[1] *Physic. Rev.*, **20** (1922), p. 300.

[2] *Proc. Nat. Acad. Sci.*, **13** (1927), pp. 43–46.

[3] *Physic. Rev.*, **27** (1926), pp. 51–67.

[4] *Ibid.*, **27** (1926), pp. 173–180.

[5] *Proc. Nat. Acad. Sci.*, **9** (1923), pp. 207–211.

[6] *Ibid.*, **7** (1921), pp. 98–107.

[7] Hall, "Conditions of Electric Equilibrium, Etc.," *Ibid.*, **11** (1925), pp. 111–116.

[8] Richardson and Compton, *Phil. Mag., London*, **24** (1912), p. 575; Hennings, *Physic. Rev., Ithaca*, **4** (1914), p. 228; Millikan, *Physic. Rev.*, **18** (1921), p. 236.

[9] Illustrations of the "Dual Theory of Metallic Conduction," *Physic. Rev.*, **28** (1926), pp. 392–417.

[10] *Ibid.*, **25** (1925), pp. 356 and 357.

LAGRANGIAN FUNCTIONS AND SCHRÖDINGER'S RULE

By H. Bateman

California Institute of Technology

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In a recent paper Schrödinger[1] has extended a rule, used by writers in the theory of gravitation, for deriving a stress energy tensor from a Lagrangian function and has illustrated its application in the case of the tensor which he has associated with the system of equations proposed by Gordon.[2]

We shall now apply the rule to various Lagrangian functions to see if it is generally applicable. Let (a_1, a_2, a_3, a_4) be the components of a typical 4-vector on which the Lagrangian function depends and let a_{mn} denote the derivative of a_m with respect to the coördinate x_n. The rule then states that the component T_{mn} of an associated stress-energy tensor is given by

$$T_{mn} = \sum \left[\sum_{l=1}^{4} a_{lm} \frac{\partial L}{\partial a_{ln}} + \sum_{l=1}^{4} a_{ml} \frac{\partial L}{\partial a_{nl}} + a_m \frac{\partial L}{\partial a_n} \right] - \delta_{mn} L \quad \ldots\ldots \quad (1)$$

where

$$\delta_{mn} = 0 \quad m \neq n$$
$$= 1 \quad m = n$$

and the summation $\sum$ extends over all the four vectors of type a.

Let us now apply this rule to the Lagrangian function

$$L = \frac{1}{2}(E^2 - H^2) \quad (2)$$

in which

$$\mathbf{E} = \mathbf{e} + \mathbf{e}^*, \quad \mathbf{h} = \mathbf{h} + \mathbf{h}^*$$
$$\mathbf{e} = -\frac{1}{c}\frac{\partial \mathbf{a}}{\partial t} - \nabla\phi, \quad \mathbf{h} = \text{curl } \mathbf{a} \tag{3}$$
$$\mathbf{e}^* = -\text{curl } \mathbf{b}, \quad \mathbf{h}^* = -\frac{1}{c}\frac{\partial \mathbf{b}}{\partial t} - \nabla\omega.$$

The variational principle

$$\delta \int\int\int\int L \, dxdydzdt = 0 \tag{4}$$

is found to give rise to the Maxwellian equations

$$\text{curl } \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad \text{div } \mathbf{E} = 0$$
$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \text{div } \mathbf{H} = 0 \tag{5}$$

but Schrödinger's rule, as it stands, does not give a symmetric stress-energy tensor. We, therefore, modify the rule by writing

$$S_{mn} = \frac{1}{2}(T_{mn} + T_{nm}) \quad \ldots\ldots \tag{6}$$

and regarding S_{mn} as the components of the stress-energy tensor. It is now found that this tensor is the difference[3] of the tensors associated with the fields ($\mathbf{e}$, $\mathbf{h}$) and ($\mathbf{e}^*$, $\mathbf{h}^*$), respectively, in other words

$$S_{mn} = s_{mn} - \overset{*}{s}_{mn}.$$

If $e^* = 0$, $h^* = 0$, we get the correct stress-energy tensor associated with the field (e, h), but if $e = 0$, $h = 0$, we get $-\overset{*}{s}_{mn}$. This anomaly may be avoided by taking as our Lagrangian function

$$L = \frac{1}{2}(E_0^2 - H_0^2)$$

where

$$\mathbf{E}_0 = \mathbf{e} + i\mathbf{e}^*, \qquad \mathbf{H}_0 = \mathbf{h} + i\mathbf{h}^*$$

but the use of a complex Lagrangian function seems undesirable.

An interesting system of equations may be obtained by using the Lagrangian function

$$L = \frac{1}{2}(E^2 - H^2) + \lambda(a.b - \phi\omega)$$

in which λ is an arbitrary constant. In this case both electricity and magnetism are present.[4]

Gordon's equations relate to the case in which electricity is distributed throughout space and its distribution and motion are governed by certain equations which are quite hard to solve. The equations are interesting because the principles of the conservation of energy and momentum are consequences of the equations derived from the Lagrangian principle and these equations include two equations satisfied by conjugate complex functions ψ and $\bar{\psi}$ which closely resemble the wave equations used by de Broglie and Schrödinger.

When there is no field[5] these wave-equations are of type

$$\square^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = k^2\psi$$

where k is a constant and the question arises whether these equations possess any solutions which are finite and continuous throughout all space.

The question is immediately answered in the affirmative by the consideration of solutions of type

$$\psi = e^{i(lx + my + nz - wt)}.$$

When $k = 0$, there are certainly other solutions, for if a is real and $\sigma^2 > 1$ the wave-function

$$\psi = \frac{1}{x^2 + y^2 + (z - iac)^2 - c^2(t - i\sigma a)^2}$$
$$= \frac{x^2 + y^2 + (z + iac)^2 - c^2(t + i\sigma a)^2}{[x^2 + y^2 + z^2 - c^2t^2 + a^2c^2(\sigma^2 - 1)]^2 + 4a^2c^2[z - \sigma ct]^2}$$

does not become infinite for any real values of x, y, z and t. The real part of ψ also possesses this property.

It is easy to write down other systems of field equations giving continuous distributions of electricity and electric currents for which the principles of the conservation of energy and momentum are satisfied identically. A very simple set of equations of this type is derived from the Lagrangian function

$$L = \frac{1}{2}(E^2 - H^2) + \lambda(A^2 - \Phi^2)$$

in which λ is a constant and

$$\mathbf{H} = \text{curl}\,\mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi.$$

The associated equations are

$$\text{curl}\,\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \lambda\mathbf{A}, \quad \text{div}\,\mathbf{E} = \lambda\Phi$$

and these give

$$\mathbf{A} + \text{div}\,\frac{1}{c}\frac{\partial \Phi}{\partial t} = 0, \quad \square^2\mathbf{A} + \lambda\mathbf{A} = 0, \quad \square^2\Phi + \lambda\Phi = 0.$$

Another interesting set of equations is derived from the Lagrangian function

$$L = -\frac{1}{2}(H^2 - E^2) - \rho[\Phi - \frac{1}{c}(\mathbf{v}.\mathbf{A})] - \mathbf{p}.\mathbf{P} + \mathbf{q}.\mathbf{Q} + \frac{1}{2}\lambda^3[L^2 - \Lambda^2]$$

in which λ is a constant and

$$\mathbf{H} = \text{curl}\ \mathbf{A}, \quad \mathbf{E} = -\frac{1\partial\mathbf{A}}{c\partial t} - \nabla\Phi$$

$$\mathbf{p} = \text{curl}\ \frac{\rho\mathbf{v}}{c}, \quad \mathbf{q} = -\frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{v}) - \nabla\rho$$

$$\mathbf{P} = \text{curl}\ \mathbf{L}, \quad \mathbf{Q} = -\frac{1}{c}\frac{\partial\mathbf{L}}{\partial t} - \nabla\Lambda.$$

The associated equations are

$$\text{curl}\ \mathbf{P} - \frac{1}{c}\frac{\partial\mathbf{Q}}{\partial t} = \mathbf{A}, \quad \text{div}\ \mathbf{Q} = \Phi$$

$$\text{curl}\ \mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{\rho\mathbf{v}}{c}, \quad \text{div}\ \mathbf{E} = \rho$$

$$\text{curl}\ \mathbf{p} - \frac{1}{c}\frac{\partial\mathbf{q}}{\partial t} = \lambda^3\mathbf{L}, \quad \text{div}\ \mathbf{q} = \lambda^3\Lambda$$

and these give

$$\text{div}\ \mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$$

$$\text{div}\ (\rho\mathbf{v}) + \frac{\partial\rho}{\partial t} = 0$$

$$\text{div}\ \mathbf{L} + \frac{1}{c}\frac{\partial\Lambda}{\partial t} = 0$$

$$\Box^2\mathbf{L} = -\mathbf{A}, \quad \Box^2\Lambda = -\Phi$$

$$\Box^2\mathbf{A} = -\frac{\rho\mathbf{v}}{c}, \quad \Box^2\Phi = -\rho$$

$$\Box^2\left(\frac{\rho\mathbf{v}}{c}\right) = -\lambda^3\mathbf{L}, \quad \Box^2\rho = -\lambda^3\Lambda.$$

An interesting feature of this system of equations is that when ρ and $\rho\mathbf{v}$ have been obtained by solving the equations

$$\Box^6\left(\frac{\rho\mathbf{v}}{c}\right) = -\lambda^3\frac{\rho\mathbf{v}}{c}, \quad \Box^6\rho = -\lambda^3\rho,$$

$$\text{div}\ (\rho\mathbf{v}) + \frac{\partial\rho}{\partial t} = 0$$

the Hertzian vectors **P**, **Q** and the field vectors, E, H can be derived by differentiations and the potentials A and Φ are connected by the well-known relation

$$\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0.$$

In both of these cases the stress-energy tensor may be obtained by Schrödinger's rule.

[1] E. Schrödinger, *Ann. Phys.*, **82** (1927), p. 265. A special form of the rule is suggested immediately by Abraham's tensor. *Phys. Zeitschr.*, **13** (1912). Hilbert's rule, used in *Math. Ann.*, **92** (1924), p. 1, points to the existence of a more general rule.

[2] W. Gordon, *Zeitschr. Phys.*, **40** (1926), p. 117.

[3] The tensor thus vanishes completely when the potentials $\mathbf{a}$, ϕ, $\mathbf{b}$, ω are chosen so that $\mathbf{e} = \mathbf{e}^*$, $\mathbf{h} = \mathbf{h}^*$. It should be remarked that another tensor possessing this property may be derived from the Lagrangian function $L = \mathbf{E}\cdot\mathbf{H}$, which also gives rise to the field-equations (5).

[4] Another Lagrangian function which is consistent with the existence of both electricity, and magnetism is discussed by E. T. Whittaker, *Proc. Roy. Soc. London*, **113A** (1927), p. 496.

[5] The question whether this is the total field or only the external field may at present be left open. If there is to be no total field and consequently no electricity both ψ and $\bar{\psi}$ must be real.

TWO REMARKS ON THE WAVE-THEORY OF MECHANICS

BY F. D. MURNAGHAN AND K. F. HERZFELD

JOHNS HOPKINS UNIVERSITY

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A. *On the Degree of Arbitrariness in the Wave-Equation.*—For a dynamical system with fixed constraints, moving in a stationary conservative force field, the action W has the form $W = -Et + S(q)$; here $E = T + V$ is the total energy, both kinetic and potential, of the system and the symbol q stands for the positional coördinates of the system. The function W of (q, t) satisfies the Hamilton equation $\frac{\partial W}{\partial t} + H\left(q, \frac{\partial W}{\partial q}\right) = 0$ where H is the Hamiltonian function whose numerical value is the constant E. If we wish to set up the partial differential equation of the family of moving surfaces $F(q, t) = 0$ which is characterized by the fact that on each member of the family W has a constant value (it being understood that the dynamical system is started off from any point but always with the same value of the energy constant E) we proceed as follows. F is to be a