SOLUTIONS OF A CERTAIN PARTIAL DIFFERENTIAL EQUATION

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1. The partial differential equation

$$\frac{\partial u}{\partial t} = x\left(\frac{\partial^2 u}{\partial x^2} - u\right) \tag{1}$$

is readily seen to possess the two particular solutions

$$U_1 = xe^{-x \tanh t} \operatorname{sech}^2 t, \tag{2}$$

$$U_2 = e^{-x \coth t}. \tag{3}$$

These may be generalized by well-known methods so as to give solutions in the form of definite integrals

$$u = x\int_{-\infty}^{\infty} e^{-x \tanh z} \operatorname{sech}^2 z\, f(t - z)\, dz \tag{4}$$

$$u = \int_{0}^{\infty} e^{-x \coth z}\, g(t - z)\, dz, \tag{5}$$

in which $f(t)$ and $g(t)$ are suitable arbitrary functions. The first of these integrals can be expanded in the form of a power series

$$u = \sum_{n=0}^{\infty}{}^{1}_{1} \frac{x^{n+1}}{n!}\, c_n(t) \tag{6}$$

in which

$$c_n(t) = \int_{-\infty}^{\infty} \tanh^n z \operatorname{sech}^2 z\, f(t - z)\, dz. \tag{7}$$

To obtain the general solution expressible as a power series of ascending powers of x we must add to (6) a constant multiple of the particular solution e^x.

The solution (4) admits the formal expansion

$$u = f(t)l_0(x) - f'(t)l_1(x) + f''(t)l_2(x) - \dots \tag{8}$$

where

$$l_n(x) = \frac{x}{n!}\int_{-\infty}^{\infty} e^{-x \tanh z} \operatorname{sech}^2 z \,.\, z^n dz. \tag{9}$$

When x is real

$$|l_n(x)| < \frac{|x|e^{|x|}}{n!}\int_{-\infty}^{\infty} \operatorname{sech}^2 z\, |z|^n dz$$

$$= 8\,|x|e^{|x|}\left[\frac{1}{2^{n+1}} - 2\,\frac{1}{3^{n+1}} + 3\,\frac{1}{4^{n+1}} - \ldots\right]$$

$$< 8\,|x|e^{|x|}\,\frac{1}{2^{n+1}}. \qquad (10)$$

Hence if $f(t) = e^{\nu t}$ the series (8) certainly converges when $|\nu| < 2$ and represents a solution of type $u = e^{\nu t}F_\nu(x)$.

This function $F_\nu(x)$ may be found by first calculating the integral

$$\int_0^{\infty} e^{-\lambda x} F_\nu(x)dx \quad (\lambda > 1) \qquad (11)$$

which is found to be equal to

$$\frac{1}{\lambda^2-1}\left(\frac{\lambda+1}{\lambda-1}\right)^{\frac{\nu}{2}} \int_{-\infty}^{\infty} \frac{e^{-\nu s}\,ds}{\cosh^2 s}. \qquad (12)$$

We may therefore write (with the aid of a well-known inversion formula)

$$F_\nu(x) = \frac{1}{2\pi i}\int_C e^{\lambda x}\frac{d\lambda}{\lambda^2-1}\left(\frac{\lambda+1}{\lambda-1}\right)^{\frac{\nu}{2}} \int_{-\infty}^{\infty} \frac{e^{-\nu s}\,ds}{\cosh^2 s}, \qquad (13)$$

where, if $x > 0$, C is a contour which starts from $-\infty$ encircles the origin in the counter clockwise direction and returns to $-\infty$ so as to enclose the points -1 and $+1$ in a simple loop. Dropping the numerical factor depending only on ν we may easily verify that the contour integral

$$H_\nu(x) = \frac{1}{2\pi i}\int_C e^{\lambda x}(\lambda+1)^{\frac{\nu-2}{2}}(\lambda-1)^{-\frac{\nu+2}{2}}\,d\lambda \qquad (14)$$

satisfies the differential equation

$$x\frac{d^2H}{dx^2} = (x+\nu)H \qquad (15)$$

even when ν is not restricted by the inequality $|\nu| < 2$; it is, in fact, essentially the solution which Professor Kármán said he had found when he mentioned the differential equation to the author as one requiring further study.

It is easily seen that the function $H_\nu(x)$ satisfies the two recurrence relations

$$(\nu - 2)H_{\nu-2}(x) + (\nu + 2)H_{\nu+2}(x) = 2(\nu + 2x)H_\nu(x) \quad (16)$$

$$(\nu + 2)H_{\nu+2}(x) - (\nu - 2)H_{\nu-2}(x) = 4x\ H'_\nu(x) \quad (17)$$

which are very much like those satisfied by the function $k_\nu(x)$ discussed elsewhere.[1] Another function which satisfies the recurrence relations (16) and (17) is obtained by noticing that the definite integral (5) gives a solution of the form $e^{\nu t}h_\nu(x)$ where

$$h_\nu(x) = \int_0^\infty e^{-x \coth z - \nu z}\, dz. \quad (18)$$

We have supposed in the derivation of these relations that $x > 0$, $\nu > 2$. The integral certainly has a meaning when $\nu = 2$ but is divergent when $\nu = 0$. To complete the definition of the function $h_\nu(x)$ we shall write for $x > 0$

$$h_0(x) = \int_0^\infty (e^{-x \coth z} - e^{-x})dz, \quad (19)$$

observing that $h_0(x)$ is not a solution of the partial differential equation (1) but that a particular solution is given by

$$u = h_0(x) + t\, e^{-x}. \quad (20)$$

An interesting addition formula for the function $h_0(x)$ is obtained by making use of the series

$$e^{-y \coth z} = k_0(y) - k_2(y)e^{-2z} + k_4(y)e^{-4z} - \dots \quad (21)$$

obtained in my former paper. With the aid of (18) and (19) we find that when $y > 0$, since $k_0(y) = e^{-y}$

$$S \equiv h_2(x)k_2(y) - h_4(x)k_4(y) + h_6(x)k_6(y) - \dots$$
$$= \int_0^\infty e^{-x \coth z}\, [e^{-y} - e^{-y \coth z}]dz$$

the integration of the series term by term over the infinite range being justified by Dini's theorem (Bromwich's Infinite Series, 1st. ed., p. 455). Hence

$$S = e^{-y}\int_0^\infty (e^{-x \coth z} - e^{-x})dz - \int_0^\infty [e^{-(x+y) \coth z} - e^{-(x+y)}]dz$$
$$= k_0(y)h_0(x) - h_0(x + y)$$

$$\therefore \quad h_0(x + y) = k_0(y)h_0(x) - k_2(y)h_2(x) + k_4(y)h_4(x) - \dots \quad (22)$$

This is a particular case of a more general formula

$$h_\nu(x + y) = k_0(y)h_\nu(x) - k_2(y)h_{\nu+2}(x) + k_4(y)h_{\nu+4}(x) - \dots \quad (23)$$

which is easily proved with the aid of (18).

This relation may be used to prove the formula (s an integer)

$$\int_0^\infty h_\nu(x+y)k_{2s}(y)dy = (-)^s\left[h_{\nu+2s}(x) - \frac{1}{2}h_{\nu+2s-2}(x) - \frac{1}{2}h_{\nu+2s+2}(x)\right] \qquad s>0 \tag{24}$$

$$= \frac{1}{2}\left[h_\nu(x) - h_{\nu+2}(x)\right] \qquad s = 0$$

A process of summation applied to the last formula gives

$$h_\nu(x) = \int_0^\infty m_{\nu-1}(x+y)k_0(y)dy \tag{25}$$

where

$$m_\nu(x) = \int_0^\infty e^{-x\coth z - \nu z}\operatorname{cosech} z.\, dz \tag{26}$$

$$k_0(y) = e^{-y} \tag{27}$$

$$x>0, \qquad y>0.$$

2. It should be mentioned that the differential equation (15) is a limiting case of the equation

$$4z\frac{d^2K}{dz^2} = 4(a-1)\frac{dK}{dz} + (z+2b)K \tag{28}$$

which Kummer[2] solves by means of a definite integral

$$K(a, b, z) = \int_0^{\frac{\pi}{2}}(\cos\nu)^{a-1}\cos\left(\frac{z}{2}\tan\nu + b\nu\right)d\nu \tag{29}$$

using a method which is only applicable when $a > 1$. When $z = 2x$, $a = 1$, Kummer's function $K(a, b, z)$ becomes a constant multiple of the function which we have called $k_{-b}(x)$. In fact,

$$K(1, -\nu, 2x) = \frac{\pi}{2}k_\nu(x). \tag{30}$$

Kummer expresses his function $K(a, b, z)$ in terms of the confluent hypergeometric function

$$\phi(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \dots \tag{31}$$

which is now usually denoted by the symbol $F(\alpha, \gamma; z)$. The actual relation is

$$e^{\frac{z}{2}} K(a, b, z) = A\phi\left(\frac{b-a+1}{2}, b - a, x\right) + Bz^a \phi\left(\frac{b+a+1}{2}, 1+a, z\right)$$

where

$$A = \frac{\pi\Gamma(a)}{2a\,\Gamma\left(\frac{a-b+1}{2}\right)\Gamma\left(\frac{a-b+1}{2}\right)} \tag{32}$$

$$B = -\frac{\pi\cos\frac{a\pi - b\pi}{2}}{2a\,\Gamma(a+1)\sin(a\pi)}$$

This relation evidently fails to hold when $a = 1$.

The differential equation (15) is also a special case of an equation

$$x\frac{d^3y}{dx^3} = (a-1)\frac{d^2y}{dx^2} + (b+x)\frac{dy}{dx} + cy, \tag{33}$$

which Kummer finds to be the equation satisfied by the definite integral

$$y = \int_0^{\frac{\pi}{2}} (\sin\nu)^{c-1}(\cos\nu)^a \cos(x\tan\nu + b\nu)d\nu. \tag{34}$$

Equation (15) is, in fact, obtained by putting $a = 1$, $b = \nu$, $c = 0$, $H = dy/dx$. It should be noticed, moreover, that Kummer's formula

$$2\int_0^{\frac{\pi}{2}} (2\cos\nu)^m \cos[x\tan\nu + (m+2n)\nu]d\nu$$

$$= e^{x}\sin(n\pi)\int_0^1 u^m(1-u)^{n-1}e^{-\frac{2x}{u}}du \tag{35}$$

reduces, when $m = 0$, to the form

$$\pi k_{2n}(x) = 2\int_0^{\frac{\pi}{2}} \cos[x\tan\nu + 2n\nu]d\nu \tag{36}$$

$$= e^x\sin(n\pi)\int_0^1 (1-u)^{n-1}e^{-\frac{2x}{u}}du \quad (x > 0)$$

and furnishes an immediate proof of the theorem that $k_{2n}(x) = 0$ when n is a positive integer and $x > 0$.

It may be mentioned that Giuliani[3] has considered the two functions

$$U(a, b, x) = \int_0^{\frac{\pi}{2}} (\cos \nu)^{a-1} \cos (x \tan \nu) \cos b\nu . d\nu$$
$$V(a, b, x) = \int_0^{\frac{\pi}{2}} (\cos \nu)^{a-1} \sin (x \tan \nu) \sin b\nu . d\nu \qquad (37)$$

and finds that they satisfy one differential equation of the fourth order. An equation

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + (1 - x) \frac{\partial u}{\partial x} - u,$$

which is readily reducible to (1), has been discussed from a different standpoint by H. Faxén and has been used in connection with Ole Lamm's theory of diffusion and sedimentation in the ultracentrifugal separator.[4]

The solution (3) is readily deducible from M'Arthur's result[5] that the function

$$y = (1 - t)^{-p} \exp. \left(\frac{xt}{1 - t}\right)$$

is a particular solution of the partial differential equation

$$x \frac{\partial^2 y}{\partial x^2} + (p + x) \frac{\partial y}{\partial x} = t \frac{\partial y}{\partial t}.$$

The addition theorem (23) corresponds to the addition theorem

$$k_\nu (x + y) = k_\nu(x)k_0(y) + k_{\nu-2}(x)k_2(y) + k_{\nu-4}(x)k_4(y) + \dots.$$
$$x > 0, y \geqq 0,$$

for the k-function.

[1] H. Bateman, "The k-Function, a Particular Case of the Confluent Hypergeometric Function," *Trans. Amer. Math. Soc.*, Oct., 1931.

[2] E. E. Kummer, "De integralibus quibusdam definitis et seriebus infinitis," *J. Math.*, **17,** 228–242 (1837); **20,** 1–10 (1840).

[3] G. Giuliani, "Aggiunte ad una memoria de Kummer," Battaglini's, *Giornale Mat.*, **26,** 234–250 (1888).

[4] H. Faxén, "Über eine Differentialgleichung aus der physikalische Chemie," *Arkiv for Matematik, Astronomi och Fysik,* Bd. 21B, No. 3 (1929).

[5] Neil M'Arthur, "Note on the Polynomials Which Satisfy the Differential Equation $xy'' + (c - x)y' - ay = 0$," *Proc. Edinburgh Math. Soc.*, **38,** 27–33 (1920).