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Nomination Processes and Policy Outcomes*

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ABSTRACT

We provide a set of new models of three different processes by which political parties nominate candidates for a general election: nominations by party leaders, nominations by a vote of party members, and nominations by a spending competition among potential candidates. We show that more extreme outcomes can emerge from spending competition than from nominations by votes or by party leaders, and that non-median outcomes can result via any of these processes. When voters (and potential nominees) are free to switch political parties, then median outcomes ensue when nominations are decided by a vote but not when nominations are decided by spending competition.

Nominations are a critical part of many elections. While the modeling of elections is extensive, there are no systematic studies of how the specifics of the nomination process affect election outcomes. In this paper we develop and analyze three simple models of prominent nomination processes, all within the same basic election setting. We show that the differences in nomination process can have a large impact on the election outcome.

In particular, we consider a setting in which two competing political parties simultaneously nominate candidates (out of their respective memberships) for an election. If elected, a candidate chooses his or her most preferred policy from a one-dimensional set of potential policies. All voters (who compose the parties and are thus also the potential

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candidates) have single-peaked preferences. The vote over the two nominees is by majority rule. The three different nomination processes we model are as follows:

1. A party leader, who is a member of the party (and thus one of the potential candidates), unilaterally chooses the party's nominee.
2. Party members vote over who should be the party's nominee.
3. Party members compete for the nomination by spending. The party member who spends (or is willing to spend) the most money wins the nomination.

Our models of these processes should prove to be useful beyond the current paper, especially when models of nomination processes become part of more general election models.

In each case we define an equilibrium to be a pair of nominees, one for each party, such that the following is true:

1. *Nomination by party leaders:* neither party leader would want to change his or her nominee, given the nominee put forth by the other party and anticipating the eventual election against the other party's nominee.
2. *Nomination by a vote of party members:* there is no other party member who would defeat the party's nominee in a majority vote of the party's members, anticipating the eventual election against the other party's nominee.
3. *Nomination by spending competition:* no other party member would be willing to spend more than the party's nominee in order to secure the party nomination, anticipating the eventual election against the other party's nominee.

As we vary the nomination process, the main characterizations of the election outcomes are as follows. We first analyze the nomination by party leaders. In this case we show that the winner can come from either party, but lies between the overall median and the leader of the party that contains the median. The outcome can range anywhere between these points. We then show that nominations by party vote are equivalent to situations where nominations are made by party leaders, but where the party leaders are the medians of the parties. This provides an intuitive relationship between nominations by a party vote and nominations by party leaders. This then implies that the election outcome when nominations are by a vote by party members always lies between the overall median and the median voter of the party which contains the overall median voter. In contrast, the outcome under spending competition is not constrained to any particular interval. Depending on the intensity of voters' preferences, the outcome can be almost anywhere. Elections by spending competition differ more dramatically from the other nomination processes, have more complicated equilibrium existence issues, and depend on the preferences of various party members in complex and subtle ways. In particular, nominations by spending competition can lead to extremist nominees from either or both parties, and can lead to extreme policy outcomes. Finally, we show that endogenizing party membership leads to a convergence to the median in the case of nomination by votes, while if nominations are by spending competition, extremist outcomes can ensue even with endogenous parties.

Although we model a simple one-dimensional left–right spectrum with single-peaked voter preferences, we show that the median voter's preferences do not always determine

the outcome.[1] Our analysis offers a different explanation from other models exhibiting non-median outcomes, as we show that incorporating parties and nominations into the electoral model can create non-median outcomes.[2] Alternative models with non-median outcomes include analyses that expand the dimensions of the outcomes (e.g. Hinich 1977), include valence, (Aragones and Palfrey 2002, Groseclose 2001), have more than two candidates (e.g. Hotelling 1929, Palfrey 1984), have citizen candidates who run at a cost (Besley and Coate 1997, Osborne and Slivinsk 1996), are based on probabilistic voting (e.g. Coughlin 1992), or focus on candidate signaling and character (Callander 2005, Kartik and McAfee 2006). In our model it is candidates' willingness to spend to influence the nomination and ultimately the election that drives the outcome away from the median.

Nomination processes are the focus of empirical work by Gerber and Morton (1998), who show that differences in the laws governing electoral primaries can have an effect on the outcome. They examine the consequences of different primary laws across states in the United States and show that closed primaries can lead to more extreme nominations, while semi-closed primaries (allowing voters to declare a party on election day and for independents to vote in a primary) lead to even more moderate nominees than completely open primaries (where strategic voting across parties can occur). Our model is one where party members are the only ones who vote, and so it is a closed system. However, the differences between nomination by party leadership and nomination by party members' vote can be seen as reflecting different degrees of closure. Moreover, once we endogenize party membership, we move closest to a semi-closed system. In that case we find that the outcome converges to the overall median, which is consistent with their finding that semi-closed systems are the most moderate. Our analysis of nomination by spending competition is harder to connect to their classification.

The rest of the literature that has examined primaries and nomination processes has focused on other aspects, such as relating the nomination process to party structure (e.g. Epstein 1986, Jewell 1984, Ranney 1975), or modeling information dispersion and acquisition through primaries (e.g. Bartels 1988, Callander 2005, Meirowitz 2005). Thus, our work presents a first systematic modeling of how nomination procedures relate to electoral outcomes.

THE GENERAL MODEL

Our model is related to a citizen–candidate framework,[3] but one where the citizens cannot simply decide to run but must be nominated through their parties. There are

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[1] In fact, observed political outcomes often deviate significantly from the median. For example, Stone and Rapoport (1994) show that the candidates competing for and winning U.S. Presidential nominations cover a wide range of political ideologies. (See also Aldrich 1980, Arterton 1977, and Gurian 1993 for more discussion of the nomination process.)

[2] Independent work by Serra (2006) also shows a model with primaries and non-median outcomes. However, that model is very different from any of ours, having two Downsian candidates in each party, with uncertainty over voter preferences, or incumbency or dogmatic preferences generating nonmedian outcomes.

[3] See Osborne and Slivinski (1996) and Besley and Coate (1997).

n voters, and voter i's preferences are represented by a utility function u_i: $[0, 1] \to \mathbb{R}$ Voters have single-peaked preferences over the interval [0, 1], and the peak of voter i is denoted x_i.

Without loss of generality, we order voters by their labels, so that $x_1 \leq x_2 \leq \ldots \leq x_n$. To keep things simple, assume that n is odd. Also, assume that no voter is indifferent between any distinct candidates i and j.

Voters are divided into two parties, P_1 and P_2, that partition $\{0,1, \ldots ,n\}$. In the first part of the paper we analyze what happens when the two parties are fixed; later we return to study party formation. We use a notation of P_ℓ and $P_{-\ell}$ to indicate a generic party ℓ and its competitor.

In general, we allow for arbitrary party structures, so that it could be that the parties are not simply left and right parties, but overlap. For instance, it could be that one party has some left- and right-minded voters, and the other party has some centrists. This means that it is possible for some voters to vote for the other party in the final outcome. We say that there is *no overlap* in parties if for each $\ell \in \{1, 2\}$ and any i and $j \in P_\ell$, there does not exist any $k \in P_{-\ell}$ such that $x_i \leq x_k \leq x_j$.

Let $M = (n + 1)/2$ be the overall median voter out of $P_1 \cup P_2$, and let M_ℓ denote a median of party ℓ.[4] Let $W[i, j]$ denote the majority winner among any two candidates i and j. Given that a candidate is identified with his or her ideal point, we abuse notation and let $u_i(j)$ denote $u_i(x_j)$, or the utility that i gets if j wins the overall election. Finally, let

$$d_i(j, k) = u_i(j) - u_i(k). \tag{1}$$

This is the difference in utility between what i gets if j is the overall winner versus what i gets if k is the overall winner.

The political process is as follows:

1. Each party (simultaneously) nominates one of its members to serve as its candidate.
2. Voters vote for one of the two candidates, and a candidate is elected by majority rule with ties broken by a fair coin toss.
3. The policy outcome is the elected candidate's most preferred policy.

We carefully model the nomination processes in (1) through equilibrium definitions, where everyone anticipates the election and outcome in (2) and (3). Given just two parties, it is a (weakly) dominant strategy for each voter to vote for his or her preferred candidate in (2). (3) is motivated by a standard argument that candidates cannot credibly commit to follow any policy other than their most preferred policies.[5]

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[4] One of the parties will have two medians, and we are explicit in cases where that matters.

[5] It is sufficient when voters have well-defined expectations regarding what policy each candidate would implement before the nomination process takes place.

NOMINATIONS WITH A FIXED PARTY STRUCTURE

Here we analyze what happens when the distribution of voters across the two parties is fixed. As discussed above, we model three different processes for the ways that parties nominate a candidate.

1. A party leader (one of the party members) unilaterally chooses the candidate.
2. Party members vote over who should be their candidate.
3. Party members compete for the nomination by spending, with the nominated candidate being the party member who spent the most.

Each of these requires a corresponding definition of equilibrium.

EQUILIBRIUM DEFINITIONS FOR THE THREE NOMINATION PROCEDURES

The definitions of equilibrium for each of the nomination procedures are as follows.

Equilibrium with Nominations by Party Leaders

An *equilibrium* in the case of nominations by party leaders is a pair of nominations, denoted $Nom(P_1) \in P_1$ and $Nom(P_2) \in P_2$, such that for each party ℓ, $W[Nom(P_\ell), Nom(P_{-\ell})]$ is preferred by the leader of party ℓ to $W[x, Nom(P_{-\ell})]$, for any $x \in P_\ell$.

This definition requires that neither party leader can benefit by changing his or her nomination.

Equilibrium with Nominations by a Vote of Party Members

An *equilibrium* in the case of nominations by a vote of party members is a pair of nominations $Nom(P_1) \in P_1$ and $Nom(P_2) \in P_2$ such that there does not exist any $x \in P_\ell$ such that $W[x, Nom(P_{-\ell})]$ is preferred by a strict majority of voters in P_ℓ to $W[Nom(P_\ell), Nom(P_{-\ell})]$.[6]

This definition requires that a party's nominee not be beaten in a head-to-head vote with some other potential nominee, given the other party's nomination. Thus, the nominee of a party must be a sort of internal Condorcet winner, given that voters anticipate the eventual election and overall outcome. This yields some intuitive interactions between the parties' nominees, as candidates who appeal to the party in the abstract might still be defeated for the nomination if they lack a chance of winning the subsequent election. Even though most of the interesting interaction under nomination by voting is between candidates that are viable given anticipations of what the other party

[6] We note that this definition is related to Duggan's (2001) definition of "group stable" equilibrium, which he defines for abstract games played between groups of players.

will do, we still find that parties' nominees can drift away from the party and overall median voters.

Equilibrium with Nominations by Spending Competition

An *equilibrium* in the case of spending competition by party members is a pair of nominations $i = Nom(P_1) \in P_1$ and $k = Nom(P_2) \in P_2$ such that

$$u_i(W[i, k]) - u_i(W[j, k]) \geq u_j(W[j, k]) - u_j(W[i, k]) \tag{2}$$

for all $j \in P_1$ and

$$u_k(W[k, i]) - u_k(W[h, i]) \geq u_h(W[h, i]) - u_h(W[k, i]) \tag{3}$$

for all $h \in P_2$.

This definition captures competition by candidates through spending. It requires that a party's nominee would not be beaten by some other nominee from the same party in a head-to-head spending competition, given the other party's nomination. That is, the party's nominee would be willing to outspend any challenger in order to keep the nomination. Here, for instance, $u_i(W[i, k]) - u_i(W[j, k])$ represents the maximum that i is willing to spend in order to win the nomination instead of having j win it, given that k is the nominee of party 2. The definition is somewhat subtle since how much a candidate would be willing to spend can depend on whom they are bidding against. A candidate might be willing to spend more to defeat a candidate who differs drastically from his or her own stance, than to defeate a candidate who is closer in stance.

This definition captures the essential aspect of competition by spending, namely how much different candidates would be willing to pay in order to gain a nomination, without getting caught up in a detailed model of the process itself. One could explicitly model this via an auction process. One natural process would be an "all-pay" auction, where each candidate spends as he or she wishes and the winner is the candidate who spends the most. An equilibrium of an alternating move version of that auction where candidates are aware of each other's willingness to pay corresponds to the equilibrium we define here. That is, a candidate who is willing to spend more than every other candidate would win the auction by spending a minimal amount as no other candidates would want to spend given that they anticipate eventually being outspent. Our setting is slightly more complicated, as a candidate's willingness to spend depends on who he or she is bidding against, but the equilibrium is an extension of that where there are private values. We provide more details in the appendix.

The important difference between nomination by spending competition and the other nomination processes is that intensity of preferences matter under spending competition, while it is only ordinal and not cardinal preferences that matter in the party leadership and voting nomination settings. This is what allows for a wide variety of outcomes under this setting, depending on how much different candidates are willing to spend to win office. Also, there are some other effects that arise, as candidates might seek the nomination even though they would lose the subsequent election in cases where they wish to prevent another nominee from obtaining office.

NOMINATION BY PARTY LEADERS

We now characterize equilibrium under each of the nomination procedures, starting with the case of a nomination by party leaders.

Example 1 Multiple Equilibria Under Party Leaders, No Overlap. There are seven voters, $N = \{1, \ldots, 7\}$, and two parties that partition N as follows: $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5, 6, 7\}$. The voters' ideal points are ordered by their labels.

First, note that, in this example, the winner will come from P_2 regardless of who the leaders are. This follows since if 3 is nominated, 3 will win against any nominee from P_1, and all members of P_2 prefer 3 to either nominee of P_1.

In this example there are multiple equilibria, but all equilibria have the same outcome: the winner is the member of P_2 who is most preferred by the leader of P_2 out of those who beat 2. The winner must always lie in the interval between 4 (the median) and the leader of P_2. For example, if the leader of P_2 is 3, then 3 is the outcome. Note that here we already see the multiplicity of equilibria; P_1 is willing to nominate either 1 or 2, as it is irrelevant. Either nomination leads to the same outcome. If the leader of P_2 is 4, then 4 is the equilibrium outcome. If the leader is 5, then the outcome is either 4 if 2 beats 5, but is 5 if 5 beats 2. If the leader is 6, then the outcome is in $\{4, 5, 6\}$, and is the highest indexed member of this set that beats 2.

Some features of this example generalize. We find that there may be a multiplicity of equilibria, but that they always lie in a well-defined interval between the overall median and the party leader of the party containing the overall median.

Proposition 1 *There always exists an equilibrium under a nomination by party leaders. The winning candidate in any equilibrium lies in the interval between (and including) the overall median voter and the leader of the party which contains the overall median voter.*

The proof appears in the appendix.

The fact that the winner always comes from the interval between the overall median, M, and the leader k of the party that contains M is relatively straightforward. If the winner came from the other side of the median from k, then k could improve by nominating M. If the winner came from the other side of k, then k could improve by nominating himself or herself. The more specific details of the equilibrium structure are complicated and there is no simple formula. We can derive a simple characterization in the case of no overlap.

Proposition 2 *If there is no overlap in parties, then there is a unique equilibrium winner. The winning candidate comes from the party that contains the overall median, and the outcome is that party's leader's most preferred member from the set of those who beat all members of the other party.*

The proof is straightforward, following the logic of Example 1, and is left to the reader. The idea is that each party leader prefers its bordering member to any candidate of the other party. The larger party (the one with the median) then necessarily wins, as its leader has a nomination available that will beat all candidates of the other party, and that

he or she prefers to any nomination of the other party. The rest of the proposition then follows easily.

It is important to emphasize that, even in the case where the parties have no overlap and split so that one party includes all voters up to the median and the other has all voters from the median onward, the outcome might not be the median. As a simple example, consider a society with three voters, and party 1 is voter 1, and party 2 is voters 2 and 3 with 3 being the leader. If voter 2 prefers 3 to 1, then the outcome will be that voters 1 and 3 are nominated and 3 wins. So the median is not the outcome, even in this most central case.

While the case with no overlap produces a unique winner, things are more complicated when there is overlap in parties. In that case there can exist multiple equilibrium outcomes, and, depending on the configuration of parties, the winning nominee can come from either party. To get some feeling for this, consider the following example.

Example 2 Multiple Equilibria Under Party Leaders. There are seven voters, $N = \{1, \ldots, 7\}$, and two parties that partition N as follows: $P_1 = \{2, 3, 6\}$ and $P_2 = \{1, 4, 5, 7\}$. The voters' ideal points are ordered by their labels. The party leaders are 6 and 7. Let preferences be such that $W[i, 5] = i$ unless $i = 6$ or $i = 7$.

There is an equilibrium where the nominees are 6 and 7. There is also an equilibrium where the nominees are 3 and 4. This is an equilibrium even though both leaders would prefer the other equilibrium.[7] Note that these two equilibria have different parties winning. Note also that the set of equilibria is not connected in the sense that there is no equilibrium where 5 is the winner. The only equilibrium outcomes are 4 or 6.

We can refine the set of equilibria using strong equilibrium. Then, we end up with the selection of equilibria where the winner lies between the peaks of the party leaders. We provide the details of this refinement in the appendix.

NOMINATION BY A VOTE OF PARTY MEMBERS

We now turn to nomination processes by a vote of party members. As we show below, nominations by a vote of party members are equivalent to having nominations by party leaders where the party leaders are the medians of the parties.

Example 3 Nomination by Voting. Reconsider Example 1 where there are seven voters, $N = \{1 \ldots, 7\}$, and two parties, $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5, 6, 7\}$. The voters' ideal points are ordered by their labels.

In the case where 5 beats 2 in an election, then the unique equilibrium outcome and nominee from P_2 is 5, while there are two equilibria in that P_1 can nominate either 1 or 2. To verify this, it is enough to check that 5 would be the nominee of party 2 regardless of party 1's nomination. Voters 5, 6 and 7 prefer to have 5 nominated than either 3 or 4 (either of whom would win in the subsequent election against either candidate from

[7] Note that this is an equilibrium in undominated strategies given that 1 beats 6 (as 1 beats 5).

party 1), and so it is clear that 5 would defeat 3 and 4 for the nomination, regardless of party 1's nomination. So consider a nominee of 6 or 7. If that nominee would win against the nominee of party 1, then 3, 4 and 5 would all rather have 5 nominated. If that nominee would lose against the nominee of party 1, then 5, 6 and 7 would all prefer to have 5 nominated. This leaves 5 as the equilibrium nomination from party 2 in all equilibria.

If 2 beats 5, then one can verify that all equilibria have P_2 nominate 4, who wins the subsequent election.

We now show that at least one equilibrium always exists and relate the equilibrium structure under voting to the nominations by party leaders.

Proposition 3 *There always exists an equilibrium under a vote by party members. The set of equilibria coincides with that where the median voter in a party is a "party leader".*[8] *The winning candidate lies between the overall median and the median*[9] *of the party containing the overall median.*

The proof appears in the appendix.

The intuition for a party acting as if the median were a party leader is much more subtle than it would seem. For example, note that it is *not* always true that, given a comparison between two arbitrary candidates, if the median prefers one to the other then so does a majority. It is possible, when comparing candidates from opposite sides of the median, that the median's preferences are in the minority.[10] Nonetheless, the claim is true. To understand this, consider the nomination of one party taking the nomination of the other party as given.[11] The set of possible nominees who could defeat the nominee of the other party is either (a) an interval including the median of the party, or (b) an interval lying entirely to one side of the party median (which then must be on the side of the other party's nominee). In case (a), where the set of viable nominees includes the party median, then the party median would be preferred to the nominee from the party by a majority of the voters of the party, as the comparison would always boil down to a comparison of the party median and some other outcome. In that case the party median is the only possible nominee in response to the other party's nominee. If instead case (b) applies and the interval is entirely on one side of the median (the same side of the party median as the other party's nominee), then any two viable nominees from that interval both lie on the same side of the party median and so a majority of the party will have preferences that agree with the party median's preferences.

Although there could be a discontinuity here when the median voter changes from one party to another, as this can move the winning candidate from one side of the median to the other, in many cases the change will not be very substantial. For instance, if each

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8 Given that one party will have two medians, this refers to a union of the sets of equilibria where each one of the two medians is party leader.

9 This is the furthest median voter of the party, if there are an even number of voters.

10 For example, voters other than the median may prefer candidates to their right over candidates to their left, while the median's preferences run in the other direction.

11 Consider the case where the first party has a single median and see the appendix for the case with two medians.

party has a fairly dense set of potential nominees near the median voter, then the eventual winner must be very close to the median. The discontinuity here comes from the finite set of potential nominees and the fact that we do not allow a candidate to do anything other than institute his or her most preferred policy.

While the nomination by party voting allows for non-median outcomes overall, the chosen candidate still comes from a well-defined interval between the overall median and the median of the party containing the overall median. As we shall now see, the equilibrium looks very different when we consider nominations by party spending.

NOMINATION BY SPENDING COMPETITION

We begin the analysis of nomination by spending competition with some examples. First, we show an equilibrium where there is an extreme outcome in terms of each party's nominee and the overall winner.

Example 4 Nomination by Spending Competition. Again, reconsider Example 1 where there are seven voters, $N = \{1, \ldots, 7\}$, and two parties, $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5, 6, 7\}$. The voters' ideal points are ordered by their labels.

Note first, that there are preference configurations where the nominee of P_2 is 3, even though all other members of party 2 would prefer to nominate 4, and even though that nominee does not lie between the overall median and the median of P_2 (in contrast to the case of nomination by voting). For example, if $d_3(3, i) > d_i(i,3)$ for all $i > 3$, then 3 wins the nomination of P_2 and the overall election.

It is also possible to have extremists from both parties nominated. For instance, suppose that all members of P_2 prefer any member of P_2 to any member of P_1. In this case the nominee of party 2 will win the election and so it is as if there were just one party and spending competition among its members. If $d_7(7, i) > d_i(i,7)$ for each $i \in \{3, 4, 5, 6\}$, then the unique equilibrium outcome would be that 7 wins the nomination and then the overall election. As the nominee from P_1 is irrelevant, we could see extreme nominees from both parties.

This example shows the contrast between nomination by spending competition and nomination by voting. Under spending competition the outcome could be any member of P_2, while in the voting case it would have to be either 4 or 5.

While the possible outcomes under nominations by spending competition are more varied than under nominations by voting, we can still say something about the outcome, at least in the case where there is no overlap in the parties which is a very natural case to consider.

Proposition 4 *If there is no overlap in parties, then any equilibrium winner under nomination by spending competition is from the party containing the median, and is a candidate who defeats all candidates from the other party.*

The proof again appears in the appendix, but is easy to explain. In this case all members of the party containing the median prefer the candidate k closest to the other party to

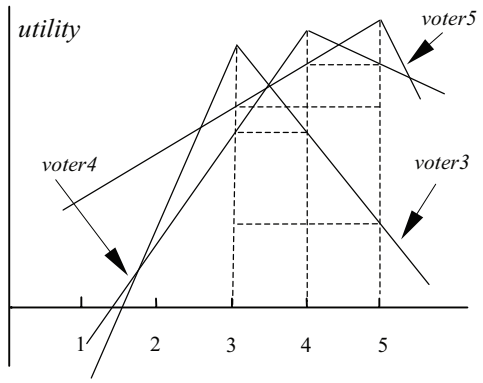

Figure 1. Preferences generating an intra-party cycle

any nominee of the other party. This means that any candidate willing to outspend k must also be able to win the election.

Proposition 4 does not mention the issue of existence. This is because of another contrast between nomination under spending competition and the other nomination procedures. Under spending competition an equilibrium need not always exist, as shown in the next example. In fact, the example shows nonexistence even in the no overlap case.

Example 5 Non-existence of Equilibrium Under Party Spending. There are five voters $N = \{1, \ldots, 5\}$ and two parties, $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5\}$. Consider the utility functions in Figure 1 for voters 3, 4 and 5. Every member of P_2 prefers any member of P_2 to any member of P_1. So, it is clear that the nominee of P_1 is irrelevant. Let $d_4(4, 3) > d_3(3, 4)$. Then 3 cannot be the nominee as 3 would be outspent by 4. Also, let $d_5(5, 4) > d_4(4, 5)$. Then 4 cannot be the nominee as 4 would be outspent by 5. This leaves only 5 as the potential nominee. However, if $d_3(3, 5) > d_5(5, 3)$, then 5 cannot be the nominee either. Thus, there are situations where there is no equilibrium.

The nonexistence of equilibrium in the case of spending competition follows from the fact that intensities of preferences matter and might not be ordered across party members in any nice way.

Sufficient Conditions for Existence Under Party Spending with No-Overlap in Parties

We have seen that an equilibrium may not exist under nominations by spending competition, even in a five-voter[12] world with single-peaked preferences and no overlap in parties. We now look for sufficient conditions on preferences for an equilibrium to exist.

[12] One could even simplify the example further having only one party, and reduce it to a three-voter world.

In the case of no overlap, an intuitive condition is sufficient to rule out the cycle exhibited in the above example and to restore existence. We abuse notation and let $i < j$ denote that x_i is to the left of x_j.

Let us say that preferences satisfy the *extremist* condition if $d_i(i, k) \geq d_j(j, k)$ whenever $i \leq j \leq k$ or $i \geq j \geq k$. This condition says that if one voter is willing to spend a given amount to move the outcome in a given direction (say to the left), then voters further to the left would be willing to spend at least as much for the same change. Under this condition there is a consistent ordering to the intensity of voters' preferences and this is enough to avoid the cycles from the example above and guarantee existence.

The extremist condition is clearly very strong, and one would expect to find many settings where it fails. However, as we see from Example 5, something on the order of this condition is needed to establish equilibrium existence. There are cases where the extremist condition is satisfied. For instance, if preferences are Euclidean (so that utility is just the opposite of the distance between the outcome and the peak, as is often assumed in the literature), then the condition is clearly satisfied.

Proposition 5 *If there is no overlap in parties and the extremist condition is satisfied, then there exists an equilibrium under nomination by spending competition.*

The proof of the proposition is constructive and appears in the appendix. The idea is that, under the extremist condition, the relevant candidates are only extreme ones. We have to be a bit careful, as the relevant ones in some cases need to be defined relative to those who win against nominees of the other party.

Sufficient Conditions for Existence Under Party Spending: the General Case

When there is an overlap in parties, cycles turn out to be surprisingly robust to preference restrictions. Even the nice ordering of preferences under the extremist condition fails to be sufficient to guarantee existence. In fact, we show that equilibria fail to exist even under stronger preference restrictions. We examine two preference restrictions: First, a "strong extremist" property (that is a strengthening of the extremist condition), and second, an ordered preference intensities condition. The failures of these two conditions to guarantee existence helps illustrate another condition, which we call the "directional party" condition, which ensures existence.

Preferences satisfy the *strong extremist* condition if for all players i, j, k such that $i \leq j \leq k$ and all alternatives h, t with $i \leq h \leq t \leq k$,

1. $d_i(h, t) > d_k(t, h)$ implies $d_i(h', t') > d_j(t', h')$ for all $i \leq h' \leq t' \leq j$ and,
2. $d_k(t, h) > d_i(h, t)$ implies $d_k(t', h') > d_j(h', t')$ for all $j \leq h' \leq t' \leq k$.

The strong extremist condition says that if one voter i has more intense preferences than another voter k regarding pairs of candidates in between those two (h and t), then voter i has more intense preferences than some other voter j who lies in the same direction as k, over pairs of alternatives between i and j. This, again, is a strong condition that imposes some consistency on preferences to rule out cycles. Similar to the extremist condition, while it is strong and only satisfied in special cases, it is satisfied by Euclidean preferences that are directly proportional to distance between an alternative and a voter's peak. Even

with this strengthening of the extremist condition, there are situations where no equilibrium exists, provided there is overlap between the parties.

Example 6 Non-existence of Equilibrium Under the Strong Extremist Condition. There are seven voters with ideal points at locations: $x_1 = 0$, $x_2 = 1$, $x_3 = 3$, $x_4 = 6$, $x_5 = 7$, $x_6 = 9$, $x_7 = 10$. Voters' preferences are distance-based, so they prefer candidates who are closer to their ideal points to those farther away. Two parties partition N as follows: $P_1 = \{1, 3\}$ and $P_2 = \{2, 4, 5, 6, 7\}$.

We suppose that the strong extremist condition is satisfied in terms of preference intensities and the following are true:[13]

$d_7(7, 2) > d_2(2, 7)$
$d_1(2, 3) > d_3(3, 2)$
$d_2(3, 6) > d_6(6, 3)$.

Let us show that there is no equilibrium. We start by showing that there is no equilibrium with 1 as the nominee of P_1. Every candidate in P_2 beats 1. Thus, by the strong extremist condition, the only candidates for nomination from P_2 are 2 and 7. The nominee for P_2 must then be 7, since $d_7(7, 2) > d_2(2, 7)$. However, if 7 is nominated by P_2, then both 1 and 3 in P_1 would rather have 3 be nominated over 1. Thus, it is impossible to have an equilibrium with 1 as the nominee of P_1. So, let us consider 3 as the nominee of P_1; 2 cannot be the nominee of P_2, as then $d_1(2, 3) > d_3(3, 2)$ implies that 1 would outbid 3 for the nomination of P_1. So the nominee of P_2 must come from $\{4, 5, 6, 7\}$. It cannot be 6, since 2 would outbid 6 given that $d_2(3, 6) > d_6(6, 3)$. By the strong extremist condition, this also means that it cannot be 5 or 4 for the same reason. So we are left with 7. However, if 7 is nominated, then 3 wins; 6 would then wish to outbid 7 (and 7 would be happy to be outbid). Thus there is no equilibrium.

Suppose now that we can order the intensity of candidate preferences. Preferences satisfy the *ordered preference intensity* condition if every distinct pair of voters i and j can be ordered in terms of preference intensity such that either $|d_i(h, k)| > |d_j(h, k)|$ (for all $h \neq k$)[14] or $|d_j(h, k)| > |d_i(h, k)|$ (for all $h \neq k$). Notice that having more intense preferences is a transitive relationship. Even this strong a condition is not enough to guarantee existence.

Example 7 Non-existence of Equilibrium when Preference Intensities are Ordered. There are seven voters with ideal points $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 7$, $x_5 = 8$, $x_6 = 9$, $x_7 = 11$, and who prefer outcomes closest to their own peaks. Two parties partition N as follows: $P_1 = \{1, 4, 5, 6, 7\}$ and $P_2 = \{2, 3\}$. Preference intensities are ordered so that $2 > 3 > 7 > 1 > 6 > 5 > 4$, where "$i > j$" means "$i$ has more intense preferences than j".

We now check that there is no equilibrium. No equilibrium can support the nomination of voter 2 in P_2 without the nomination of 7 in P_1 because 7 could win the final election and has the most intense preferences in P_1. But the pair (7, 2) is not an equilibrium either

[13] These three relationships are consistent with the strong extremist condition.

[14] It would be more natural to require this only when h and k lie to one side of i and to one side of j, but even under this very strong condition equilibria fail to exist.

since voter 2 would be outspent by voter 3, as 3 is the best outcome that 2 can rationally expect given the next round. Following the same logic, (7, 3) is not an equilibrium because 7 would be outspent by 4, 5 or 6. Furthermore, in each of (4, 3), (5, 3), (6, 3), voter 1 would outspend these other potential nominees from P_1 as he or she has the most intense preferences in P_1 after 7. Finally, voter 2 would not let voter 3 win the nomination under (1, 3), so that cannot be an equilibrium.

These last two examples suffer similar cycling issues: we first begin to move in one direction, but then someone on the opposite side breaks the directional trend by stealing the nomination, and starts a cycle. The following condition is sufficient to prevent cycling, thus implying equilibrium existence.

Preferences satisfy the *directional-party condition* if for each party ℓ, either

1. $d_i(h, t) \geq d_j(t, h)$ for all $i \in P_\ell$ and $j \in P_\ell$ and $h, t \in N$ such that $i \leq h < t \leq j$, or
2. $d_i(h, t) \leq d_j(t, h)$ for all $i \in P_\ell$ and $j \in P_\ell$ and $h, t \in N$ such that $i \leq h < t \leq j$.

The directional party condition says that there is a consistent direction with respect to which a party's preferences can be ordered. Either it is always voters more to the left who care at least as much as voters to the right, or vice-versa. Again, this condition is very strong, but satisfied when preferences are Euclidean (the opposite of the distance between an alternative and the voter's peak).

Proposition 6 *If preferences satisfy the directional-party condition, then an equilibrium under nomination by spending competition exists.*

The proof is in the appendix, and uses an algorithm that identifies an equilibrium under the directional party condition.

Our results have shown that nominations by party leaders and party vote have an interesting relationship, in that nominations by party vote look as if the party median was a party leader. These then lead to outcomes lying between the overall median and the leader (or median) of the party containing the overall median. In both cases equilibria exist. In contrast, the case of spending competition brings in preference intensity which leads to a wider variety of possible outcomes, as well as existence problems.

We now turn to endogenizing the parties. This is important in order to understand how anticipated outcomes will affect incentives for voters to switch parties and try to affect the overall outcome.

ENDOGENOUS PARTIES

Interestingly, it turns out that, with nominations by voting, endogenizing parties leads to median outcomes, while under nomination by spending competition it is still possible to get extreme outcomes in both nominations and the overall winner.[15]

[15] In this section we do not consider endogenous parties with party leaders, as it is not so clear how to properly define equilibrium in that case (e.g., who are the leaders if a leader switches parties?). Moreover, we already see an interesting contrast between the voting and spending competition cases, which is our more central focus.

Equilibrium with Endogenous Parties

Consider a partition of the population into two parties, (P_1, P_2), with the possibility that one of these is empty. We say that $(P'_1, P'_2,)$ is adjacent to (P_1, P_2) if there exists i such that $(P'_1, P'_2,) = (P_1 \backslash \{i\}, P_2 \cup \{i\})$ or $(P'_1, P'_2,) = (P_1 \cup \{i\}, P_2 \backslash \{i\})$. Thus, adjacent pairs of parties are those where the only difference is that one voter has switched parties.

An equilibrium with endogenous parties is a pair of parties (P_1, P_2), with the possibility that one is empty, that partition the set of voters, and a pair of nominations that form an equilibrium $(Nom(P_1), Nom(P_2))$,[16] as well as a specification of an equilibrium $(Nom(P_1^*), Nom(P_2^*))$ for every adjacent partition into two parties (P_1^*, P_2^*), such that:

$$u_i(W[Nom(P_\ell), Nom(P_{-\ell})]) \geq u_i(W[Nom(P_\ell \backslash \{i\}), Nom(P_{-\ell} \cup \{i\})]), \tag{4}$$

for each P_ℓ and $i \in P_\ell$. A party structure together with specifications of (equilibrium) nominations for that party structure and all adjacent ones is in equilibrium if no member of one party wishes to switch to the other party, anticipating the equilibrium that would ensue.[17]

ENDOGENOUS PARTIES AND NOMINATION BY VOTING

We first revisit nominations by party voting. Consider the following example.

Example 8. Every Equilibrium Outcome is the Median with Endogenous Parties, but not with Exogenous Parties. There are seven voters, $N = \{1, \ldots, 7\}$, and two parties that partition N as follows: $P_1 = \{1, 2, 3, 7\}$ and $P_2 = \{4, 5, 6\}$. Let 6 beat 3 in an election. One equilibrium *when these are exogenous parties* is (3, 5), with candidate 5 winning. This is not, however, part of an equilibrium with endogenous parties. Candidate 4, the median, can join P_1. With the new lineup of $P'_1 = \{1, 2, 3, 4, 7\}$ and $P'_2 = \{5, 6\}$, (4, 5) is an equilibrium (with either exogenous or endogenous parties). Let us check that $P'_1 = \{1, 2, 3, 4, 7\}$ and $P'_2 = \{5, 6\}$, (4, 5) is part of an equilibrium with endogenous parties. Clearly, candidate 4 would not wish to switch, as 4 wins the election. Candidates 1, 2, 3 and 7 would have no effect on the outcome by switching to P_2 as it is still an equilibrium to have 4 nominated by P_1 against 5 from P_2; and candidates 5 and 6 would have no effect on the outcome by switching to P_1 as it is then still an equilibrium to have 4 nominated against the remaining candidate in P_2.

This feature that the median is the winner is not just an artifact of this example, but is true of all equilibria under nominations by voting when parties are endogenous.

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[16] In the case where one of the parties is empty, then its nomination is ignored, and the other party's nominee wins the election by default.

[17] One might consider other sorts of equilibrium definitions, where coalitions of voters can separate and form new parties, etc. That is certainly of interest, but beyond the scope of our analysis, given the complications introduced by handling three or more parties.

Proposition 7. *When nominations are by votes, then in every equilibrium with endogenous parties $W[Nom(P_1), Nom(P_2)] = M$. Moreover, such an equilibrium exists.*

The proof is in the appendix. The intuition is roughly as follows. Suppose the outcome were not the median. Then we know from Proposition 3 that it lies between the median and the median of the party containing the median. It must then be that the other party (not having the median) has a majority which would prefer the median over the current outcome. Then, by switching, the median would be nominated and win. This last part takes some proof, as one has to worry about what possible other equilibria could arise if the median switched parties, and one has to show that the only possibility is to have the median nominated.

While the outcome is necessarily the median once parties are endogenized under nominations by voting, the parties can still have a variety of configurations. For instance, it could be that the equilibrium is to have the median alone in one party, or instead at the other extreme to have all voters in the same party. What is tied down is that, unless one of the nominees is the median, then the party structure will turn out to be unstable. This emphasizes that the *equilibrium party structure cannot be separated from what the equilibrium nominees are*. It could be that parties are stable with one pair of nominees, but not with another.

ENDOGENOUS PARTIES AND NOMINATION BY SPENDING COMPETITION

We now turn to endogenizing parties under spending competition. Here it turns out that non-median outcomes are possible, as we now show.

Example 9. Existence of Extreme Equilibrium Outcomes with Endogenous Parties. There are five voters $N = \{1, \ldots, 5\}$, and two parties that partition N as follows: $P_1 = \{1, 3\}$ and $P_2 = \{2, 4, 5\}$. Voters' ideal points are ordered by their labels. Moreover, assume that $d_1(2, 3) > d_3(3, 2)$, and $d_2(i, j) > d_h(k, t)$ for all $h \in \{3, 4, 5\}$, and all i, j, k, t such that $2 \geq i > j$ and $2 \geq k > t$.

For P_1 and P_2 above, (1, 2) is a pair of nominations that form an equilibrium where the general winner is voter 2. Let us check that there is some specification of equilibria for each possible switching of some voter, so that no voter would desire to switch parties. If voter 1 switches party then P_1 only consists of voter 3, the median. In this case, regardless of the nominee from P_2, the final winner is voter 3, and voter 1 is made worse off. If instead voter 3 switched parties, then voter 1 would become the only possible nomination in P_1. In P_2, voter 2 outbids any member, so he or she is nominated as part of any equilibrium. Voter 3 is not strictly better off since voter 2 is still the general winner. It is clear that voter 2 will not gain by switching parties, regardless of the equilibrium specification. So, we are left only to consider what happens if voter 4 (or 5) switches parties. Here, (1, 2) is still an equilibrium because then 4 (5) does not want to outspend 1 as he or she would still lose to 2 (and 3 still does not want to outspend 1 given that $d_1(2, 3) > d_3(3, 2)$); and voter 2 continues to outbid the members of his or her party.

Example 9 shows that, in contrast to nominations by voting, nomination by spending can provide non-median outcomes that are robust to party switching. Just as with fixed parties, there are issues with equilibrium existence, but the directional party condition is again sufficient to guarantee existence.

Proposition 8. *Suppose that nominations are by spending competition. If preferences satisfy the directional party condition and are in the same direction for each party, and $N \geq 5$, then an equilibrium with endogenous parties exists.*[18]

The proof of the proposition involves an explicit construction of the two parties and nominations, putting the two most extreme voters (in terms of the directional preference) in different parties. For instance, if the lowest indexed voters are those who have stronger preferences under the directional preference condition, then the constructed equilibrium parties have 1 and 3 together in one party and 2 and 4 together in the other, with any allocation of the remaining voters between the parties. 1 and 2 are nominated and 2 wins the election. None of the remaining voters can switch the outcome by changing parties. 2 clearly has no gain from changing, and if 1 changes parties, then 3 wins the nomination and the election, which cannot be improving for 1.

Example 9 and the proof of Proposition 8 show us that, even with endogenous parties, it is possible to have extreme outcomes under nomination by spending competition. This contrasts with nominations by party votes, where Proposition 7 shows a median outcome. This makes the point that how nominations are conducted can have a big impact on election outcomes, and that if spending plays a substantial role in the nomination process, then outcomes can differ dramatically from a pure voting setting.

CONCLUDING REMARKS

We have seen that the nomination process is important in determining the outcome of elections, even in a simple single-peaked world. When parties of fixed configurations vote over their nominees, the outcomes that emerge from the election are as if the party medians were party leaders, so the outcomes lie between one of those median peaks and the overall median, but can differ from the overall median. The divergence from the median depends on the specific configurations of parties and voters' ideal points. If parties are endogenous, then the outcome must be the overall median voter. Depending on preferences, a wider range of outcomes are possible under nominations by spending competition, even when parties are endogenous. There it is a very different process in which indensity of preference determines the outcome. Our analysis provides insight into the diversity of

[18] In the case where N = 3, there need not always exist an equilibrium. For instance, suppose that 1 cares most, then 2, then 3, where 2 is the median. Suppose also that 1 beats 3 in an election. If 1 and 2 are in the same party, then the nomination of that party must be 1 (regardless of whether 3 is present). That is not stable as then 2 would rather switch parties and win the nomination and then the election. It is also not stable to have 1 and 2 in separate parties, as then 1 would like to join the party that 2 is in, to win that nomination and the overall election.

outcomes that can occur even in settings where the election is well ordered on one dimension and there are only two parties. This suggests that it is important to model nomination processes in order to understand electoral outcomes, even in the starkest settings.

There is much room for further research, and important ways in which the analysis should be extended. We close by mentioning a few of the most obvious directions for further study.

First, we have modeled extreme versions of nomination processes, where either there are party leaders, there is a vote among party members, or there is simply a spending competition among party members. Reality is, of course, more complex, and involves combinations of these three elements. Party leadership has some discretion in identifying potential nominees, the electorate has substantial input, and spending by potential nominees can also clearly have an effect. Identifying how these different influences interact is of interest.[19]

Second, our analysis has been confined to elections of single representatives or officials from two party settings. While this has wide application (even beyond the United States), it is also important to understand nomination processes in multiparty systems, as well as things like selections of party lists and platform design and their influence on electoral competition.

Along with multiparty analyses it would also be important to allow for independent voters who are not affiliated with any party. Our results for fixed party structures are easily modified to accommodate the existence of independent voters. For the cases of party leaders and spending competition, there is no change in the statement of the results as the independent voters would simply be incorporated through the determination of the eventual voting outcome (the W function). In the case of nominations by a party vote, there would be a small modification to the statement of the results. Consider a case where there are independent voters who cannot run as their own candidate or participate in the nomination process of either party. These voters only enter the political process by voting for one of the two candidates in the overall election, and are thus subsumed the W function. Our analysis only requires changes in the case where the general median is an independent voter. There, it is important that a unique Condorcet winner exist among all potential candidates from either party.[20] That party member then plays a central role in locating the final winner; the existence part of the results remains unchanged. For example, the second part of Proposition 1 then reads: The winning candidate in any equilibrium lies in the interval between (and including) the Condorcet winner among all party members and the leader of the party which contains the Condorcet winner. While this indicates how independent voters can be addressed when they are fixed, allowing voters to choose whether or not to join a party would be another interesting avenue to follow.

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[19] Related to this, a referee suggested examining situations where party leaders are chosen endogenously, potentially by a vote. In our setting this maps indirectly into choices of candidates, and thus looks like voting over the nominee; but with richer institutional detail this could be an interesting variation to consider.

[20] Take the closest party member to the left of the overall median and the closest one to the right (even if they are from the same party). If one beats the other then that agent is the overall Condorcet winner among potential candidates.

Third, general forms of stability with endogenous parties, where one allows either more than two parties or more than one voter to change at a time, face substantial existence hurdles. Nonetheless this needs to be investigated, as in situations where two parties are nominating extreme candidates, there are strong incentives for centrist voters to split off and form their own party. This again points to an interest in the modeling of multiple party systems, even for the understanding of two party systems.[21] Although modeling party formation has generally been a difficult task and there is a paucity of workable models, it is such a important aspect of electoral competition that it begs for further analysis.

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[21] There are economies of scale and other aspects of parties (branding, reputation, etc.) that may make it hard to form new parties (or even to switch parties), and so our endogenous party equilibrium analysis may still be a good starting point. But understanding party formation more generally is clearly important.

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APPENDIX

Justifying the Equilibrium in a Spending Competition via an All-Pay Auction

Consider the following all-pay auction.[22, 23]

Time proceeds in discrete periods $t \in \{1, 2, \dots\}$. Candidates alternate in their moves in the order of their indices. Spending amounts start at 0 and fall on a grid with increments of $\varepsilon > 0$. A candidate who is called upon to move can choose to raise his or her spending to any higher feasible amount or to leave it unchanged. If a candidate does not match the current highest spending amount on his or her turn, then he or she is out of the auction.[24] The auction ends at the first time where there is a single candidate remaining, or where candidates have each exhausted their budgets. The candidates each have a budget of B. In the case where candidates each hit their budget, the winner is the last candidate.

We consider a case where $u_i(W[h, k]) - u_i(W[j, k])$ is not a positive multiple of ε for any $i, j, h,$ and k, $u_i(j)$ is not a multiple of ε for any i, and no "ties" occur (so that each $u_i(j)$ and $u_i(j) - u_i(k)$ takes a distinct value across all i, j and any $k \neq j$).[25]

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[22] While this has some specific features to make the analysis relatively easy, some other variations in terms of tie-breaking and rules for dropping out of the auction can be handled with additional arguments.

[23] The timing of this auction provides for different conclusions from simultaneous or sealed-bid auction models of electoral contests (e.g. Meirowitz 2006).

[24] On the first round, the current highest spending amount is considered to be $\varepsilon > 0$, so that a candidate is dropped from the auction if he or she does not initiate some minimal spending. The last mover has a slight advantage in that if the other players have not spent anything, then that candidate wins without spending the ε.

[25] Note that generically, ties only occur in a situation where the outcomes are the same, so for instance if $W[i, k] = W[j, k] = k$, and in that case the eventual election outcome is not dependent on which of i or j wins.

We show the following claim when $|P_1| = 3$. The extension to more candidates holds by an extension with an inductive argument, although we do not include a proof.

Claim 1 *Let $|P_1| = 3$ and suppose that there is a candidate $i = Nom(P_1) \in P_1$ and a candidate $k = Nom(P_2) \in P_2$ such that*

$$u_i(W[i, k]) - u_i(W[j, k]) > u_j(W[j, k]) - u_j(W[i, k]) \tag{5}$$

for all $j \in P_1$, and that also

$$u_j(W[h, k]) - u_j(W[i, k]) \leq u_h(W[h, k]) - u_h(W[i, k]) \tag{6}$$

for any $h \in P_1$ and $j \in P_1$ both distinct from i. Then there is a small enough ε and a large enough B such that there is a unique subgame perfect equilibrium outcome (anticipating $Nom(P_2) = k$) where i wins the above-described all-pay auction and spends at most ε.

Before moving to prove the claim, let us discuss the importance of (6). It is illustrated in the following example.

Example 10 Critical Spending by a Non-Winning Candidate. In this example, $i = 3$ satisfies (5). Yet, in every equilibrium candidate 2 wins, and candidate 1 campaigns even though 1 loses. Candidate 1 campaigns in order to drag candidate 2 into the race.

Let $W[1, \mathrm{k}] = k$, $W[2, k] = 2$, $W[3, k] = 3$. Let ε be in units and consider the following preferences.

1. Candidate 1's preferences are $u_1(k) = -1.4$, $u_1(2) = 6.5$, $u_1(3) = 0.3$.
2. Candidate 2's preferences are $u_2(k) = 0.2$, $u_2(2) = 8.5$, $u_2(3) = 7.6$.
3. Candidate 3's preferences are $u_3(k) = 0.4$, $u_3(2) = 1.5$, $u_3(3) = 4.5$.

Let us provide the insight to why even though $i = 3$ satisfies (5), in every equilibrium candidate 2 wins. Note that if candidate 1 spends 5 at the first opportunity, then candidate 3 will surely end up dropping out of the auction. Also, candidate 2 will respond to outbid candidate 1, as candidate 2 is willing to spend 8.3 to change the winner from 1 to 2. Thus, if candidate 1 spends 5 at the first opportunity, candidate 2 will respond, candidate 3 will then drop out and candidate 2 will win. This means that it costs at most 5 for candidate 1 to ensure that candidate 2 wins. The equilibrium then cannot be that candidate 1 will win.

Effectively, even though there is a candidate who is willing to outspend every other candidate in a head-to-head race (5), that is not enough to guarantee that the candidate wins. If there is some other candidate who strongly prefers to see yet another candidate win, there are situations where the head-to-head winner does not prevail. That is ruled out by condition (6).

To prove Claim 1, we make use of the following observation: by Proposition 2 in Dekel, Jackson, and Wolinsky (2006), for large enough B and small enough ε there is a unique equilibrium outcome of this game if there are only two candidates, i and j, in the spending competition.[26] That outcome is the candidate for whom

[26] As emphasized by Dekel, Jackson, and Wolinsky (2006), this result is a variation on a result originally shown by Leininger (1991), who examines ε-equilibria, in a slightly different auction.

$$u_i(W[i, k]) - u_i(W[j, k]) > u_j(W[j, k]) - u_j(W[i, k])$$

if there is such a candidate.[27] This also holds in any subgame where we start with large enough remaining budgets, and a current standing bid such that if it is i's turn to move i does not have to bid more than $u_i(W[i, k]) - u_i(W[j, k])$ in order to stay in the auction.

Proof of Claim 1 Let us label the candidates in P_1 as 1, 2, 3. We need to verify the claim for each choice of i, because the game is not fully symmetric due to the starting order of moves.

First, note that the game is finite and, given the distinct payoffs, there are only equal payoffs between actions at a node if the outcome is the same across actions. This implies that there is a unique equilibrium outcome.[28]

Let $v = max_{j \neq i, j \in P_1} u_j(W[j, k]) - u_j(W[i, k])$.

Let us show that there is a large enough budget so that, in any subgame where i moves last, and there is at least that budget remaining, then i wins without any additional bidding. Consider a subgame where i moves last and the current starting bid is the smallest increment larger than v larger than the current bid of any $j \leq i$. (Note that in this case the total budget must be at least the current starting bid.) Then it is clear that i will win, as all other bidders will drop out at their turns by the definition of v and condition (6). Let us now proceed by induction. Suppose that i wins at no additional cost in any subgame where i moves last and the current starting bid is at least $k\varepsilon$ larger than the current bid of any $j \leq i$ for some $k > 0$, and the remaining budget is at least some B_k. We show that the same is true when the remaining budget is at least $B_k + v + \varepsilon$. If the other bidders bid, they must expect that there is an outcome other than i winning, as otherwise, by dropping out, the first bidder would be sure that i would win in the continuation by the observation above and would save whatever payment. Likewise, the second bidder would then drop out. So, if another player bids, she must be expecting some $j \neq i$ to win. Supposing that the equilibrium continuation is to have some other bidder bid with an expectation of j winning, it follows that no bidder raises his or her bid by more than $u_j(W[j, k]) - u_j(W[i, k])$ (recalling (6)). But if i next bids the minimal increment larger than $u_j(W[j, k]) - u_j(W[i, k])$, it follows from the induction step that i will win, and from (5) that i gains more than $u_j(W[j, k]) - u_j(W[i, k])$. This contradicts j winning in the continuation, and the claim is established.

Next, we show that if $i = 1$ and there is a large enough budget, it follows that i will win with a bid of ε. Note that if i bids ε, then, by the claim above, i will be the last mover and will win in the continuation with no additional payment. If i does not bid, then some other candidate will win. By assumption, i strictly prefers to win (and by more than ε).

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[27] Otherwise $W[i, k] = W[j, k] = k$ and then the first mover drops out. The tie-breaking rule we state here is slightly different than that in Dekel, Jackson, and Wolinsky (2006); but in a way that actually makes things slightly easier to see.

[28] There is a possibility that $W[h, k] = W[\ell, k]$ and there is indifference over which of h or ℓ wins, so here we mean eventual outcome of the election against k. Note, however, that by (5), such indifference cannot involve i.

Finally, consider the case where $i = 2$. We show that i will win with a bid of ε. If 1 were to bid, it must that 1 expects some $j \neq i$ to win, as otherwise 1 could drop out and have $i = 2$ win in the continuation (by the observation above). It must then also be that 1 bids no more than $u_1(W[j, k]) - u_1(W[i, k]) \leq u_j(W[j, k]) - u_j(W[i, k]) < u_i(W[i, k]) - u_i(W[j, k])$. By matching 1's bid, by the claim above $i = 2$ wins in the continuation, which is strictly improving for 2 compared to any outcome where j wins; which is a contradiction. Thus, 1 drops out. Then it is clear that $i = 2$ bids ε and 3 drops out in the continuation. ■

PROOFS OF THE PROPOSITIONS

Proof of Proposition 1 Let D_ℓ and $D_{-\ell}$ respectively be the leaders of parties ℓ and $-\ell$. Denote by $[(Nom(P_\ell), Nom(P_{-\ell}))$ the pairs of nominations. Without loss of generality, assume $M \in P_\ell$.

Suppose $D_\ell \geq M$. First, we show that the winning candidate in equilibrium lies in $[M, D_\ell]$. By way of contradiction, suppose the winner, call it W^*, is to the left of (less than) M. If D_ℓ nominates M, then $W[W^*, M] = M$ and so D_ℓ is strictly better off by single-peakedness. Outcome W^* could not be supported in equilibrium, a contradiction. If $W^* > D_\ell$, then from a similar argument, D_ℓ is better off nominating himself or herself because $W[W^*, D_\ell] = D_\ell$, a contradiction.

Secondly, we prove existence. If $D_\ell = M$, then it is always an equilibrium for D_ℓ to nominate himself or herself and for $D_{-\ell}$ to choose arbitrarily a nominee in $P_{-\ell}$. If $D_\ell > M$, then take $\hat{x}$ which is defined as the closest point to D_ℓ in $P_\ell \cap [M, D_\ell]$ such that $W[y, \hat{x}] = \hat{x}$ for all $y \in P_{-\ell}$. If $\hat{x} = D_\ell$, then (D_ℓ, y) with any $y \in P_{-\ell}$ is an equilibrium. If $\hat{x} \neq D_\ell$, then for all $x \in P_\ell \cap (\hat{x}, D_\ell]$, there exists $y \in P_{-\ell}$ such that $W[x, y] = y$ (for if this were not true, x would be closer to D_ℓ which violates the definition of $\hat{x}$). Define $x^* \equiv \min(P_\ell \cap (\hat{x}, D_\ell])$. Let $y^* \in P_{-\ell}$ be the closest point to $D_{-\ell}$ in $P_{-\ell}$ such that $W[x^*, y^*] = y^*$. Note that $W[x, y^*] = y^*$ for all $x \in (\hat{x}, D_\ell]$. Now, if $y^* \in (\hat{x}, D_\ell]$, then (x^*, y^*) is an equilibrium because the candidates in P_ℓ that could defeat y^* would make D_ℓ strictly worse off;, and so x^* is a best-response for D_ℓ. By definition, y^* is the best nomination for $D_{-\ell}$ when $Nom(P_\ell) = x^*$. But, if $y^* < \hat{x}$, then $(\hat{x}, y^*)$ is an equilibrium because D_ℓ is indifferent between all the alternatives in $P_{-\ell}$ while $\hat{x}$ is D_ℓ's best choice when facing y^*.

Now suppose $D_\ell < M$ and let X be the set of voters' peaks. Consider the dual $(X', >')$ of $(X', >)$ where i's peak in X' is greater than j's if and only if it is smaller than j's in X. The above argument completes the proof as $D_\ell >'$ M in X'. ■

Proof of Proposition 3: First, we prove that a pair of nominations is an equilibrium under a vote by party members if and only if this pair is an equilibrium with nomination by medians as party leaders. Then we show existence and conclude.

Let us first show that if a pair of nominations is an equilibrium with medians as party leaders, then it is an equilibrium under nomination by voting. So, let (one of) the medians of each party be a party leader: $D_\ell = M_\ell$ and $D_{-\ell} = M_{-\ell}$. Suppose $(Nom(P_\ell), Nom(P_{-\ell})) = (i, j)$ is an equilibrium with medians as party leaders. This means that

$W[i, j] \supseteq_{M_\ell} W[x, j]$ for all $x \in P_\ell$. If $W[M_\ell, j] = M_\ell$, then it must be that $i = M_\ell$. In that case, for any x, since M_ℓ is a median of the party and preferences are single peaked, there is not a strict majority of the party that prefers $W[x, j]$ to M_ℓ, and so it remains an equilibrium nomination for ℓ under voting. So consider the case where $W[M_\ell, j] = j$. There, it must be that either j lies between the overall median and M_ℓ, or on the other side of the median from M_ℓ. This means that for any x (including i), $W[x, j]$ lies to the same side of M_ℓ as j. In that case a (weak) majority has the same preferences as M_ℓ over the pair $W[i, j]$ and $W[x, j]$. Thus, if $W[i, j] \supseteq_{M_\ell} W[x, j]$, then this is true for at least a weak majority of member of party P_ℓ and so no other nominee would defeat i as a nominee. Since ℓ was arbitrary, any (i, j) which is an equilibrium with medians as party leaders is an equilibrium under a nomination by voting.

To see the converse, consider an equilibrium (i, j) under nomination by voting. By means of contraction, suppose that this is not an equilibrium for any choice of medians as party leaders. That is, suppose there exists a party ℓ such that i is not the choice of the party median(s) in response to j. As argued above, the only possible outcomes as a function of the nominations of party ℓ either include at least one of the party medians, or all lie on the same side of the party median(s) as j. Consider the former case first, that is, $W[M_\ell, j] = M_\ell$ where M_ℓ is a median of P_ℓ. In that case, the median closest to $W[i, j]$ would also defeat j.[29] Therefore, i must be the median closest to $W[i, j]$; otherwise a strict majority of P_ℓ would prefer that median to $W[i, j]$, contradicting that (i, j) is an equilibrium. But if i is a median and $W[i, j] = i$, then he or she must be the choice of a party median in response to j, a contradiction. Second, consider the latter case where neither median would win against j. There, all of the party members to the opposite side of the party median(s) to j have the same preferences as the party median(s) over all the possible outcomes since all possible outcomes are to one side of the party median(s). In that case it must be that, if i is not defeated by a strict majority, then there is no other nomination that the median (or either median if there is more than one) would prefer to i. Agent i is thus the choice of a median of P_ℓ, a contradiction.

By Proposition 1 we know that there exists an equilibrium under nominations by any pair of leaders, and so there exists one where the medians are party leaders. Therefore, by the first part of the proof, an equilibrium exists under vote by party members. The third part of our claim follows immediately from Proposition 1. ■

Proof of Proposition 4 Without loss of generality, let P_2 be the party containing the median. Suppose to the contrary of the claim, that the winner j was from P_1. Let k be the member of P_2 closest to P_1. Since there is no-overlap and $M \in P_2$, k would defeat any member of P_1. So, that the winner j is in P_1 implies $Nom(P_2) \equiv i \neq k$. That is, some i losing to j outspends k. But $d_k(k, j) > d_i(j, k)$, as it must be that $d_i(j, k) < 0$ and $d_k(k, j) > 0$. This is a contradiction, because no such i would outbid k.

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[29] We know both M_ℓ and $W[i, j]$ defeat j (at least weakly). Since the set of winners against any candidate is a connected set, the median closest to $W[i, j]$ also beats j.

Next suppose that the winner $Nom(P_2)$ could be beaten by some member of P_1. A similar argument as the one just given reaches a contradiction. For $Nom(P_2)$ to win, a member of P_1 losing to $Nom(P_2)$ would have to outspend a member who could defeat $Nom(P_2)$, a contradiction since there is no overlap. ■

Proof of Proposition 5: Without loss of generality, let P_1 contain the median and lie to the left, and order voters by their labels. Let k be the minimal labeled voter in P_2. Let S_1 be the subset of voters in P_1 who would beat k in the election (and this set is non-empty given that the median is in this set). Let $k = Nom(P_2)$. Note that all voters in $P_1 \backslash S_1$ prefer any nominee from S_1 to k and so will not wish to outbid any nominee in S_1; and changing the nominee from P_2 (given that $Nom(P_1) \in S_1$) will not change the outcome. Thus, to complete the specification of an equilibrium, it is enough to find a nominee from S_1 that would not be outbid by any other nominee from S_1. Consider the two extreme candidates from S_1, and label them i and j. If $d_i(i, j) \geq d_j(j, i)$, then set $Nom(P_1) = i$ and otherwise set $Nom(P_1) = j$. ■

Proof of Proposition 6: With directional parties, there are two cases: either (I) preference intensities for both parties (weakly) increase in the same direction, or (II) preference intensities for the parties increase in opposite directions.

We show that for both cases an equilibrium can be found.

Case I. Without loss of generality, assume that preference intensity in both parties (weakly) increases as the candidates move leftward. Now, choose $1 \equiv \min P_\ell$ and $2 \equiv \min P_{-\ell}$, the leftmost candidates from each party. If $1 \leq M$ and $2 \leq M$, then it is straightforward to check that $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$ is an equilibrium. However if (say) $2 > M$, then pick the leftmost candidate from Party ℓ who can defeat candidate 2 in a pairwise election. In this case, $M \in P_\ell$ and so such a candidate exists. Call this candidate 3. It is straightforward to check that $(Nom(P_\ell) = 3, Nom(P_{-\ell}) = 2)$ is an equilibrium.

Case II. Let C_ℓ be the direction set of party ℓ, which contains all candidates on the side of the median corresponding to the direction of that party's increasing preferences. Wlog, assume that party ℓ's preference intensities increase for candidates to the right and party $-\ell$'s preferences are increasing to the left. Formally, $C_\ell = \{i \in P_\ell : M \leq i\}$ and $C_{-\ell} = \{i \in P_{-\ell} : i < M\}$. Furthermore, assume wlog that $M \in P_\ell$.

Case IIa. $C_{-\ell} = \varnothing$. Let $2 = \min C_{-\ell}$ be the candidate from $C_{-\ell}$ that is closest to the median. If $C_\ell \backslash [M, 2] \neq \varnothing$, then choose the candidate closest to 2 in that set and call him or her 1. Then $Nom(P_\ell) = 1$, $Nom(P_{-\ell}) = 2)$ is an equilibrium. Otherwise, if $C_\ell / [M, 2] = \varnothing$, then choose the candidate from C_ℓ that is closest to 2, call him or her 1, and notice $(Nom(P_\ell) = 1$, $Nom(P_{-\ell}) = 2)$ is an equilibrium.

Case IIb: $\breve{C}_{-\ell} \neq \varnothing$. Let $2 = \max \breve{C}_{-\ell}$ be the candidate from $C_{-\ell}$ that is closest to the median. Denote by 1 the candidate from C_ℓ that is furthest from the median *and* can defeat candidate 2.[30] Now, if $C_{-\ell} \cap [M, 1] = \varnothing$, then $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$ is

[30] Such a candidate can always be found since it is possible to choose the median.

an equilibrium; otherwise, let $3 = \max C_{-\ell} \cap [M, 1]$ and note $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 3)$ is an equilibrium.

Proof of Proposition 7: We prove that for every equilibrium partition into parties (P_1, P_2) it must be that $W[Nom(P_1), Nom(P_2)] = M$.

Suppose to the contrary, that $W^* = W[Nom(P_1), Nom(P_2)] \neq M$ in some equilibrium. Without loss of generality, suppose that $M \in P_1$, where $M_1 \leq M \leq M_2$. The possible alignments for W^* can be divided into two distinct cases.

(1) $W^* \leq M$. Since $M \in P_1$, we know from our characterization of equilibrium that $M_1 \leq W^* < M$. Now let $P'_1 = P_1/\{M\}$ and $P'_2 = P_2 \cup \{M\}$. If $(Nom(P'_1), Nom(P'_2)) = (i, M)$ for any i, then M would win and so there would be a profitable deviation for M contradicting equilibrium. Thus, it must be that $(Nom(P'_1), Nom(P'_2)) = (i, j)$, where $j \neq M$. In particular, $j \in (M, M_2]$ without loss of generality, which we explain next. We know that $W[i, j] \in (M, M_2]$, by our characterization of equilibrium. So, if $j = W[i, j]$, then $j \in (M, M_2]$. Now suppose $i = W[i, j]$. Note that the median of P'_1 is to the left of M_1, as removing members to the right of M_1 (such as M) moves the median to the left. So, for (i, j) to be a pair of nominations in equilibrium, $j > M$ because otherwise a majority in P'_1 would always be strictly better off nominating M_1 over i. In equilibrium, j must be such that any nomination which a majority in P'_1 prefers to $i = W[i, j]$ also loses to j. Therefore, if (i, j) is an equilibrium where $j > M_2$, then there exists $j^* \in (i, M_2]$ such that (i, j^*) is also equilibrium. As a result, $j \in (M, M_2]$ without loss of generality. Now we show that $(Nom(P'_1), Nom(P'_2)) = (i, j)$ with $j \in (M, M_2]$ cannot be an equilibrium. First, $Nom(P_1)$ beats j, or else j would have been the nominee of P_2 since $j \in (M, M_2]$ is preferred to W^* by a majority in P_2. Second, $Nom(P_1) \leq M$. To see why, notice that if $Nom(P_1) = W^*$ then $Nom(P_1) \in [M_1, M]$, and if $Nom(P_2) = W^*$ then $Nom(P_1) > M$ would imply that a majority in P_2 is strictly better off nominating M_2 rather than $Nom(P_2)$, a contradiction. Now we have two cases:

Case I. $Nom(P_1) \geq M_1$: Then, $M_1 \leq Nom(P_1) \leq M$. Recall that the median of P'_1 lies to the left of M_1. Therefore, a majority in P'_1 prefers $Nom(P_1)$ to $W[i, j] \in (M, M_2]$, and since $Nom(P_1) = W[Nom(P_1), j]$, (i, j) cannot be an equilibrium.

Case II. $Nom(P_1) < M_1$: Because $W^* \in [M_1, M]$, $W^* = Nom(P_2)$. For $(Nom(P_1), Nom(P_2))$ to be a pair of nominations in equilibrium, $Nom(P_1)$ must be such that any nomination which a majority in P_2 prefers to $Nom(P_2)$ loses to $Nom(P_1)$. Thus, if $(Nom(P_1), Nom(P_2))$ is an equilibrium where $Nom(P_1) < M_1$, then there exists $\text{Nom}^*(P_1) \in [M_1, M]$ such that $(Nom^*(P_1), Nom(P_2))$ is also equilibrium. We have seen in case I that it is not possible either.

(2) $W^* > M$. This cannot be since the outcome must lie between the party median of the party containing M and M, and so must lie between M_1 and M.

To complete the proof, we argue that there exists a partition into parties with the median as the outcome, and corresponding equilibria (for all possible adjacent party structures). To see this, choose parties with no overlap such that the median is the most extreme voter in one of the parties. Let h be the voter immediately to the right of the median and t be the voter immediately to the left of the median. If h defeats t, then have

the median be in the party that contains t (and nominations be M and h), and otherwise have the median be in the party that contains h (and nominations be M and t). Regardless of the deviation by any voter, let the median be nominated. ■

Proof of Proposition 8: Without loss of generality, suppose that preference intensity increases leftwards (left directional parties). Since $N \geq 5$, there exists a partition of N into (P_1^*, P_2^*) such that $\min P_1^* < \min P_2^* < M$ and no $I \in N$ is such that $\min P_1^* < i < \min P_2^*$. Let $c_1 = \min P_1^*$ (i.e. the leftmost voter in P_1^*) and $c_2 = \min P_2^*$. By the algorithm in the proof of Proposition 6, (c_1, c_2) is an equilibrium of the nomination process and so $((P_1^*, P_2^*), (c_1, c_2))$ may be an equilibrium with endogenous parties. We prove next that it actually is an equilibrium. First, take any voter $x > c_2$. If x switches party, then the algorithm predicts that (c_1, c_2) is still an equilibrium. Therefore, x cannot be strictly better off in all the equilibria of the game with partition $(P_x \backslash \{x\}, P_{-x} \cup \{x\})$. Secondly, if c_1 changes party, then $Nom(P_1^* \backslash \{c_1\}) > c_2$ because c_1 and c_2 are the leftmost candidates in each party. Since $c_1 > c_2$, by single-peakedness this cannot benefit c_1 as it could only push the final winner to the right. Finally, $W[c_1, c_2] = c_2$ and thus there is no equilibrium that could make c_2 strictly better o§ after switching. ■

NOMINATION BY PARTY LEADERS AND STRONG EQUILIBRIA

A strong equilibrium in the case of nominations by a vote of party leaders is a pair of nominations $Nom(P_1) \in P_1$ and $Nom(P_2) \in P_2$ such that:

(1) The pair is an equilibrium in the case of nominations by a voter of party leaders.

(2) There does not exist any pair of nominees $(i; j)$ where $i \in P_1$ and $j \in P_2$ such that $W[i, j]$ is preferred to $W[Nom(P_1), Nom(P_2)]$ by the leader of P_1 and the leader of P_2.

The idea is that the party leaders cannot get a better outcome by agreeing to change strategies.

Returning to Example 2, there are seven voters, $N = \{1, \ldots, 7\}$, and two parties that partition N as follows: $P_1 = \{2, 3, 6\}$ and $P_2 = \{1, 4, 5, 7\}$. The voters' ideal points are ordered by their labels. The party leaders are 6 and 7. Let preferences be such that $W[i, 5] = i$ unless $i = 6$ or $i = 7$.

The equilibria are (6, 7) and (3, 4). However,(3, 4) is not a strong equilibrium because both party leaders prefer $W[6, 7] = 6$ to $W[3, 4] = 4$.

Proposition 9. *If the pairs of nominees (i, j) and (i', j') are both strong equilibria in the case of nominations by a vote of party leaders, then $W[i, j] = W[i', j']$.*

Proof of Proposition 9: The possible locations of party leaders can be divided into two cases.

(1) Party leaders are on the same side of the median. Let D_ℓ and $D_{-\ell}$ respectively be the leaders of parties ℓ and $-\ell$. Without loss of generality, assume that $M \in P_\ell$, and $D_{-\ell} < D_\ell \leq M$. We know that $W[i, D_\ell] = D_\ell$ is an equilibrium outcome whenever $i < D_\ell$, and we will show that D_ℓ is the only strong equilibrium outcome. Suppose that W^* is a strong equilibrium outcome different from D_ℓ. Then $W^* \in [D_\ell, M]$, since whenever

$W^* < D_\ell$, D_ℓ can improve the outcome by nominating himself, and whenever $W^* > M$, D_ℓ can improve the outcome by nominating M. So $W^* \in [D_\ell, M]$. But, then both $D_{-\ell}$ and D_ℓ would prefer that $i < D_\ell$ and D_ℓ are their respective parties.nominees. Thus, the outcome $W^* \neq D_\ell$ is not supportable as a strong equilibrium, which is a contradiction.

(2) Party leaders are on opposite sides of the median. Without loss of generality, assume that $M \in P_\ell$, and $D_{-\ell} < M < D_\ell$. We will show that whenever $D_{-\ell} < M < D_\ell$, there is always exactly one equilibrium outcome, and hence only one strong equilibrium outcome. Recall, from the proof of Proposition 1, that $\hat{x}$ is defined as the closest candidate to D_ℓ in $P_\ell \cap [M, D_\ell]$ such that $W[\hat{x}, y] = \hat{x}$ for all $y \in P_{-\ell}$. First of all, we know that for any equilibrium outcome W^*, $W^* \in [\hat{x}, D_\ell]$; otherwise D_ℓ could strictly improve the outcome. Trivially, if $\hat{x} = D_\ell$, then the only possible equilibrium outcome is $W[D_\ell, nom(P_{-\ell})] = D_\ell$. Now, let $\hat{x} \neq D_\ell$, and (as in the proof of Proposition 1), define $x^* \equiv \min(P_\ell \cap (\hat{x}, D_\ell])$ and $y^* \in P_{-\ell}$ as the closest point to $D_{-\ell}$ in $P_{-\ell}$ such that $W[x^*; y^*] = y^*$. Whenever $y^* \in [D_{-\ell}, M]$, D_ℓ's best-response is to nominate $\hat{x}$ which, by definition, defeats all of $P_{-\ell}$. So, in this case, the only equilibrium outcome is $W^* = \hat{x}$. Suppose instead that $y^* \in [\hat{x}, D_\ell]$. Then, $W[x^*, y^*] = y^*$ is the only possible equilibrium outcome. ■