PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

Volume 10 SEPTEMBER 15, 1924 Number 9

THE DERIVATION OF ELECTROMAGNETIC FIELDS FROM A BASIC WAVE-FUNCTION

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Communicated July 10, 1924

1. Derivation of a Logarithmic Wave Function.—Electromagnetic fields may be derived from wave-functions in at least two ways that are analytically distinct.

In the first place four wave-functions satisfying a divergence relation may be chosen as the components of a 4-vector and field-vectors derived from these four electromagnetic potentials in the usual way. The four potentials may in their turn be derived by differential operations from the components of a 6-vector whose components may be taken to be any six wave-functions. This method is a generalization of the well-known methods of Fitzgerald and Hertz;[1] it has the disadvantage that the wave-functions cannot be chosen arbitrarily if magnetic poles are to be excluded.

Secondly, a standard type of definite integral may be adopted for the representation of a wave-function and suitable variations made in the limits and arbitrary functions that will give six wave-functions capable of representing the components of the field-vectors E and H in an electromagnetic field.

The second method has been adopted in only a few cases[2] and there is still much to be learned regarding it. The method will be studied here in connection with a definite integral of type

$$V = \int W f(u) du \qquad (1)$$

in which

$$W = \frac{1}{X^2 + Y^2 + Z^2 - c^2T^2}$$

and $X = x - \xi(u), \quad Y = y - \eta(u).$
$Z = z - \zeta(u), \quad T = t - u.$

The quantity c represents the velocity of light and is supposed to be constant. The function $f(u)$ is arbitrary.

When u is constant the quantity W is a wave-function which we shall regard as basic since it is the analogue of the fundamental potential function $1/r$ of electrostatics.

When ξ, η, ζ and u are all real the function W has singularities spread over the double cone. $X^2 + Y^2 + Z^2 = c^2 T^2$ which may be represented geometrically in a Minkowski world in which x, y, z and ct are running coordinates. The locus of the singularities will be called the *skeleton* of a wave-function.

A wave-function W without singularities may be obtained by making u complex and complex integration has been used by Conway[3] and Herglotz[4] to derive the Liénard electromagnetic potentials from the basic wave-function. The components of the field vectors in the field of an electric pole have not, however, been represented by contour integrals of the standard type. An attempt will be made here to use standard integrals taken along real paths to represent the components of the field vectors in a fundamental type of electromagnetic field.

If we wish to integrate $W f(u)$ along a real path we must clearly avoid values of u for which W is infinite, i.e., for which

$$X^2 + Y^2 + Z^2 = c^2T^2$$

It is known that when $\xi'^2(u) + \eta'^2(u) + \zeta'^2(u) < c^2$ there is one such value of u for which $u \leq t$. We shall denote this value by τ. $\forall$ And so for every $P(x, y, z, t)$ the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \xi(\tau)]^2 = c^2 [t - \tau]^2 \qquad (1a)$$

defines the corresponding $\tau((x, y, z, t)$.

An integral between constant limits is useless, for the quantity τ depends on x, y, z and t and these quantities may be chosen so as to bring τ within any previously assigned range of integration.

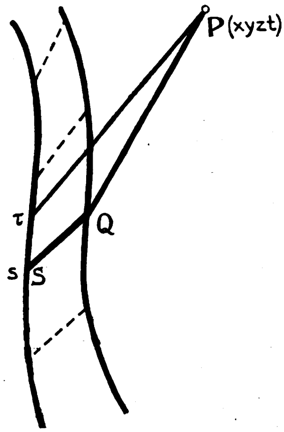
We must work, then, with limits, one of which at least depends on x, y, z and t. Now it has been shown by one of us that the integral (1) is indeed a wave-function if taken between the limits $-\infty$ and s where s is defined by the equations.

$$[x - \xi(s) - l(s)]^2 + [y - \eta(s) - m(s)]^2 + [z - \zeta(s) - n(s)]^2 = c^2 [t - s - p(s)]^2$$
$$s < s + p(s) < t$$
$$[l(s)]^2 + [m(s)]^2 + [n(s)]^2 = c^2[p(s)]^2$$

and that the dangerous values which of u for W is infinite generally lie outside the range $(-\infty, s)$ though τ may coincide with s in certain cases.

Regarding $\xi(s)$, $\eta(s)$, $\zeta(s)$ as the rectangular coördinates at time s of a moving point S, and $\xi(s) + l(s)$, $\eta(s) + m(s)$, $\zeta(s) + n(s)$ as the coördinates at time $s + p(s)$ of an associated moving point, Q, the quantity s defined by the above equations may be interpreted geometrically as follows:

Let the world lines of S and Q be drawn according to Minkowski's scheme and let P be the world point (x, y, z, t). Let a light-line be drawn from P in the direction in which t decreases to meet the world line of Q and from the point of intersection draw a light-line in the direction in which t decreases so as to meet the world line of S.



The value of t at the point of intersection is the value of s defined by the above equations and this value is unique if the velocities of S and Q are always less than c.

Thus s is defined by means of a broken light-line from P to S and is clearly less than τ, which is defined by means of a direct light-line from P to S.

There is one case, however, when $s = \tau$ and this occurs when P lies on a prolongation of a light-line drawn from S to Q in the direction in which t increases. For such a point P the integral is infinite at the upper limit of the integral and the wave-function V is generally infinite; it may be finite, however, if $f(s) = O$.

The skeleton of V is thus a "wing" made up of light-lines starting from the world line of Q and proceeding in the direction in which t increases. The breadth of the wing may be made as small as we please by choosing $f(u)$ so that it differs from zero only for a short range of values of u.

Now our object is to carry the limit s close up to the value τ and to obtain logarithmic expressions analogous to those which occur in the theory of Cauchy's principal values of integrals. In order to obtain finite expressions we shall introduce a second point R associated with S, the coördinates of R at time $\sigma + \omega(\sigma)$ being

$$\xi(\sigma) + \lambda(\sigma),\ \eta(\sigma) + \mu(\sigma),\ \zeta(\sigma) + v(\sigma)$$

Instead of $-\infty$ we now use a second limit σ defined by the equations

$$[x - \xi(\sigma) - \lambda(\sigma)]^2 + [y - \eta(\sigma) - \mu(\sigma)]^2 + [z - \zeta(\sigma) - v(\sigma)]^2 = c^2[t - \sigma - \omega(\sigma)]^2$$
$$\sigma < \sigma + \omega(\sigma) < t$$
$$[\lambda(\sigma)]^2 + [\mu(\sigma)]^2 + [v(\sigma)]^2 = c^2[\omega(\sigma)]^2$$

Moving the world lines of Q and R close up to that of S the limiting value of the integral is

$$V = -\frac{f(\tau)}{2M}\log\frac{L}{\Lambda}$$

where

$$\left.\begin{aligned} L &= Xl(\tau) + Ym(\tau) + Zn(\tau) - c^2Tp(\tau) \\ \Lambda &= X\lambda(\tau) + Y\mu(\tau) + Zv(\tau) - c^2T\omega(\tau) \\ M &= X\xi'(\tau) + Y\eta'(\tau) + Z\zeta'(\tau) - c^2 T \\ X &= x - \xi(\tau). \qquad Y = y - \eta(\tau) \\ Z &= z - \zeta(\tau), \qquad T = t - \tau \end{aligned}\right\} \quad (3)$$

and primes denote differentiations with respect to τ.

Since we are concerned only with the ratios of l, m, n, p, to λ, μ, v, ω we can ignore the fact that in the above construction $l, m, n, p. \lambda, \mu, v, \omega$ are eventually made infinitesimally small and we may regard them as finite functions of τ. It is easily verified that the function V thus defined is a wave-function and that its skeleton consists of two wings issuing from S.

2. *Derivation of Electromagnetic Fields from the Logarithmic Wave-Function.*—Our object now is to derive a set of wave-functions which can be used as the components of two field-vectors E and H satisfying the Maxwellian equations

$$\text{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \text{div}\,\mathbf{E} = o$$

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \text{div}\,\mathbf{H} = o$$

In the first place it should be remarked that if E and H are wave-functions of types

$$\mathbf{E} = \mathbf{e}f(\tau), \qquad \mathbf{H} = \mathbf{h}\,f(\tau) \quad (4)$$

where $\tau(x, y, z, t)$ is defined by equation (1a) and the function $f(\tau)$ can be chosen arbitrarily, the conditions

$$\left.\begin{aligned} \Delta\tau \times \mathbf{h} &= \frac{1}{c}\frac{\partial\tau}{\partial t}\mathbf{e} \qquad & \Delta\tau.\,\mathbf{E} = o \\ \Delta\tau \times \mathbf{e} &= -\frac{1}{c}\frac{\partial\tau}{\partial t}\mathbf{h} \qquad & \Delta\tau.\,\mathbf{H} = o \end{aligned}\right\} \quad (5)$$

must be satisfied. These conditions are sufficient to make wave-functions of type (4) solutions of Maxwell's equations provided we add the equations which express that E and H are wave-functions for all forms of the function $f(\tau)$.

The conditions (5) are analogous to the conditions of compatibility which occur in the theory of the propagation of waves of discontinuity.[5] The geometrical meaning of the conditions is that a world-plane through P representing the direction of the 6-vector (e, h) contains the light-line from S and is tangential to the light-cone whose vertex is at S. This indicates a method of obtaining the components of the field-vectors.

Using a geometrical picture in the Minkowski space in which x, y, z and ict are rectangular coördinates of a point we displace the points (l, m, n, icp), $(\lambda, \mu, v, ic\omega)$ by means of small rotations through the same angle $\delta\theta$ about the planes of yz, zx, xy, xt, yt and zt in turn. In this way we obtain six components of types

$$H_x = \frac{f(\tau)}{M}\left[\frac{nY - mZ}{L} - \frac{vY - \mu Z}{\Lambda}\right]$$

$$E_x = \frac{cf(\tau)}{M}\left[\frac{pX - lT}{L} - \frac{\omega X - \lambda T}{\Lambda}\right]$$

where X, Y, Z, T have the same values as in (3).

The skeleton of the field thus specified consists of two wings issuing from S and made up of electric charges radiated along light-lines drawn from different positions of S. Whenever a positive charge is radiated in one direction a compensating negative charge is radiated in another direction. By suitably choosing $f(\tau)$ the breadth of each wing may be made effectively as small as we please. In this fundamental type of field each component is a limiting form of a standard wave-function of type

$$\int_{\sigma}^{s} Wf(\tau)d\tau - \int_{\sigma'}^{s'} Wf(\tau)d\tau$$

The field is regarded as fundamental on account of the simplicity of the field-vectors and the atomistic character of its skeleton. Moreover all the fields that are of particular interest in physics may be derived from this type by superposition, the field of an electric pole being obtained by an operation analogous to differentiation in which the wings of two fields cancel one another out.

When $\xi = \eta = \zeta = l = m = \lambda = \mu = o$, $\begin{matrix} n = cp \\ v = -c\omega \end{matrix}$ one wing may be cancelled out by means of the operation $\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}$ while both wings may be cancelled out by means of the operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}$ or the equivalent operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The resulting field is that of a Hertzian dipole whose varying moment arm is in the direction of OZ. When the logarithmic wave-function

$$V = \frac{1}{r} \log \frac{r + z}{r - z}$$

is taken as one component of a Hertzian vector the resulting field has a skeleton composed of two wings but these wings are the loci of radiated dipoles which are gradually extinguished as they travel with velocity c. This gradual extinction may be regarded as the result of a secondary radiation of dipoles. The field is certainly not as simple as the field which is regarded here as fundamental.

[1] For references see H. Bateman, "*Electrical and Optical Wave Motion*," p. 7.

[2] *L. c.*, pp. 12, 113.

[3] *Proc. London Math. Soc.* (Ser. 2), **1** (1903).

[4] *Gött. Nachr.*, (1904).

[5] O. Heaviside, "*Electrical Papers*," Vol. 2, p. 405; A. E. H. Love, *Proc. London Math. Soc.* (Ser. 2), **1** (1903), p. 37; P. Duhem, *Comptes Rendus*, **131** (1900), p. 1171.

THE INFLUENCE ON SECONDARY X-RAY SPECTRA OF PLACING THE TUBE AND RADIATOR IN A BOX

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Communicated August 6, 1924

A number of experiments have been reported recently[1] on the secondary X-ray spectra produced in secondary radiators composed of chemical elements of low atomic weight by the K series lines from a molybdenum target tube. These spectra show strong lines having the wave-lengths of the primary rays and in addition other strong lines having wave-lengths several hundredths of an Ångström longer. The second set of lines of longer wave-lengths seem to occupy about the same positions in the spectrum no matter what the chemical element composing the secondary radiator may be. Their positions are in substantial agreement with Prof. A. H. Compton's interesting theory of the transfer of energy and momentum from radiation to single electrons.

These spectra differ essentially from those obtained in recently described experiments,[2] performed in our X-ray laboratories. In our spectra the line of longer wave-length shifted its position when secondary radiators of different chemical elements were used. In every case its position agreed very well with the idea that the radiation represented by it was produced by the bombardment of the photoelectrons due to the primary rays against