are assumed to be the fields calculated from (13) and (14) with $q_\ell = \rho\, dy'\, dz'$. This approximation is good as long as the point of interest y, z is far enough away from the circuit so that the irregularities of the surface are unimportant and as long as $(z - z') > \frac{dz'}{2}$ and $(y - y') > \frac{dy'}{2}$ since the fields external to a circular cylinder of charge are exactly the same as if all the charge were concentrated on the axis of the cylinder. When $(z - z')$ and $(y - y')$ approach zero (13) and (14) approach infinity. This does not represent the physical situation accurately. The fields inside a circular cylinder of charge increase linearly from zero at the center of the cylinder to a maximum at the edge. When $(z - z') < k\frac{dz'}{2}$ and $(y - y') < k\frac{dy'}{2}$ the electric fields will be approximated by a linear function that goes from zero when $(y - y') = 0$ and $(z - z') = 0$ to the values given by (13) and (14) when $(y - y') = k\frac{dy'}{2}$ and $(z - z') = k\frac{dz'}{2}$ where k is some fraction which will be selected with the aid of some sample calculations. Since the difficulty just discussed is associated with the finite size of the elements dy' and dz' it introduces no complication into the derivation of the equations. It will be discussed further in the section concerned with the numerical solution of the equations.

With the aid of equation (13) it is now possible to determine the driving term $\partial i_1/\partial t$ of the circuit equation in terms of an integral over the charge in the electron stream. The total displacement current flowing into the circuit at z due to the charge in the beam is

$$\frac{\partial i}{\partial t}(z,t) = w\frac{\partial^2}{\partial t^2}\left[D_y(\tfrac{d}{2}, z, t)\right] \quad (15)$$

where $D_y(\frac{d}{2}, z, t)$ is the displacement flux terminating on the circuit at



THE EQUIVALENT CIRCUIT OF A TRANSMISSION LINE DRIVEN BY A DISTRIBUTED CURRENT.

FIG. 2

z and w is the width of the actual circuit in the x-direction. From (13)

$$d\,D_y(\tfrac{d}{2},z,t) = \frac{\rho(y',z',t)}{2d}\left\{\frac{\cos\frac{\pi}{2}\,y'}{-\sin\frac{\pi}{2}\,y' + \cosh\frac{\pi}{d}(z-z')}\right\} dz'dy' \qquad (16)$$

and defining

$$F = \frac{\cos\frac{\pi}{d}\,y'}{-\sin\frac{\pi y'}{d} + \cosh\frac{\pi}{d}\,(z-z')} \qquad (17)$$

$$\left[\frac{w\,\partial^2 D_y(\frac{d}{2},y,t)}{\partial t^2}\right]_1 = \left[\frac{w}{2d}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2}\rho(y',z',t)\,F\,dy'dz'\right]_1 . \qquad (18)$$

Retardation effects have been neglected in writing (18), the flux at $\frac{d}{2}, z$ is assumed to have propagated instantaneously from y', z'. This is permissible because the wave velocity and the electron velocity are much less than the velocity of light and because the function F decreases rapidly with increase in $(z - z')$ so that as the effect of retardation becomes more important, the strength of the retarded fields becomes less. The subscript 1 on brackets of (18) indicate that the fundamental frequency component of this equation is to be taken in accordance with the assumptions made in obtaining the circuit equation. The equation for the circuit voltage can now be written

$$\frac{\partial^2 V}{\partial t^2} - v_o^2\frac{\partial^2 V}{\partial z^2} = \pm\frac{v_o Z_o w}{2d}\left[\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2}\rho(y',z',t)\,F\,dy'\,dz'\right]_1 . \qquad (19)$$

The expressions for the space charge fields E_{sy} and E_{sz} of (11) and (12) can be obtained in a similar fashion. Define

$$Y = \left\{\frac{\left[\sin\frac{\pi y}{d}\cos\frac{\pi y'}{d}\right]\left[-\sin\frac{\pi y'}{d}\sin\frac{\pi y}{d} + \cosh\frac{\pi}{d}(z-z')\right] - \sin\frac{\pi y'}{d}\cos\frac{\pi y'}{d}\cos^2\frac{\pi y}{d}}{\left[-\sin\frac{\pi y}{d}\sin\frac{\pi y'}{d} + \cosh\frac{\pi}{d}(z-z')\right]^2 - \cos^2\frac{\pi y'}{d}\cos^2\frac{\pi y}{d}}\right\} \qquad (20)$$

and

$$Z = \left\{ \frac{\cos\frac{\pi y'}{d}\cos\frac{\pi y}{d}\ \sinh\frac{\pi}{d}(z - z')}{\left[-\sin\frac{\pi y}{d}\sin\frac{\pi y'}{d} + \cosh\frac{\pi}{d}(z - z')\right]^2 - \cos^2\frac{\pi y'}{d}\cos^2\frac{\pi y}{d}} \right\} . \tag{21}$$

Then

$$E_{sy}(y,z,t) = \frac{1}{2\epsilon d}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho(y',z',t)\ Y\ dy'\ dz' \tag{22}$$

and

$$E_{sz}(y,z,t) = \frac{1}{2\epsilon d}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho(y',z',t)\ Z\ dy'\ dz' . \tag{23}$$

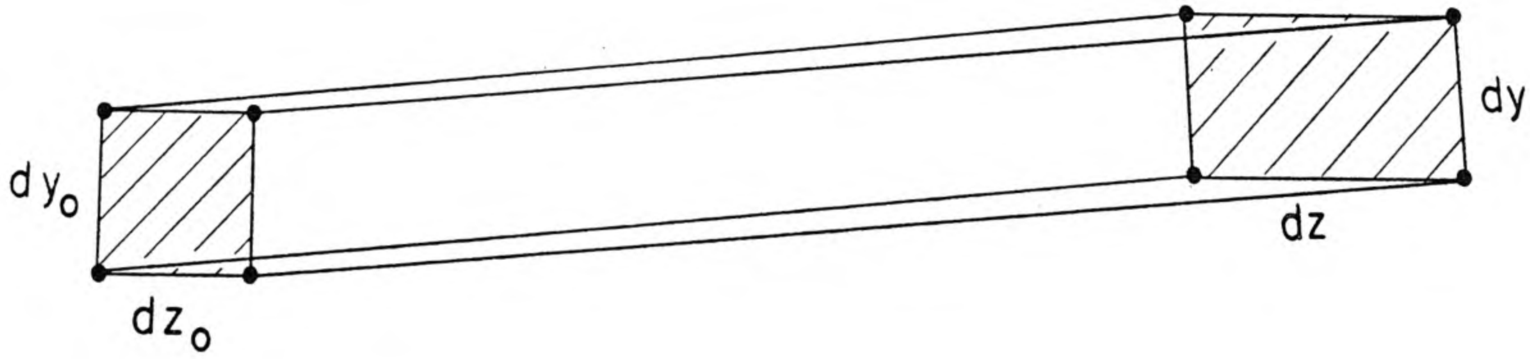
The fact that it is possible to have electron cross-overs in a nonlinear problem of this type means that the electron stream can no longer be treated as a fluid; the Lagrange rather than the Eulerian equation of continuity must be used. The Lagrange equation of continuity in two dimensions is

$$\rho_o\ (y_o,z_o,0)\ \left|dy_o\ dz_o\right| = \rho(y,z,t)\ \left|dy\ dz\right| . \tag{24}$$

This equation simply states that all the charge initially in the region $dy_o\ dz_o$ at $y_o,z_o,0$ is contained in the region $dy\ dz$ at y,z,t . This relation is illustrated in Figure 3. If the electrons at the corners of the element $dy_o\ dz_o$ end on the corners of the element $dy\ dz$ after a time t then the total charge in the element $dy_o\ dz_o$ must be the same as the total charge in the element $dy\ dz$. The area elements can be related to give

$$\rho(y,z,t) = \rho_o(y_o,z_o,0) \begin{vmatrix} \frac{\partial y_o}{\partial y} & \frac{\partial y_o}{\partial z} \\ \frac{\partial z_o}{\partial y} & \frac{\partial z_o}{\partial z} \end{vmatrix} . \tag{25}$$

In order to apply (25) the trajectory equations for a representative set of electrons must be known; that is, y and z must be known as a function of time for the particular electron that was at y_o,z_o at $t = 0$.



TYPICAL AREA ELEMENTS IN THE APPLICATION OF THE LaGRANGE CONTINUITY EQUATION

FIG. 3

$$y = y(y_o, z_o, t) \tag{26}$$

$$z = z(y_o, z_o, t) \ . \tag{27}$$

These are exactly the quantities that are obtained from integration of the force equations.

There is a complication in the use of (25) when there are electron cross-overs. y_o and z_o are not single-valued functions of y and z and the right-hand side of (25) should be written as a sum over the different branches of the functions. Rather than use (25) and integrate over y' and z', however, it is easier to use (24) and convert the integrals in (19), (22) and (23) into integrals over the initial positions and use (26) and (27) to convert F, Y, and Z into functions of the initial positions because y and z are always single-valued functions of y_o and z_o. Equations (19), (22) and (23) become

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{v_o^2}\frac{\partial^2 V}{\partial t^2} = \mp \frac{Z_o w}{2d\, v_o}\left[\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho_o(y_o', z_o', 0)\, F\, dy_o'\, dz_o'\right]_1 \tag{28}$$

$$E_{sy}(y,z,t) = \frac{1}{2\epsilon d}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho_o(y_o', z_o', 0)\, Y\, dy_o'\, dz_o' \tag{29}$$

$$E_{sz}(y,z,t) = \frac{1}{2\epsilon d}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho_o(y_o', z_o', 0)\, Z\, dy_o'\, dz_o' \quad . \tag{30}$$

Equations (11), (12), and (28) together with (29) and (30) are the equations describing the system.

THE SYSTEM EQUATIONS IN REDUCED VARIABLES

The choice of variables used to describe the problem has been straightforward. These variables are not a practical choice for numerical work, a set of reduced variables has been chosen to simplify the numerical computations and eliminate duplications in the work. These variables have been chosen to give agreement with the small signal analyses of crossed field devices as much as possible. Other variables have been chosen similar to the variables of the large signal analysis of the ordinary traveling-wave tube. The variables are listed here with some of the reasons for their choice. The system equations are then written in terms of the new variables.

The voltage on the circuit is assumed to be of the form

$$V(y,z,t) \equiv A(z) \sinh \beta_e (y + \tfrac{d}{2}) \frac{2 I_o Z_o \Phi^2}{D} \cos \phi(z,t)$$

$$= K A(z) \frac{\sinh \beta_e (y + \frac{d}{2})}{\sinh \beta_e d} \cos \phi \quad . \tag{31}$$

$A(z)$ is the amplitude function and it is assumed to vary slowly with z . I_o is the total current in the unperturbed electron beam defined as a positive number. Z_o is defined by (3). D is defined by the equation

$$D^2 = \frac{\omega}{\omega_c} \frac{I_o Z_o \Phi^2}{2V_o} \alpha \quad . \tag{32}$$

V_o is a voltage corresponding to the energy of the electrons at the average drift velocity

$$V_o = \frac{u_o^2}{2\eta} = \frac{1}{2\eta} \frac{E_o^2}{B_o^2} \quad . \tag{33}$$

Φ is a factor which relates the impedance at the beam entrance position to the impedance at the circuit

$$Z(y_o) = Z_o \frac{\sinh^2 \beta_e (y_o + \frac{d}{2})}{\sinh^2 \beta_e \frac{d}{2}} = Z_o \Phi^2 \quad . \tag{34}$$

α is the magnitude of the ratio of the r.f. electric field in the y direction to that in the z direction at the beam entrance position y .

$$\alpha = \left| \frac{E_y(y_o)}{E_z(y_o)} \right| = \coth \beta_e d \; (y + \tfrac{d}{2}) \quad . \tag{35}$$

ω_c is the cyclotron angular frequency

$$\omega_c = \eta B_o \quad . \tag{36}$$

The phase of the voltage wave

$$\phi(z,t) \equiv \omega(\frac{z}{u_o} - t) - \theta(z) \tag{37}$$

defines $\theta(z)$ as the negative of the difference between the actual phase of the wave and the phase of a wave traveling with a velocity equal to the average drift velocity of the electrons. It is anticipated that θ will be a slowly varying function of z. A direct consequence of (37) is

$$\left.\frac{\partial}{\partial t}\right|_z = -\omega \left.\frac{\partial}{\partial \phi}\right|_z . \tag{38}$$

The normalized y and z coordinates are

$$r = 2y/d \tag{39}$$

and

$$s = D\,\beta_e z \quad . \tag{40}$$

ϕ_o^i is the phase of the circuit wave that the i^{th} "electron" sees when it is at $z = 0$ and r_o^i is the normalized y position of the i^{th} electron when it is at $z = 0$.

$$\phi_o = \frac{\omega z_o}{u_1(y)} \tag{41}$$

where $u_1(y)$ is the z velocity of the electrons as they enter the interaction region. Equation (41) assumes that the z velocity of the electrons is a constant for $z < 0$. Equation (37) can be written

$$z^i \equiv \frac{u_o}{\omega}\left[\phi(s, r_o^i, \phi_o^i) + \theta(s)\right] + u_o\, t^i \tag{42}$$

in which form it represents the z position of the i^{th} electron in terms of the phase of the voltage wave on the circuit as seen by the i^{th} electron when it is at z.

$$\frac{dy^i}{dt} \equiv 2u_o\, Dp^i\, (r_o^i, s, \phi_o^i) \tag{43}$$

and

$$\frac{dz^i}{dt} = u_o \left[1 + 2Dq^i\ (r_o^i, s, \phi_o^i)\right] \tag{44}$$

define the velocity components of the electrons. When a quantity is written as a function of r_o^i, s, ϕ_o^i as in some of the preceding equations it simply means that at some position s these quantities are different for electrons that started in different phases and at different r positions.

Equations (39) and (41) can be used to convert the left-hand side of (24) to

$$\rho_o(y_o, z_o, 0)\ |dy_o\ dz_o| = \frac{d}{2\omega}\, u_1(y_o)\ \rho_o(y_o, 0, \phi_o)\ |dr_o\ d\phi_o|\ . \tag{45}$$

$u_1(y_o)\ \rho_o(y_o, 0, \phi_o)$ must be independent of ϕ_o since the stream is unmodulated at $z = 0$. Let

$$i_1 = |u_1(y_o)\ \rho_o(y_o)| \tag{46}$$

be the current density in the unmodulated electron stream and

$$i_2 = I_o / \tau w \tag{47}$$

be the average current density in the unmodulated electron stream. τ is the beam thickness in the y direction at $z = 0$ and w is the beam width in the x direction.

$$\rho_1 = \frac{i_2}{u_o} \tag{48}$$

is a sort of average charge density in the electron stream and

$$\omega_p^2 = \left|\frac{\eta\,\rho_1}{\epsilon_o}\right| \tag{49}$$

is a plasma frequency. A space charge parameter

$$g = \frac{\omega_p^2}{8\omega\ \omega_c D} \tag{50}$$

is defined similar to those used by Tien[8] and Rowe[9] in the large signal analysis of the ordinary traveling wave tube and easily related to the parameter used by Gould[4] in the small signal analysis of beam type crossed-field devices.

$$u_o = v_o\,(1 + Db) \tag{51}$$

defines a velocity injection parameter b, similar to that used by Pierce and others in various traveling-wave tube analyses. u_o in (51) is the average drift velocity of the electrons, the actual injection velocities are specified by the initial values of p^i and q^i.

THE CIRCUIT EQUATION IN REDUCED VARIABLES

With the aid of (38) the equation for the circuit voltage can be written

$$\frac{\partial^2 V(z,t)}{\partial z^2} - \frac{\omega^2}{v_o^2}\,\frac{\partial^2 V(z,t)}{\partial \phi^2} = \mp\left[\frac{\omega^2\, w\, Z_o}{2v_o d}\,\frac{\partial^2}{\partial \phi^2}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho_o(y_o',z_o')F\, dy_o' dz_o'\right]. \tag{52}$$

Equation (31) defining the circuit voltage can be used to rewrite the terms on the left-hand side of (52).

$$\frac{\partial V}{\partial z} = KD\,\beta_e\left\{\frac{dA}{ds}\cos\phi - A\sin\phi\,\frac{d\phi}{ds}\right\} \tag{53}$$

$$\frac{\partial \phi}{\partial s} = \frac{1}{D} - \frac{d\Theta}{ds} \tag{54}$$

$$\frac{\partial^2 V}{\partial z^2} = (D\beta_e)^2\,K\left\{\frac{d^2}{ds^2}\cos\phi - \frac{2dA}{ds}\sin\phi\,\frac{\partial\phi}{\partial s} - A\cos\phi\left(\frac{\partial\phi}{\partial s}\right)^2 - A\sin\phi\,\frac{\partial^2\phi}{\partial s^2}\right\} \tag{55}$$

$$\frac{\partial^2 V}{\partial \phi^2} = -\,A\,K\cos\phi \quad . \tag{56}$$

The right-hand side of (52) can be expanded in a Fourier series in ϕ since for constant z, ϕ is directly proportional to t. If only the fundamental component of this expansion is retained the right-hand side of (52) becomes

$$\pm \frac{\omega^2 w Z_o}{2\pi v_o d} \left[\left\{ \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-d/2}^{d/2} \rho_o F \, dy_o' \, dz_o' \cos \phi'(z,t) \, d\phi' \right\} \cos \phi + \left\{ \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-d/2}^{d/2} \rho_o F \, dy_o' \, dz_o' \sin \phi'(z,t) \, d\phi' \right\} \sin \phi \right] \tag{57}$$

$\sin \phi'$ and $\cos \phi'$ have been written outside of the y_o', z_o' integrals because ϕ' is not a function of y_o' or z_o'. Since $\sin \phi$ and $\cos \phi$ are orthogonal functions the coefficients multiplying $\sin \phi$ and $\cos \phi$ in the circuit equation can be equated individually, yielding two equations for the circuit.

$$(D\beta_e)^2 K \left\{ \frac{d^2 A}{ds^2} - A\left(\frac{1}{D} - \frac{d\Theta}{ds}\right)^2 \right\} + A K \frac{\omega^2}{v_o^2} = \pm \frac{\omega^2 w Z_o}{2\pi v_o d} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-d/2}^{d/2} \rho_o F \, dy_o' \, dz_o' \cos \phi' \, d\phi' \, , \tag{58}$$

$$(D\beta_e)^2 K \left\{ A \frac{d^2 \Theta}{ds^2} - 2 \frac{dA}{ds} \left(\frac{1}{D} - \frac{d\Theta}{ds}\right) \right\} = \pm \frac{\omega^2 w Z_o}{2\pi v_o d} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-d/2}^{d/2} \rho_o F \, dy_o' \, dz_o' \sin \phi' \, d\phi' \, . \tag{59}$$

Equation (51) defining the velocity injection parameter b can be used to reduce (58) and (59) to

$$\frac{d^2A}{ds^2} - A\left[\left(\frac{1}{D} - \frac{d\Theta}{ds}\right)^2 - \left(\frac{1+Db}{D}\right)^2\right] =$$

$$\pm\left(\frac{1+Db}{D}\right)\frac{u_o w}{d 4\pi I_o \Phi^2 \sinh \beta_e d}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho_o F \, dy_o' \, dz_o' \cos\phi' \, d\phi' \qquad (60)$$

$$A\frac{d^2\Theta}{ds^2} - 2\frac{dA}{ds}\left(\frac{1}{D} - \frac{d\Theta}{ds}\right) =$$

$$\pm\left(\frac{1+Db}{D}\right)\frac{u_o w}{d\, 4\pi I_o \Phi^2 \sinh \beta_e d}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho_o F \, dy_o' \, dz_o' \sin\phi' \, d\phi' . \qquad (61)$$

Equations (39), (40), (45), (46), and (47) can be used to convert the right-hand sides of (60) and (61) to integrals over ϕ_o' and r_o' .

$$\frac{d^2A}{ds^2} - A\left[\left(\frac{1}{D} - \frac{d\Theta}{ds}\right)^2 - \left(\frac{1+Db}{D}\right)^2\right] =$$

$$\mp\left(\frac{1+Db}{D}\right)\frac{1}{8\pi \Phi^2 \sinh \beta_e d \, \beta_e \tau}\int_0^{2\pi}\int_{\phi_o'-\pi}^{\phi_o'+\pi}\int_{-1}^{1} \frac{i_1}{i_2} F \, dr_o' \, d\phi_o' \cos\phi \, d\phi \qquad (62)$$

$$A\frac{d^2\Theta}{ds^2} - 2\frac{dA}{ds}\left(\frac{1}{D} - \frac{d\Theta}{ds}\right) =$$

$$\mp\left(\frac{1+Db}{D}\right)\frac{1}{8\pi \Phi^2 \sinh \beta_e d \, \beta_e \tau}\int_0^{2\pi}\int_{\phi_o'-\pi}^{\phi_o'+\pi}\int_{-1}^{1} \frac{i_1}{i_2} F \, dr_o' \, d\phi_o' \sin\phi \, d\phi . \qquad (63)$$

Equations (62) and (63) are the circuit equations in reduced variables. The range of the integral in ϕ_o' has been changed from $-\infty$ to ∞ to $\phi_o' - \pi$ to $\phi_o' + \pi$. That is, only charge that started within phase $\pm\pi$ of the charge that is at z at time t is effective in producing field at z . After the beam is bunched the electrons may not be in the same order as at starting so that in the numerical work the integral is carried out to include those electrons which are actually nearest to the electron of interest. This will be discussed in the section concerning numerical solutions. Chopping off the integral is an assumption that is based on the fact that F is a rapidly decreasing function of $z - z_o'$. The same assumption will be made in the integrals involving Y and Z .

THE EQUATIONS OF MOTION IN REDUCED VARIABLES

The equations of motion (11) and (12) can now be written in terms of the new variables. The acceleration in the y-direction is

$$\frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{ds}{dt}\frac{\partial}{\partial s}\left(\frac{dy}{dt}\right) = 2\beta_e D^2 u_o^2 \left[1 + 2\,Dq\right]\frac{\partial p}{\partial s} \tag{64}$$

and the acceleration in the z-direction is

$$\frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{ds}{dt}\frac{\partial}{\partial s}\left(\frac{dz}{dt}\right) = 2\beta_e D^2 u_o^2 \left[1 + 2\,Dq\right]\frac{\partial q}{\partial s} \quad . \tag{65}$$

Equations (11) and (12) become

$$2\beta_e D^2 u_o^2 \left[1 + 2\,Dq\right]\frac{\partial p}{\partial s} = \eta\,\beta_e\,K\,A\,\frac{\cosh \beta_e\,(y + \frac{d}{2})}{\sinh \beta_e d}\cos\phi$$

$$+ \eta\,E_o - \eta\,B_o\,u_o\left[1 + 2Dq\right] - \frac{\eta}{2\epsilon d}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho_o\,Y\,dy_o'\,dz_o' \tag{66}$$

and

$$2\beta_e D^2 u_o^2 \left[1 + 2Dq\right]\frac{\partial q}{\partial s} = \eta\,K\,\beta_e d\,\frac{\sinh \beta_e\,(y + \frac{d}{2})}{\sinh \beta_e d}\left\{\frac{dA}{ds}\cos\phi - A\sin\phi\left[\frac{1}{D} - \frac{d\theta}{ds}\right]\right\}$$

$$+ 2\eta\,u_o\,D\,B_o p - \frac{\eta}{2\epsilon d}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2} \rho_o\,Z\,dy_o'\,dz_o' \quad . \tag{67}$$

Equations (66) and (67) can be reduced to

$$\left[1 + 2Dq\right] \frac{\partial p}{\partial s} =$$

$$\frac{\omega_c}{D\omega}\left[\frac{A}{\alpha} \cosh \frac{\beta_e d}{2} (1+ r) \cos \phi - q + g \int_{\phi_o'-\pi}^{\phi_o'+\pi} \int_{-1}^{1} \frac{i_1}{i_2} Y \, dr_o' \, d\phi_o'\right] \quad (68)$$

$$\left[1 + 2Dq\right] \frac{\partial q}{\partial s} =$$

$$\frac{\omega_c}{D\omega}\left\{ \frac{dA}{ds} \cos \phi - A \sin \phi \left[\frac{1}{D} - \frac{d\Theta}{ds}\right]\right\} \frac{D}{\alpha} \sinh \frac{\beta_e d}{2}(1+ r)+p + g \quad (69)$$

$$\int_{\phi_o'-\pi}^{\phi_o'+\pi} \int_{-1}^{1} \frac{i_1}{i_2} Z \, dr_o' \, d\phi_o' \quad .$$

It should be recognized that p, q, r and ϕ in (68) and (69) are functions of s, r_o and ϕ_o. In the integral terms i_1 is a function of r_o'; i_2 is a constant; and Y and Z are functions of r_o, r_o', ϕ_o, ϕ' and s.

Two additional equations are necessary to complete the problem, equations relating r and ϕ to the velocities. By definition $dy/dt \equiv 2u_o Dp$, and for a particular electron

$$\frac{dy}{dt} = \frac{ds}{dt}\frac{\partial y}{\partial s} = D\omega\left[1 + 2 Dq\right]\frac{\partial y}{\partial s} \quad . \quad (70)$$

Therefore,

$$\frac{\partial r}{\partial s} = \frac{4}{\beta_e d}\,\frac{p}{1 + 2Dq} \quad . \quad (71)$$

The equation for ϕ is obtained by taking the time derivative of (42) and setting it equal to (44).

$$\frac{dz}{dt} = \frac{u_o}{\omega}\left[\frac{ds}{dt}\left\{\frac{\partial \phi}{\partial s} + \frac{d\Theta}{ds}\right\}\right] + u_o = \frac{1}{\beta_e D}\,\frac{ds}{dt}$$

$$\frac{\partial \phi}{\partial s} + \frac{d\Theta}{ds} = \frac{2q}{1 + 2Dq} \quad . \quad (72)$$

Equations (62), (63), (68), (69), (71) and (72) comprise the general working equations for a beam type crossed-field device.

THE SYSTEM EQUATIONS FOR SMALL D .

If $D \ll 1$ so that terms of first order and higher in D may be neglected, the equations above reduce to

$$\frac{d\Theta}{ds} + b = \mp \frac{1}{16\pi A \Phi^2 \sinh \beta_e d \, \beta_e \tau} \int_0^{2\pi} \int_{\phi'_o-\pi}^{\phi'_o+\pi} \int_{-1}^{1} \frac{i_1}{i_2} F \, dr'_o \, d\phi'_o \cos\phi \, d\phi \tag{73}$$

$$\frac{dA}{ds} = \pm \frac{1}{16\pi \Phi^2 \sinh \beta_e d \, \beta_e \tau} \int_0^{2\pi} \int_{\phi'_o-\pi}^{\phi'_o+\pi} \int_{-1}^{1} \frac{i_1}{i_2} F \, dr'_o \, d\phi'_o \sin\phi \, d\phi \tag{74}$$

$$\left[1 + 2Dq\right] \frac{\partial p}{\partial s} = \frac{\omega_c}{D\omega} \left[\frac{A}{\alpha} \cosh \frac{\beta_e d}{2} (1+r) \cos\phi - q + g \int_{\phi'_o-\pi}^{\phi'_o+\pi} \int_{-1}^{1} \frac{i_1}{i_2} Y \, dr'_o \, d\phi'_o\right] \tag{75}$$

$$\left[1 + 2Dq\right] \frac{\partial q}{\partial s} = \frac{\omega_c}{D\omega} \left[-\frac{A}{\alpha} \sinh \frac{\beta_e d}{2} (1+r) \sin\phi + p + g \int_{\phi'_o-\pi}^{\phi'_o+\pi} \int_{-1}^{1} \frac{i_1}{i_2} Z \, dr'_o d\phi'_o\right] \tag{76}$$

$$\frac{\partial r}{\partial s} = \frac{4}{\beta_e d} \frac{p}{1 + 2\,Dq} \tag{77}$$

$$\frac{\partial \phi}{\partial s} + \frac{d\Theta}{ds} = \frac{2q}{1 + 2\,Dq} \quad . \tag{78}$$

The terms containing p and q multiplied by D have been retained in these equations because it is possible for these terms to be large even if D is small if the electrons are poorly injected into the focussing fields. A different choice of variables which recognizes the fact that the motion in the absence of r.f. fields is in general some form of cycloidal motion would result in simpler equations for such cases. Such a set of variables will be discussed in a future report.

THE THIN BEAM EQUATIONS

The case for which the beam thickness in the transverse direction is so small that all electrons at a given z position are acted on by the same fields, has received considerable attention in the small signal case.[1,3,4] If the electron stream can be considered thin so that the integral in r_o reduces to multiplication by a constant, $i_1 = i_2$ and $\int d\, r_o' = 2\tau/d$ the equations for small D reduce to

$$\frac{d\Theta}{ds} + b = \mp \frac{1}{8\pi\, \beta_e d\, \Phi^2 \sinh \beta_e d\, A} \int_0^{2\pi} \int_{\phi_o'-\pi}^{\phi_o'+\pi} F\, d\phi_o' \cos\phi\, d\phi \tag{79}$$

$$\frac{dA}{ds} = \pm \frac{1}{8\pi\, \beta_e d\, \Phi^2 \sinh \beta_e d} \int_0^{2\pi} \int_{\phi_o'-\pi}^{\phi_o'+\pi} F\, d\phi_o' \sin\phi\, d\phi \tag{80}$$

$$\left[1 + 2\,Dq\right] \frac{\partial p}{\partial s} = \frac{\omega_c}{D\omega} \left[\frac{A}{\alpha} \cosh \frac{\beta_e d}{2}(1+r) \cos\phi - q + \frac{S}{2\,\beta_e d} \int_{\phi_o'-\pi}^{\phi_o'+\pi} Y\, d\phi_o'\right] \tag{81}$$

$$\left[1 + 2Dq\right] \frac{\partial q}{ds} = \frac{\omega_c}{D\omega} \left[- \frac{A}{\alpha} \sinh \frac{\beta_e d}{2}(1+r) \sin\phi + p + \frac{S}{2\beta_e d} \int_{\phi_o'-\pi}^{\phi_o'+\pi} Z\, d\phi_o'\right] \tag{82}$$

$$\frac{\partial r}{\partial s} = \frac{4}{\beta_e d} \frac{p}{1 + 2Dq} \tag{83}$$

$$\frac{\partial \phi}{\partial s} + \frac{d\Theta}{ds} = \frac{2q}{1 + 2Dq} \quad . \tag{84}$$

The constant S in equations (81) and (82) has been defined similar to that used by Gould[4]

$$S = \frac{\sigma_o}{2\epsilon_o\, B_o\, u_o\, D} \tag{85}$$

where $\sigma_o = I_o/u_o w$ is a surface charge density.

A DISCUSSION OF THE PROCEDURE FOR SOLVING THE SMALL D THIN BEAM EQUATIONS

The small D thin beam equations (79), (80), (81), (82), (83), and (84) are to be integrated numerically with respect to s. It has been pointed out previously that the terms involving Dp and Dq have been retained to allow for cases in which the electron trajectories would not be straight lines in the absence of circuit and space charge fields. The discussion now will be confined to cases in which Dq and Dp can be assumed to be negligible. Cases where this assumption is not permissible will be discussed later when a more appropriate set of variables will be introduced. The equations of motion can now be written

$$p = \frac{A}{\alpha} \sinh \frac{\beta_e d}{2} (1+r) \sin \phi - \frac{S}{2\beta_e d} \int_{\phi_o'-\pi}^{\phi_o'+\pi} Z \, d\phi_o' \tag{86}$$

$$q = \frac{A}{\alpha} \cosh \frac{\beta_e d}{2} (1+r) \cos \phi + \frac{S}{2\beta_e d} \int_{\phi_o'-\pi}^{\phi_o'+\pi} Y \, d\phi_o' \tag{87}$$

$$\frac{dr}{ds} = \frac{4p}{\beta_e d} \tag{88}$$

$$\frac{d\phi}{ds} + \frac{d\Theta}{ds} = 2q \quad . \tag{89}$$

Before going any further the system equations will be written in finite difference forms. The equations of motion become

$$p_{m+1}^{i} = \frac{A_m}{\alpha} \sinh \frac{\beta_e d}{2} (r_m^i + 1) \sin \phi_m^i - \frac{\pi S}{N\beta_e d} \sum_{j \neq i} Z_m^{ij} \tag{90}$$

$$q_{m+1}^{i} = \frac{A_m}{\alpha} \cosh \frac{\beta_e d}{2} (r_m^i + 1) \cos \phi + \frac{\pi S}{N\beta_e d} \sum_{j \neq i} Y_m^{ij} \tag{91}$$

$$r_{m+1}^{i} = r_m^i + \frac{2h}{\beta_e d} (p_{m+1}^i + p_m^i) \tag{92}$$

$$\phi_{m+1}^{i} = \phi_m^i + \Theta_{m-1} - \Theta_m + h(q_{m+1}^i + q_m^i) \tag{93}$$

$$A_{m+1} = A_{m-1} \mp \frac{h}{4N\,\beta_e d\,\Phi^2 \sinh \beta_e d} \Bigg[\sum_{i=2}^{N-1} (\phi_m^{i+1} - \phi_m^{i-1}) \sin \phi_m^i \sum_{j=1}^{N} F_m^{ij}$$

$$+ \Big((\phi_m^1 - \phi_m^{N-1} + 2\pi) \sin \phi_m^N \Big) \sum_{j=1}^{N} F_m^{Nj}$$

$$+ \Big((\phi_m^2 - \phi_m^N + 2\pi) \sin \phi_m^1 \Big) \sum_{j=1}^{N} F_m^{1j} \Bigg] \tag{94}$$

$$\Theta_{m+1} = \Theta_{m-1} - 2bh \pm \frac{h}{A_m\, 4N\,\beta_e d\,\Phi^2 \sinh \beta_e d} \Bigg[\sum_{i=2}^{N-1} (\phi_m^{i+1} - \phi_m^{i-1}) \cos \phi_m^i \sum_{j=1}^{N} F_m^{ij}$$

$$+ \Big((\phi_m^1 - \phi_m^{N-1} + 2\pi) \cos \phi_m^N \Big) \sum_{j=1}^{N} F_m^{Nj}$$

$$+ \Big((\phi_m^2 - \phi_m^N + 2\pi) \cos \phi_m^1 \Big) \sum_{j=1}^{N} F_m^{1j} \Bigg] \tag{95}$$

$$F_m^{ij} = \frac{\cos t_m^{ij}}{\cosh s_m^{ij} - \sin t_m^{ij}} \tag{96}$$

$$Y_m^{ij} = \frac{\left[\cosh s_m^{ij} - \sin t_m^{ij} \sin r_m^i\right] \sin r_m^i \cos t_m^{ij} - \sin t_m^{ij} \cos t_m^{ij} \cos^2 r_m^i}{\left[\cosh s_m^{ij} - \sin t_m^{ij} \sin r_m^i\right]^2 - \left[\cos t_m^{ij} \cos r_m^i\right]^2} \tag{97}$$

$$Z_m^{ij} = \frac{\cos t_m^{ij} \cos r_m^i \sinh s_m^{ij}}{\left[\cosh s_m^{ij} - \sin t_m^{ij} \sin r_m^i\right]^2 - \left[\cos t_m^{ij} \cos r_m^i\right]^2} \;. \tag{98}$$

The following definitions have been made in writing the above equations.

$s = mh \qquad m = 0,1,2,3,\cdots \qquad h = \text{constant}$ (99)

N is the number of "electrons" representing the beam.

Y_m^{ij}, Z_m^{ij}, F_m^{ij} are the values of Y, Z and F computed when the electron numbered i is at position m and the electron numbered j is at its corresponding position.

The integration rule used for A and Θ is

$$f(m + 1) = f(m - 1) + 2h\, f'(m) \quad . \tag{100}$$

The integration rule used for r and ϕ is

$$f(m + 1) = f(m) + h \left[\frac{f'(m) + f'(m + 1)}{2}\right] \quad . \tag{101}$$

The integration rule used in forming $\int_0^{2\pi}\left[\int_{\phi_o'-\pi}^{\phi_o'+\pi} F\, d\phi_o'\right] \sin\phi\, d\phi$ is simply the trapezoidal rule. It appears in a somewhat more complicated form than usual in (94) and (95) because the increments in ϕ are not equal and because the fact that ϕ is periodic with period 2π has been used in writing (94) and (95). t_m^{ij} and s_m^{ij} require a little more discussion. Their definitions will be given here and explained during the discussion of the procedure for solving the equations

$$t_m^{ij} = r_m^j \tag{102}$$

$$s_m^{ij} = \frac{\pi}{\beta_e d}\left[\phi_m^j - \phi_m^i\right] \quad . \tag{103}$$

The general idea involved in solving the above equations is as follows. Assume that at $m = 0$ the beam is unmodulated and that the amplitude A_o and the phase Θ_o associated with the wave on the circuit are known. At this plane in space the electron beam passing by sees a voltage which varies sinusoidally with time (or phase) $V_o \propto A_o \cos\phi$. If the electron beam is considered as N separate lumps of charge for each complete period in phase of the voltage wave then these lumps of charge can be identified by the average phase ϕ_o^i of the voltage wave as the i^{th} lump of charge passes the plane $m = 0$. The average force on this lump of charge due to the circuit voltage can then be used to find the acceleration of this particular lump of charge and similarly for all N lumps of charge. The $(N+1)^{th}$ lump of charge that comes by will be acted on by exactly the same field as the first lump of charge (under steady state conditions) so that there is no need to consider more than N "electrons" distributed over a range of 2π in phase. The lumps of charge will be referred to as electrons in the remainder of this discussion. With this knowledge of the acceleration of each electron it is possible to find the time it takes for the electrons to reach the next plane in z, $m = 1$. Since each of these electrons is accelerated in a different manner they will not arrive at $m = 1$ at the equally spaced intervals in time (or phase) that they arrived at $m = 0$. In addition, the presence of the electrons causes the phase velocity of the circuit wave to vary as a function of m. The quantity that must be known in order to compute the acceleration of the electrons at $m = 1$ is the phase of the voltage wave, ϕ_1^i, that the electrons see at $m = 1$. From this and the new voltage amplitude the time necessary to get to $m = 2$ can be computed. As far as the electrons are concerned the information that is being obtained is the time when each electron of a representative set is at a given z position and the velocities and y position of these electrons when they are at that position. The time is obtained in terms of the phase of the voltage wave seen by the electrons rather than the actual time since it is the phase that is important to the motion of the electrons.

Nothing was said about the space charge force terms in the preceding discussion. It is clear, however, that since the position of the electrons is only known up to the plane at which the force is being computed it is impossible to compute the space charge forces exactly. In order to compute the space charge force on electron i when it is at a particular plane in z it would be necessary to know the position of all other electrons at that time. The procedure that is used to obtain these positions approximately is one of linear extrapolation. The velocity of all the electrons is known for a particular plane in z and the time that these electrons are at this plane is also known. It is assumed that the electrons continue to travel at the same velocity after they leave this particular plane so that their position as a function of time can be computed. Using the definition of ϕ (37) the difference in time between the arrival of electron i and electron j at position m is

$$t_m^j - t_m^i = \frac{1}{\omega}(\phi_m^i - \phi_m^j) \quad . \tag{104}$$

Electron j is traveling with z velocity

$$\frac{dz^j}{dt} = u_o\left[1 + 2\,Dq_m^j\right] \approx u_o \tag{105}$$

and y velocity

$$\frac{dy^j}{dt} = 2u_o\, Dp_m^j \approx 0 \tag{106}$$

so that to a first approximation electron j is located a distance

$$z^j - z^i = \frac{u_o}{\omega}\,(\phi_m^j - \phi_m^i) \tag{107}$$

away from electron i when electron i is at m and at r position r_m^j. This information is contained in equations (102) and (103) for t_m^{ij} and s_m^{ij}. With this information it is now possible to compute F_m^{ij}, Y_m^{ij} and Z_m^{ij} from (96), (97), and (98). It is possible to make this simple approximation for the electron positions because for typical values of $\beta_e d$; F, Y, and Z are rapidly decreasing functions of $z^j - z^i$ so that as the approximation becomes worse the relative effects of these electrons becomes less. As a matter of fact, it would not be practical to include all electrons in computing the space charge forces so that only electrons within a phase π of electron i are included in obtaining the fields acting on i when it is at m .

The actual procedure for solution is then:

1. A_m, Θ_m, p_m^i, q_m^i, r_m^i, and ϕ_m^i are known for $i = 1,2,\cdots,N$.
2. Select a particular electron i and another electron j and compute t_m^{ij} and s_m^{ij} from (102) and (103) and F_m^{ij}, Y_m^{ij}, Z_m^{ij} from (96), (97), and (98).

3. Repeat this for $j = 1,2,\cdots,N$, excluding the case $i = j$ from Y_m^{ii} and Z_m^{ii} since an electron produces no force on itself. The electrons j are to be the electrons within phase π of electron i.

4. Form the $\sum_j Y_m^{ij}$, $\sum_j Z_m^{ij}$ and $\sum_j F_m^{ij}$. Use (90), (91), (92), and (93) to compute p_{m+1}^i, q_{m+1}^i, r_{m+1}^i and ϕ_{m+1}^i, and store $\sum_j F_m^{ij}$ for use in computing A_{m+1} and Θ_{m+1}.

5. Repeat steps (2), (3), (4) for $i = 1,2,3, \cdots, N$.

6. Use the values of $\sum_j F_m^{ij}$ obtained in (4) to compute A_{m+1} and Θ_{m+1}. All the quantities of step (1) are now known at step (2) and the cycle can be repeated.

CONCLUSION

A set of equations describing the nonlinear interaction between a beam of electrons focussed by crossed electric and magnetic fields and a wave on a slow-wave circuit has been obtained. These equations have been simplified to apply to the case of a thin beam when the interaction parameter D is small and the electron trajectories are straight lines in the absence of r.f. fields. The assumptions necessary to make these equations soluble have been discussed and a procedure for numerically integrating this set of equations has been outlined.

REFERENCES

1. J. R. Pierce, Traveling-Wave Tubes, D. Van Nostrand (1950).
2. R. W. Gould, "A Field Analysis of the M-Type Backward Wave Oscillator" California Inst. of Technology, Electron Tube and Microwave Laboratory, Technical Report No. 3, September 1955.
3. M. Muller, "Traveling Wave Amplifiers and Backward Wave Oscillators", Proc. IRE 42, 1651 (1954).
4. R. W. Gould, "Space Charge Effects in Beam-Type Magnetrons", California Institute of Technology, Electron Tube and Microwave Laboratory, Tech. Report No. 5, June 1957.

5. J. Feinstein and G. S. Kino, "The Large-Signal Behavior of Crossed-Field Traveling-Wave Devices", Proc. IRE 45, 1364, (1957).

6. A. T. Nordsieck, "Theory of the Large-Signal Behavior of Traveling-Wave Amplifiers", Proc. IRE 41, 630 (1953).

7. H. C. Poulter, "Large Signal Theory of the Traveling-Wave Tube", Tech. Report No. 73, Electronics Research Laboratory, Stanford University, (1954).

8. P. K. Tien, L. R. Walker, V. M. Wolontis, "A Large Signal Theory of Traveling-Wave Amplifiers", Proc. IRE 43, 260(1955).

9. J. E. Rowe, "A Large-Signal Analysis of the Traveling-Wave Amplifier", University of Michigan, Electron Tube Laboratory, Technical Report No.19, (1955).

10. W. R. Smythe, Static and Dynamic Electricity, McGraw Hill Company, (1950), Chapter 4.

Nonr 220(13)

DISTRIBUTION OF TECHNICAL REPORTS

Chief of Naval Research 2
Navy Department - CODE 427
Washington 25, D. C.

Director, Naval Research Lab.
Washington 25, D. C.
Attn: Code 5240 1
Code 7130 1
Code 2000 6
Code 5430 1

Commanding Officer 1
Office of Naval Res.Branch Office
1000 Geary Street
San Francisco, California

Scientific Liaison Officer 25
Office of Naval Research, London
c/o Navy 100, Box 39, F.P.O.
New York, New York

Commanding Officer 2
Office of Naval Res.Branch Office
1030 E. Green Street
Pasadena, California

Commanding Officer 1
Office of Naval Research Branch Of.
The John Crerar Library Building
86 East Randolph Street
Chicago 1, Illinois

Commanding Officer 1
Office of Naval Res.Branch Office
346 Broadway
New York 13, New York

Officer-in-Charge 3
Office of Naval Research
Navy No. 100
Fleet Post Office
New York, New York

Chief, Bureau of Aeronautics EL 4 1
Navy Department EL 43 1
Washington 25, D. C. EL 45 1

Chief, Bureau of Ordnance
Navy Department Re 4 1
Washington 25, D.C. Re 9 1

Periodicals Librarian
General Library
California Inst. of Technology 1

Lincoln Laboratory
Massachusetts Inst. of Technology 1
Cambridge 39, Massachusetts

Signal Corps Resident Engineer
Electronic Defense Laboratory 1
Post Office Box 205
Mountain View, California

Cornell Aeronautical Laboratory 1
Cornell Research Foundation
Buffalo 21, New York

Director, Electronics Defense Gr. 1
Engineering Research Inst.
University of Michigan
Ann Arbor, Michigan

Georgia Inst. of Technology 1
Atlanta, Georgia
Attn: Librarian

Fred D. Wilimek 1
Varian Associates
611 Hansen Way
Palo Alto, California

John Dyer 1
Airborne Instrument Laboratory
Mineola, L.I., New York

Bell Telephone Laboratories
Murray Hill, New Jersey
Attn: Librarian 1
J. R. Pierce 1

Hughes Aircraft Company
Culver City, California 1
Attn: Mr. Milek, Tech.Librarian

RCA Laboratories
Princeton, New Jersey 1
Attn: Mr.Herold and H.Johnson

Federal Tele. Laboratories
500 Washington Avenue
Nutley, New Jersey
Attn: W. Derrick 1
K. Wing 1

Technical Library
G. E. Microwave Laboratory
601 California Avenue 1
Palo Alto, California

Chief of Naval Operations Op 20X 1
Navy Department Op 421 1
Washington 25, D. C. Op 55 1

Director, Naval Ordnance Laboratory 1
White Oak, Maryland

Director, Naval Electronics Lab. 1
San Diego 52, California

Dept. of Electronics and Physics 1
U.S. Naval Post Graduate School
Monterey, California

Commander Code 366 1
Naval Air Missile Test Center
Point Mugu, California

U. S. Naval Proving Ground 1
Attention: W. H. Benson
Dahlgren, Virginia

Commander 1
U.S. Naval Air Development Center
Johnsville, Pennsylvania

Committee on Electronics 1
Research and Development Board
Department of Defense
Washington 25, D. C.

Director, National Bureau of Stds. 1
Washington 25, D. C.
Attn: Div. 14.0 CRPL, Librarian

Commanding Officer 1
Engineering Res. and Dev. Lab.
Fort Belvoir, Virginia

Ballistics Research Laboratories 2
Aberdeen Proving Ground, Maryland
Attn: D. W. H. Delsasso

Chief, Ordnance Development Div. 2
National Bureau of Standards
Connecticut Ave and VanNess St, NW
Washington 25, D. C.

Commanding Officer 1
Frankford Arsenal
Bridesburg, Philadelphia, Pa.

Columbia Radiation Laboratory 1
538 W. 120th Street
New York 27, New York

Countermeasures Laboratory 1
Gilfillan Brothers, Inc.
1815 Venice Boulevard
Los Angeles, California

The Rand Corporation 1
1700 Main Street
Santa Monica, California
Attn: Librarian

Technical Library 1
Research and Development Board
Pentagon Building
Washington 25, D. C.

The Motorola Riverside Research Lab. 1
8330 Indiana Avenue
Riverside, California
Attn: Mr. John Byrne

Chief, Bureau of Ships CODE 816 1
Department of the Navy 820 1
Washington, D. C. 840 1

Panel on Electron Tubes 1
346 Broadway (8th Floor)
New York 13, New York

Supervisor of Research Laboratory 1
Electrical Engineering Building
Purdue University
Lafayette, Indiana

W. E. Lear 1
University of Florida
Department of Electrical Eng.
Gainesville, Florida

Director, Microwave Research Inst. 1
Polytechnic Institute of Brooklyn
55 Johnson Street
Brooklyn 1, New York

Material Laboratory Library,CODE 912B
New York Naval Shipyard 1
Brooklyn 1, New York

University of Washington
Department of Electrical Engineering
Seattle, Washington
Attn: E. A. Harrison 1
A. V. Eastman 1

Thermionics Branch 5
Signal Corps Eng. Laboratories
Evans Signal Lab, Bldg. 42
Belmar, New Jersey

Commanding General
Air Research and Dev.Command
Post Office Box 1395
Baltimore 3, Maryland
Attn: RDRR 1
RDDDE-3 1
RDDDE-5 1

Commanding General WCLC 1
Wright Air Devel.Center WCLRC 1
Wright-Patterson Air Force Base
Ohio

Commanding General CRRE 1
Air Force Cambridge Research Cen.
230 Albany Street
Cambridge 39, Massachusetts

Commanding General RCRW 1
Rome Air Development Center
Griffiss Air Force Base
Rome, New York

Armed Services Tech.Information 5
Document Service Center DSC-SA
Knott Building
Dayton 2, Ohio

Director CR4582 1
Air University Library
Maxwell Air Force Base, Alabama

Chief, Western Division 1
Air Research and Devel. Command
Office of Scientific Research
P.O. Box 2035, Pasadena, Calif.

Microwave Laboratory 1
Stanford University
Stanford, California
Attn: F.V.L. Pindar

University of Michigan 1
Electron Tube Laboratory
Ann Arbor, Michigan
Attn: E. Rowe

Mr. John S. McCullough 1
Eitel-McCullough, Inc.
San Bruno, California

University of Colorado 1
Department of Electrical Engineering
Boulder, Colorado

Ramo-Wooldridge Corporation 1
Control Systems Division
P.O. Box 900B
Hawthorne, California
Attn: Librarian

Electrical Engineering Dept. 1
Princeton University
Princeton, New Jersey

National Union Radio Company 1
350 Scotland Road
Orange, New Jersey
Attn: Dr. A. M. Skellet

Dr. J. E. Shepherd 1
Sperry Gyroscope Company
Great Neck, L.I., New York

W. L. Maxson Corporation 1
460 West 34th Street
New York 1, New York
Attn: M. Simpson

Mr. H. R. Argento
Raytheon Corporation 1
Waltham, Massachusetts

Dr. E. D. McArthur
Electron Tube Laboratory
General Electric Company 1
The Knolls
Schenectady, New York

General Electric Company 1
Electronic Components Division
Power Tube Department
Microwave Laboratory at Stanford
Palo Alto, California

Johns Hopkins University 1
Radiation Laboratory
1315 St. Paul Street
Baltimore 2, Maryland
Attn: Margaret Poole, Librarian

Raytheon Corporation 1
Waltham, Massachusetts
Attn: Librarian

Cascade Research
53 Victory Lane 1
Los Gatos, California

Engineering Library 1
Stanford University
Stanford, California

Research Laboratory of Electronics 1
Massachusetts Inst. of Technology
Cambridge 39, Massachusetts

Sloane Physics Laboratory 1
Yale University
New Haven, Connecticut
Attn: R. Beringer

Mr. H J Reich
Department of Electrical Eng. 1
Yale University
New Haven, Connecticut

Electron Tube Section 1
Electrical Engineering Dept.
University of Illinois
Champaign, Illinois

Chairman, Div. of Electrical Eng. 1
University of California
Berkeley 4, California

Technical Report Collection 1
303A, Pierce Hall
Harvard University
Cambridge 38, Massachusetts

Laboratory for Insulation Research
Massachusetts Inst. of Technology 1
Cambridge 39, Massachusetts
Attn: A. von Hippel

Chief, West Coast Office 1
Signal Corps Eng. Laboratories
75 So. Grand Avenue
Pasadena, 2, California

Office of Technical Services 1
Department of Commerce
Washington 25, D. C.

Professor W. P. Dyke 1
Linfield College
McMinnville, Oregon

Additional copies for Staff and
Future Requirements 25

Stanford Electronics Laboratories 1
Stanford University
Stanford, California
Attn: Applied Electronics Laboratory
Document Library

Mr. E. C. Okress, Technical Director
Westinghouse Microwave Center
310 N. Aurora Street 1
Ithaca, New York