RELATIONS BETWEEN CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Some of the functions mentioned in a recent paper may be expressed in terms of known functions.

The function $H_{2n}(x)$, which was required to be such that

$$\int_0^\infty e^{-\lambda x} H_{2n}(x)\,dx = \left(\frac{\lambda+1}{\lambda-1}\right)^n \frac{1}{\lambda^2-1},$$

is readily seen to be given by the equation

$$H_{2n}(x) = xe^{-x} F(n+1;\ 2;\ 2x) \tag{1}$$

where $F(\alpha;\gamma;z)$ denotes Kummer's function

$$1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \ldots$$

which Kummer himself denotes by the symbol $\phi(\alpha, \gamma, z)$. Kummer's function $K(a, b, z)$ may be expressed in terms of Whittaker's function $W_{k,m}(z)$ by comparing his expression for K in terms of ϕ with the relation at the end of §16.41 of Whittaker and Watson's "Modern Analysis." When $z > 0$ the relation is

$$2^{2m}\,\Gamma(1/2 + m + k)K(2m, -2k, z) = \pi z^{m-1/2}\, W_{k,m}(z) \tag{2}$$

This relation, combined with the relation just mentioned, should take the place of equation (32) of my recent paper, which is marred by some slips. In particular,

$$k_{2n}(z)\,\Gamma(1+n) = W_{n,1/2}(2z) \tag{3}$$

and the asymptotic expansion of the function k is consequently (for $x > 0$)

$$k_{2n}(x) \sim \frac{1}{\Gamma(1+n)} e^{-x}(2x)^n \left\{ 1 + \sum_{m=1}^{\infty} (-1)^m \frac{n(n-1)^2 \ldots (n-m+1)^2(n-m)}{m!\,(2x)^m} \right\} \tag{4}$$

When n is a positive integer the series ends and we have

$$k_{2n}(x) = (-1)^{n-1}\, 2x.e^{-x} F(1-n;\ 2;\ 2x) \tag{5}$$

From the definition of the function h it is readily seen that when $a > 0$

$$\int_0^\infty e^{-ax} h_{-2n}(x)dx = \int_0^\infty e^{-ax} dx \int_0^\infty e^{-x \coth z - 2nz} dz$$

$$= \int_0^\infty \frac{e^{-2nz} dz}{a + \coth z}$$

$$= \frac{1}{2n} \cdot \frac{1}{a+1} - \frac{1}{n+1} \frac{1}{(a+1)^2} - \frac{1}{n+2} \frac{a-1}{(a+1)^3} - \dots \quad (6)$$

Now the last series is known to be equal to the integral

$$\int_0^\infty e^{-ax} k_{2n}(x)\, dx \,.\, \frac{\pi}{2} \operatorname{cosec}(n\pi)$$

and so it is readily seen that when $x > 0$, $n > 0$,

$$k_{-2n}(x) = \frac{2}{\pi} \sin(n\pi)\, h_{2n}(x). \quad (7)$$

This relation is equivalent to equation (36) of my former paper in which $k_{-2n}(x)$ should appear instead of $k_{2n}(x)$. Whether n is a positive integer or not we may write

$$h_{2n}(x) = \frac{1}{2} \Gamma(n) W_{-n, \frac{1}{2}}(2x) \qquad \left.\begin{matrix} n > 0 \\ x > 0 \end{matrix}\right\} \quad (8)$$

and the asymptotic expansion for the function h is accordingly

$$h_{2n}(x) \sim \frac{\Gamma(n) e^{-x}}{2^{n+1} x^n} \left\{ 1 + \sum_{m=1}^{\infty} (-1)^m \frac{n(n+1)^2 \dots (n+m-1)^2 (n+m)}{m!\ (2x)^m} \right\}. \quad (9)$$

This expression fails when $n = 0$ for then

$$h_0(x) = \frac{1}{2} [\Gamma'(1) - \log 2]\, e^{-x} - e^x \int_x^\infty e^{-2\xi} \log \xi \, d\xi \quad (10)$$

and the asymptotic expansion is

$$h_0(x) \sim \frac{1}{2} e^{-x} \left[\Gamma'(1) - \log(2x) - \frac{1}{(2x)} + \frac{1!}{(2x)^2} - \frac{2!}{(2x)^3} + \dots \right]. \quad (11)$$