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Abstract

A complete analysis of acoustic absorption by a spherical gas bubble is developed by the application of the classical Rayleigh method. The absorption considered is that due to the viscosity and heat conduction of the gas bubble. Specific results are presented for the S-wave scatter and absorption for the case of an air bubble in water, and the absorption effects of viscosity and heat conduction alone are calculated explicitly. The results found here are of similar magnitude to those found by Pfriem and Spitzer who used an approximate procedure.

Introduction

While the scattering of sound waves by a spherical gas bubble in a liquid has received extensive treatment by the familiar Rayleigh procedure[1], a complete analysis of the acoustic absorption does not appear to be available. This absorption is a consequence of viscosity and heat conduction in the gas bubble. Results for the absorption have been obtained by Pfriem[2] and Spitzer[3], among others, but these approaches were based on approximate models which are limited to the case of spherically symmetric bubble oscillations. It follows that the scatter and absorption is determined only for the case in which the S-wave is significant. It is found here that the long wavelength limit of the present analysis is in approximate agreement with the results of previous calculations.

The calculation here will be made first of the absorption due to the viscosity of the gas, and then the absorption due to both the viscosity and the heat conduction of the gas is determined. It is shown in this way that the thermal absorption is more important than the viscous absorption in the range of bubble sizes and acoustic frequencies of general interest. The numerical results which are given below show the scattering and absorption cross sections for an air bubble in water with viscosity only compared with the corresponding cross sections with heat conduction only.

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[1] See, for example, Lord Rayleigh, "Theory of Sound," Vol. II, p. 282, Dover Publications (New York, 1945); V. C. Anderson, J. Acous. Soc. Am., 22, 426-431 (1950).

[2] H. Pfriem, Akustische Zeitschrift, 5, 202-212 (1940).

[3] L. Spitzer, Jr., OSRD Report M05, Sec. 6.1-sr20-918 (1943).

Viscous Absorption

The gas bubble will be taken to have the equilibrium radius <u>a</u>, and a spherical coordinate system will be used with origin at the center of the bubble. The radial distance from the origin is r and the polar angle between the radius vector and the z-axis is ψ. The disturbance representing the sound wave field outside the gas bubble can be described by the velocity potential, φ, which satisfies the wave equation

$$\Delta\varphi = \frac{1}{c_w^2} \frac{\partial^2\varphi}{\partial t^2} \tag{1}$$

where c_w is the sound speed in the liquid. The appropriate solution of Eq. (1), in the presence of the gas bubble, will be composed of an incident plane wave of unit amplitude and of angular frequency ω and scattered waves:

$$\varphi = e^{i(k_w z - \omega t)} + \sum_{\ell=o}^{\infty} S_\ell h_\ell(k_w r) P_\ell (\cos\psi) e^{-i\omega t}$$

where k_w is the wave number given by $k_w = \omega/c_w$, h_ℓ is the spherical Hankel function of the first kind of order ℓ, P_ℓ is the Legendre polynomial of degree ℓ, and the S_ℓ are constants to be determined. Using the familiar expansion of a plane wave

$$e^{ikz} = e^{ikr\cos\psi} = \sum_{\ell=o}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell (\cos\psi) ,$$

where j_ℓ is the spherical Bessel function of order ℓ, one may write the superposition of incident and scattered waves as

$$\varphi = \sum_{\ell=0}^{\infty} \left[i^{\ell} (2\ell+1) j_{\ell} (k_w r) + S_{\ell} h_{\ell} (k_w r) \right] P_{\ell} (\cos\psi) e^{-i\omega t} . \tag{2}$$

The acoustic velocity field is given by $\nabla\varphi$, and the acoustic pressure is given by $-\rho_w \partial\varphi/\partial t$. If this pressure is written in the form

$$P = P_w (1 + p) , \tag{3}$$

where P_w is the mean, or unperturbed, pressure then

$$p = - \frac{\rho_w}{P_w} \frac{\partial \varphi}{\partial t} = \frac{i\omega \rho_w}{P_w} \varphi . \tag{4}$$

The mean pressure P_w is related to the equilibrium gas pressure P_o as follows:

$$P_w + \frac{2\sigma}{a} = P_o \tag{5}$$

where σ is the surface tension constant. In particular, the perturbation pressure, $P_w p$, at the bubble wall, $r = a$, is given by

$$P_w p = i\omega\rho_w \sum_{\ell=0}^{\infty} \left[i^{\ell} (2\ell+1) j_{\ell} (k_w a) + S_{\ell} h_{\ell} (k_w a) \right] P_{\ell} (\cos\psi) e^{-i\omega t} , \tag{6}$$

and the radial component of the velocity at $r = a$ is

$$v_r = \sum_{\ell=0}^{\infty} \left[i^{\ell} (2\ell+1) j'(k_w a) + S_{\ell} h'(k_w a) \right] k_w P_{\ell} (\cos\psi) e^{-i\omega t} . \tag{7}$$

Within the gas bubble, the motion of the fluid is governed by the equation of continuity

$$\frac{\partial \rho}{\partial t} + \rho\, \nabla \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \nabla \rho = 0 \quad , \tag{8}$$

by the momentum equation[4]

$$\rho \left[\frac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v} \cdot \nabla)\underset{\sim}{v} \right] = - \nabla P + \left(\frac{4}{3}\eta + \beta \right) \nabla(\nabla \cdot \underset{\sim}{v}) - \eta \nabla \times (\nabla \times \underset{\sim}{v}) \quad , \tag{9}$$

and by the equation of state

$$\frac{P}{P_o} = \left(\frac{\rho}{\rho_o} \right)^{\gamma} \quad . \tag{10}$$

In these equations ρ is the density, ρ_o is the equilibrium value of the density, $\underset{\sim}{v}$ is the fluid velocity, η is the coefficient of shear viscosity, β is the coefficient of bulk viscosity, and γ is the ratio of the specific heat of the gas at constant pressure to the specific heat at constant volume. In Eq. (9) it has been assumed that there are no body forces, and it has been assumed further that the coefficients of viscosity are constants. Eq. (10) states the gas expands and contracts adiabatically. It may also be remarked that the stress on a surface element with outward unit normal $\underset{\sim}{n}$ is[5]

$$P\underset{\sim}{n} + \frac{2}{3}\eta\, \underset{\sim}{n}(\nabla \cdot v) - 2\eta\, (\underset{\sim}{n} \cdot \nabla)\underset{\sim}{v} - \beta \underset{\sim}{n}(\nabla \cdot \underset{\sim}{v}) - \eta\, \underset{\sim}{n} \times (\nabla \times \underset{\sim}{v}) \quad . \tag{11}$$

The motion will be taken to be irrotational and then the system of equations will be linearized. If one writes

$$\rho = \rho_o\,(1 + \sigma) \quad ,$$

$$P = P_o(1 + p) \quad ,$$

$$T = T_o\,(1 + \theta) \quad ,$$

[4] S. Goldstein, "Lectures on Fluid Mechanics", pp. 89-90, Interscience Publishers, Inc. (New York, 1960).

[5] L. M. Milne-Thomson, "Theoretical Hydrodynamics", 3rd. ed., p. 538, MacMillan Co. (New York, 1955).

the linearized equations obtained by treating σ, p, θ and $\underset{\sim}{v}$ as small quantities are

$$\frac{\partial \sigma}{\partial t} = - \nabla \cdot \underset{\sim}{v} \; ; \tag{8'}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} = - \frac{P_o}{\rho_o} \nabla p + \frac{1}{\rho_o} \left(\frac{4}{3} \eta + \beta \right) \nabla (\nabla \cdot \underset{\sim}{v}) \; : \tag{9'}$$

and

$$p = \gamma \sigma \; . \tag{10'}$$

Equations (8'), (9'), and (10') lead directly to the relation

$$\frac{\partial^2 p}{\partial t^2} - c_o^2 \nabla^2 p - \nu \nabla^2 \left(\frac{\partial p}{\partial t} \right) = 0 \; , \tag{12}$$

where

$$\mu = \frac{4}{3} \eta + \beta \; ; \quad \nu = \frac{\mu}{\rho_o} \; ; \quad c_o^2 = \frac{\gamma P_o}{\rho_o} \; .$$

With the substitution

$$p (\underset{\sim}{r}, t) = p (\underset{\sim}{r}) \, e^{-i \omega t}$$

Eq. (12) becomes

$$\nabla^2 p + k^2 p = 0 \tag{13}$$

where

$$k^2 = \frac{\omega^2}{c_o^2 - i \omega \nu} \tag{14}$$

If one defines

$$k_o^2 = \frac{\omega^2}{c_o^2} , \qquad \text{and} \qquad k_\nu^2 = \frac{\omega}{\nu} ,$$

then the complex wave number of Eq. (14) may be written as

$$k^2 = \frac{k_o^2}{1 - i(k_o/k_\nu)^2} \tag{14'}$$

Since p is finite at $r = 0$, the general solution of Eq. (13) must have the form

$$p = \sum_{\ell=0}^{\infty} A_\ell \, j_\ell \, (kr) \, P_\ell \, (\cos \psi) , \qquad r \leq a . \tag{15}$$

From Eq. (9') one finds for the radial component of the velocity

$$-i\omega v_r = - \frac{P_o}{\rho_o} \frac{\partial p}{\partial r} + \frac{\mu}{\rho_o} \frac{\partial}{\partial r} \left(\frac{i\omega}{\gamma} p \right) ,$$

or

$$v_r = - \frac{i\omega}{\gamma} \left(\frac{1}{k_o^2} - \frac{i}{k_\nu^2} \right) \frac{\partial p}{\partial r} . \tag{16}$$

It follows that at $r = a$

$$v_r(a) = \sum_{\ell=0}^{\infty} A_\ell f_\ell P_\ell \, (\cos \psi) , \tag{17}$$

where

$$f_\ell = -\frac{i\omega k}{\gamma}\left(\frac{1}{k_o^2} - \frac{i}{k_\nu^2}\right) j_\ell'\ (ka)\ . \tag{17'}$$

Equation (11) gives the perturbed radial stress as

$$P_r = P_o p + \left(\frac{2}{3}\eta - \beta\right)\ \nabla\cdot\underset{\sim}{v} - 2\,\eta\frac{\partial v_r}{\partial r}$$

which can also be written

$$P_r = P_o\ \left\{\left[1+i\left(\frac{2}{3}\ \frac{k_o^2}{k_s^2} - \frac{k_o^2}{k_b^2}\right)\right]p - \frac{2\gamma}{\omega}\ \frac{k_o^2}{k_s^2}\ \frac{\partial v_r}{\partial r}\right\} \tag{18}$$

where

$$k_s^2 = \frac{\rho_o\omega}{\eta}\qquad ;\qquad k_b^2 = \frac{\rho_o\omega}{\beta}\ . \tag{19}$$

This perturbed radial stress at $r = a$ may be put in the form

$$P_r = P_o\ \sum_{\ell=o}^{\infty}\ A_\ell\, g_\ell\, P_\ell\ (\cos\psi) \tag{20}$$

where by Eqs. (15) and (17)

$$g_\ell = \left[1+i\left(\frac{2}{3}\ \frac{k_o^2}{k_s^2} - \frac{k_o^2}{k_b^2}\right) - 2i\ \frac{k_o^2}{k_s^2}\ \left(\frac{1}{k_o^2} - \frac{i}{k_\nu^2}\right)\left(k^2 - \frac{\ell(\ell+1)}{a^2}\right)\right] j_\ell\,(ka)$$

$$- 4i\ \frac{k_o^2}{k_s^2}\ \frac{k}{a}\left(\frac{1}{k_o^2} - \frac{i}{k_\nu^2}\right) j_\ell'\ (ka)\ . \tag{20'}$$

The boundary conditions at $r = a$ are that the radial component of the velocity and the radial component of the stress be continuous. Eq. (7) and (17) give the velocity relation

$$A_\ell f_\ell = S_\ell k_w h'(k_w a) + i^\ell (2\ell+1) k_w j'(k_w a) \ , \tag{21}$$

and Eqs. (6) and (20) give the stress relation

$$A_\ell P_o g_\ell = i\omega \rho_w S_\ell h_\ell (k_w a) + i\omega \rho_w i^\ell (2\ell+1) j_\ell (k_w a) \ . \tag{22}$$

These equations determine A_ℓ and S_ℓ. Solving for the scattering amplitudes S_ℓ, one gets

$$S_\ell = \frac{N_\ell}{M_\ell} \ , \tag{23}$$

where

$$N_\ell = -i^\ell (2\ell+1) j_\ell (k_w a) \left[1 + \frac{i P_o k_w}{\rho_w \omega} \frac{j'_\ell (k_w a)}{j_\ell (k_w a)} \frac{g_\ell}{f_\ell} \right] ;$$

and

$$M_\ell = h_\ell (k_w a) \left[1 + \frac{i P_o k_w}{\rho_w \omega} \frac{h'_\ell (k_w a)}{h_\ell (k_w a)} \frac{g_\ell}{f_\ell} \right] \ .$$

Since g_ℓ and f_ℓ involve spherical Bessel functions with complex arguments, it is difficult to perform an evaluation of S_ℓ for the general case. Simplifications can be made, however, when the absolute values of these arguments are small in comparison with unity in which case the spherical Bessel functions may be expanded in rapidly converging power series. For bubble sizes of general interest and for frequencies in the ordinary acoustic range, both $|ka|$ and $|k_w a|$ are small compared with unity and the expansion procedure may be employed. In this way scattering amplitudes after some algebraic manipulations are found to be in the first approximation

$$S_o = \frac{-ix}{(1-m/x)+i(x+mn/x)} \quad ; \tag{24}$$

$$S_\ell = \frac{-i^\ell (2\ell+1)\left[(1-\ell m_\ell) - i\,\ell m_\ell n_\ell\right]}{\left[1-\ell m_\ell + \dfrac{(\ell+1)m_\ell n_\ell R_\ell^2}{(2\ell+1)x^{2\ell+1}}\right] - i\left[\ell m_\ell n_\ell + \dfrac{R_\ell^2}{(2\ell+1)x^{2\ell+1}} + \dfrac{(\ell+1)m_\ell R_\ell^2}{(2\ell+1)x^{2\ell+1}}\right]} \;;$$

$$\ell \geq 1 \quad ; \tag{25}$$

where

$$x = k_w a \;; \quad m = \frac{3P_o k_w \gamma}{\rho_w \omega^2 a} \;; \quad n = \frac{k_o^2}{k_b^2} \;;$$

$$m_\ell = \frac{\gamma P_o k_o^2}{\ell \rho_w \omega^2} \;; \quad n_\ell = \frac{2\ell(\ell-1)}{k_s^2 a^2} \;; \quad R_\ell = 1\cdot 3\cdot 5 \cdots (2\ell+1) \;.$$

The total scattering cross section is determined by a familiar procedure[6] in terms of the S_ℓ's as follows

$$Q_s = \sum_{\ell=o}^{\infty} Q_s^{(\ell)} = \frac{4\pi}{k_w^2} \sum_{\ell=o}^{\infty} \frac{|S_\ell|^2}{2\ell+1}$$

and the absorption cross section is

$$Q_a = \sum_{\ell=o}^{\infty} Q_a^{(\ell)} = \frac{\pi}{k_w^2} \sum_{\ell=o}^{\infty} (2\ell+1)\left[1 - \left|1 + \frac{2(-i)^\ell S_\ell}{2\ell+1}\right|^2\right] \;.$$

[6]See, for example, P. M. Morse and H. Feshbach, "Methods of Theoretical Physics" Part II, p. 1488, McGraw-Hill Book Co. (New York, 1953).

For the extinction cross section one has

$$Q = Q_s + Q_a = -\frac{4\pi}{k_w^2} \sum_{\ell=o}^{\infty} \mathrm{Re}\left[(-i)^{\ell} S_\ell\right] .$$

The scattering and absorption cross sections which are obtained are thus

$$Q_s^{(o)} = \frac{4\pi}{k_w^2} \frac{x^2}{(1-m/x)^2 + (x+mn/x)^2} \; ; \tag{26}$$

$$Q_a^{(o)} = \frac{4\pi}{k_w^2} \frac{mn}{(1-m/x)^2 + (x+mn/x)^2} \; ; \tag{27}$$

$$Q_s^{(\ell)} = \frac{4\pi}{k_w^2} \frac{(2\ell+1)\left[(1-\ell m_\ell)^2 + \ell^2 m_\ell^2 n_\ell^2\right] x^{4\ell+2}/R_\ell^4}{\left[\frac{(1-\ell m_\ell)x^{2\ell+1}}{R_\ell^2} + \frac{(\ell+1)m_\ell n_\ell}{(2\ell+1)}\right]^2 + \left[\frac{\ell m_\ell n_\ell x^{2\ell+1}}{R_\ell^2} + \frac{1+(\ell+1)m_\ell}{(2\ell+1)}\right]^2} \; ; \tag{28}$$

$$Q_a^{(\ell)} = \frac{4\pi}{k_w^2} \frac{(2\ell+1)m_\ell n_\ell x^{2\ell+1}/R_\ell^2}{\left[\frac{(1-\ell m_\ell)x^{2\ell+1}}{R_\ell^2} + \frac{(\ell+1)m_\ell n_\ell}{(2\ell+1)}\right]^2 + \left[\frac{\ell m_\ell n_\ell x^{2\ell+1}}{R_\ell^2} + \frac{1+(\ell+1)m_\ell}{(2\ell+1)}\right]^2} . \tag{29}$$

Some features of these expressions may be pointed out. The resonance effect in the cross sections is observed with the different resonant frequencies for the various modes of excitation. It may also be noted that in the absorption cross section only the bulk viscosity contributes to the S - wave absorption. There is, of course, no absorption for the P-wave since in the present approximation the $\ell = 1$ disturbance corresponds to a translatory oscillation of the whole bubble without change of shape. There is, finally, the familiar result that for $|x| << 1$ $Q_s^{(o)}$ and $Q_a^{(o)}$ give essentially the entire scattering and absorption cross sections.

Some numerical results have been computed for the case of an air bubble in water for $P_o = 1$ atm and $T = 20^\circ C$. The bulk viscosity of air has been taken to be given by[7] $\beta = 0.57\eta$. The results are shown in Figs. 1-4.

Viscous and Thermal Absorption

When the effect of heat conduction in the gas is included, one can no longer use the adiabatic equation of state. One has instead an energy equation and and equation of state connecting pressure, density, and temperature. The gas will be taken here to be a perfect gas so that the equation of state is

$$p = \frac{R}{M} \rho T ,$$

where M is the molar weight and R is the universal gas constant. The energy equation is[4]

$$\Phi + \kappa \Delta T = \rho c_v \left[\frac{\partial T}{\partial t} + \underset{\sim}{v} \cdot \nabla T \right] + P(\nabla \cdot \underset{\sim}{v}), \tag{30}$$

where

$$\Phi = \eta \left\{ \nabla \cdot (\nabla \underset{\sim}{v}^2) + 2 \nabla \cdot \left[(\nabla \times \underset{\sim}{v}) \times \underset{\sim}{v} \right] - 2 \underset{\sim}{v} \cdot \nabla (\nabla \cdot \underset{\sim}{v}) + (\nabla \times \underset{\sim}{v})^2 - \frac{2}{3} (\nabla \cdot \underset{\sim}{v})^2 \right\}$$
$$+ \beta (\nabla \cdot \underset{\sim}{v})^2 .$$

[7] F. S. Sherman, NACA Tech. Note 3298 p. 21 (July, 1955); C. Truesdell, J. Rat. Mech. and Anal. $\underset{\sim}{2}$, p. 649 (1953).

In these equations κ is the coefficient of heat conduction and c_v is the specific heat at constant volume. The perfect gas equation of state when linearized becomes

$$p = \sigma + \theta \tag{31}$$

where p, σ, and θ have been defined in the previous section. The linearization of Eq. (30) gives

$$\Delta\theta = \frac{1}{D}\frac{\partial\theta}{\partial t} + \frac{P_o}{\kappa T_o} \nabla\cdot\underset{\sim}{v} \tag{32}$$

where $D = \kappa/\rho_o c_v$ is the thermal diffusivity. Equations (8') and (9') together with Equations (31) and (32) lead to

$$\Delta\left[c_i^2 \Delta\theta - \frac{\partial^2\theta}{\partial t^2} + \nu \frac{\partial}{\partial t} \Delta\theta\right] = \frac{1}{D}\frac{\partial}{\partial t}\left[c_a^2 \Delta\theta - \frac{\partial^2\theta}{\partial t^2} + \nu\frac{\partial}{\partial t}\Delta\theta\right] . \tag{33}$$

In Eq. (33) $c_i = (P_o/\rho_o)^{1/2}$ is the isothermal sound speed and $c_a = (\gamma P_o/\rho_o)^{1/2}$ is the adiabatic sound speed. For periodic oscillations

$$\theta(\underset{\sim}{r}, t) = \theta(\underset{\sim}{r})e^{-i\omega t}$$

Eq. (33) becomes

$$\left[1 - i(k_i/k_\nu)^2\right]\Delta^2\theta + \left[k_i^2 + i(k_t k_i/k_a)^2 + (k_t k_i/k_\nu)^2\right]\Delta\theta$$

$$+ i\, k_i^2 k_t^2 \theta = 0 \quad , \tag{34}$$

where

$$k_i^2 = (\omega/c_i)^2 \; ; \quad k_a^2 = (\omega/c_a)^2 \; ; \quad k_t^2 = \omega/D \; ;$$

and as before $k_\nu^2 = \omega/\nu$.

Equation (34) may also be written in the form

$$(\Delta + k_1^2)(\Delta + k_2^2)\theta = 0 \; , \tag{35}$$

with

$$k_{1,2}^2 = \frac{1}{2}\left[G \pm (G^2+4F)^{1/2}\right] \; , \tag{36}$$

$$F = \frac{k_t^2 k_i^2 \left[(k_i^2/k_\nu^2) - i\right]}{1 + (k_i/k_\nu)^2} \; , \tag{37}$$

and

$$G = \frac{k_i^2}{1+(k_i^2/k_\nu^2)} \left\{ 1 + (k_t/k_\nu)^2 - (k_i k_t/k_\nu k_a)^2 \right.$$

$$\left. + i\left[(k_t/k_a)^2 + (k_i/k_\nu)^2 + (k_i k_t/k_\nu^2)^2\right]\right\} \; . \tag{38}$$

Both θ and v_r are finite at the origin so that one may write the expansion of θ in the form

$$\theta = \sum_{\ell=0}^{\infty} \left[A_\ell j_\ell(k_1 r) + B_\ell j_\ell(k_2 r)\right] P_\ell(\cos\psi) \; , \quad \text{for } r \leq a \; . \tag{39}$$

At $r = a$ one has the boundary conditions, as before, which ensure the continuity of the radial components of the stress and of the velocity. It is now necessary to impose an additional boundary condition to determine the temperature field. This boundary condition will be taken here to be that the perturbed temperature, $T_o \theta$, at the bubble wall is zero at all times. That this assumption for the boundary condition is a good approximation may be shown by the following physical argument. If the thermal conductivity of the liquid is κ_L and the temperature gradient in the liquid at the bubble boundary is $(\partial T/\partial r)_L$, and if the corresponding quantities in the bubble are denoted by κ_G and $(\partial T/\partial r)_G$, then the continuity of heat flow at the interface requires that

$$\kappa_G (\partial T/\partial r)_G = \kappa_L (\partial T/\partial r)_L \quad .$$

One may observe that ordinarily $\kappa_L >> \kappa_G$ so that $(\partial T/\partial r)_L << (\partial T/\partial r)_G$. For example, for water $\kappa_L \sim 5.9 \times 10^4$ (c.g.s.) and $\kappa_G \sim 2 \times 10^3$ (c.g.s.). One may now argue that not only is the temperature gradient in the liquid a small quantity, but in addition it comes close to its constant value at infinity in a short distance into the liquid from the bubble boundary. The heat diffusion length in the liquid is $(D_L/\omega)^{1/2}$ and the heat diffusion length in the bubble is $(D_G/\omega)^{1/2}$. Here ω is the angular oscillation frequency and D_L, D_G are the thermal diffusivities of the liquid and the gas, respectively. The diffusion length, $(D_L/\omega)^{1/2}$, over which the temperature varies in the liquid is small compared with the diffusion length in the gas, $(D_G/\omega)^{1/2}$. This follows since $D_L << D_G$. For the example of water and air one has $D_L \sim 1.4 \times 10^{-3}$(c.g.s.) and $D_G \sim 0.2$ (c.g.s.). Thus one may say that the temperature gradient in the liquid is small and further the distance over which this gradient extends is small. As a consequence, the variation in the temperature at the bubble boundary must be small.

By a procedure similar to that followed to obtain Eqs. (17) and (30), one obtains for $r = a$

$$v_r = \sum_{\ell=o}^{\infty} \left[A_\ell f_\ell(1) + B_\ell f_\ell(2) \right] P_\ell (\cos\psi) \quad , \tag{40}$$

and

$$P_r = P_o \sum_{\ell=o}^{\infty} \left[A_\ell g_\ell(1) + B_\ell g_\ell(2) \right] P_\ell (\cos\psi) \quad , \tag{41}$$

where, with $\alpha = 1$ or 2,

$$f_\ell(\alpha) = i\omega \frac{\kappa T_o}{DP_o} k_\alpha \left[-\frac{1}{k_a^2} + \frac{i}{k_\nu^2} - \frac{ik_\alpha^2}{k_t^2 k_i^2} - \frac{k_\alpha^2}{k_t^2 k_\nu^2} \right] j_\ell'(k_\alpha a) \quad , \tag{42}$$

$$g_\ell(\alpha) = \frac{\kappa T_o}{DP_o} \left\{ \left[\left(\frac{k_i}{k_a}\right)^2 + \frac{2i}{3}\left(\frac{k_i}{k_s}\right)^2 - i\left(\frac{k_i}{k_b}\right)^2 - k_\alpha^2 \left(-\frac{i}{k_t^2} + \frac{2}{3}\frac{k_i^2}{k_t^2 k_s^2} - \frac{k_i^2}{k_t^2 k_b^2} \right) \right. \right.$$

$$\left. + 2i\left(\frac{k_i}{k_s}\right)^2 \left(k_\alpha^2 - \frac{\ell(\ell+1)}{a^2} \right) \left(-\frac{1}{k_a^2} + \frac{i}{k_\nu^2} - \frac{ik_\alpha^2}{k_t^2 k_i^2} + \frac{k_\alpha^2}{k_t^2 k_\nu^2} \right) \right] j_\ell(k_\alpha a)$$

$$\left. + \frac{4i k_i^2 k_\alpha}{k_s^2 a} \left[-\frac{1}{k_a^2} + \frac{i}{k_\nu^2} - \frac{ik_\alpha^2}{k_t^2 k_i^2} - \frac{k_\alpha^2}{k_t^2 k_\nu^2} \right] j_\ell'(k_\alpha a) \right\} \quad . \tag{43}$$

The boundary conditions can now be summarized as follows:

$$A_\ell j_\ell(k_1 a) + B_\ell j_\ell(k_2 a) = 0 \quad ;$$

$$A_\ell f_\ell(1) + B_\ell f_\ell(2) = S_\ell k_w h_\ell'(k_w a) + i^\ell (2\ell+1) k_w j_\ell'(k_w a) \quad ;$$

$$A_\ell g_\ell(1) + B_\ell g_\ell(2) = \frac{i\omega \rho_w}{P_o} S_\ell h_\ell(k_w a) + \frac{i\omega\rho_w}{P_o} i^\ell (2\ell+1) j_\ell(k_w a) \quad .$$

one finds S_ℓ as

$$S_\ell = \frac{N_\ell}{M_\ell} \tag{44}$$

with

$$N_\ell = k_w h_\ell'(k_w a)\left[j_\ell(k_1 a) g_\ell(2) - j_\ell(k_2 a) g_\ell(1)\right]$$

$$- \frac{i\omega\rho_w}{P_o} h_\ell(k_w a)\left[j_\ell(k_1 a) f_\ell(2) - j_\ell(k_2 a) f_\ell(1)\right] \quad , \tag{45}$$

and

$$M_\ell = -i^\ell(2\ell+1)\left\{ k_w j_\ell'(k_w a)\left[j_\ell(k_1 a) g_\ell(2) - j_\ell(k_2 a) g_\ell(1)\right]\right.$$

$$\left. - \frac{i\omega\rho_w}{P_o} j_\ell(k_w a)\left[j_\ell(k_1 a) f_\ell(2) - j_\ell(k_2 a) f_\ell(1)\right]\right\} \quad . \tag{46}$$

The cross sections are thus determined in principle for any ω in terms of S_ℓ by the general formulas for the scattering and absorption cross sections already given in the previous section.

Thermal Absorption

One may determine the effect of heat conduction alone in the scattering and absorption of sound waves, by omitting the contributions of the viscous terms in the general expressions just found. Of interest for this case are the limits of Eqs. (42) and (43) as k_ν, k_s, and k_b become infinitely large:

$$f_\ell(\alpha) = \frac{i\omega\kappa T_o}{DP_o} k_\alpha \left[-\frac{1}{k_a^2} - \frac{i k_\alpha^2}{k_t^2 k_i^2}\right] j_\ell'(k_\alpha a) \quad ; \tag{47}$$

$$g_\ell(\alpha) = \frac{\kappa T_o}{DP_o}\left[\left(\frac{k_i}{k_a}\right)^2 + i\left(\frac{k_\alpha}{k_t}\right)^2\right] j_\ell(k_\alpha a) \quad ; \tag{48}$$

where k_1^2 and k_2^2 are now given by simplified expressions

$$k_{1,2}^2 = \frac{1}{2}\left[G \pm (G^2 + 4F)^{1/2}\right] \tag{49}$$

with

$$F = -i\, k_t^2\, k_i^2 \quad , \tag{50}$$

and

$$G = k_i^2\left[1 + i\left(\frac{k_t}{k_a}\right)^2\right] \quad . \tag{51}$$

Explicit expressions for the cross sections will be given only for the case $|k_w a| << 1$ in which the important contribution is from the S-wave alone. It may be observed that in the frequency range of general interest one has $|k_a/k_t|^2 << 1$, and the following approximations may be made:

$$k_1^2 \approx i\gamma\, k_t^2\left[1 - i\,\frac{(\gamma - 1)}{\gamma}\,\frac{k_a^2}{k_t^2}\right] \quad ;$$

$$k_2^2 \approx k_a^2 \quad .$$

Then, after some manipulations, the cross sections are found to be:

$$Q_s^{(o)} = \frac{4\pi}{k_w^2}\,\frac{x^2}{(1 - \frac{m'}{x})^2 + (x + \frac{m'n'}{x})^2} \quad , \tag{52}$$

and

$$Q_a^{(o)} = \frac{4\pi}{k_w^2}\,\frac{m'n'}{(1 - \frac{m'}{x})^2 + (x + \frac{m'n'}{x})^2} \tag{53}$$

where

$$m' = \frac{3P_o k_w \gamma}{\rho_w \omega^2 a} \frac{N'}{M'} \quad , \tag{54}$$

$$M' = \left[1 + \frac{36(\gamma-1)^2}{d^4}\right] (\cosh d - \cos d) + \frac{6(\gamma-1)}{d} (\sinh d - \sin d)$$

$$+ \frac{18(\gamma-1)^2}{d^2} (\cosh d + \cos d) - \frac{36(\gamma-1)^2}{d^3} (\sinh d + \sin d) \quad , \tag{55}$$

$$N' = (\cosh d - \cos d) + \frac{3(\gamma-1)}{d} (\sinh d - \sin d) \quad , \tag{56}$$

$$d = (2\gamma)^{1/2} (k_t a) \quad , \tag{57}$$

and

$$n' = \frac{3(\gamma-1) \left[d(\sinh d + \sin d) - 2 (\cosh d - \cos d) \right]}{d^2(\cosh d - \cos d) + 3 (\gamma-1) d (\sinh d - \sin d)} \quad . \tag{58}$$

For $d < 1$, it is found, approximately, that

$$m' = \frac{3P_o k_w}{\rho_w \omega^2 a} \quad , \tag{54'}$$

and

$$n' = \frac{\gamma-1}{15} (k_t a)^2 \tag{58'}$$

while for $e^d >> 1$, it is found that

$$m' = \frac{3\gamma P_o k_w}{\rho_w \omega^2 a} \tag{54''}$$

and

$$n' = \frac{3(\gamma-1)}{(2\gamma)^{1/2}} \frac{1}{k_t a} \tag{58''}$$

It may be remarked that the resonance frequencies as obtained from (54') and (54''), show a shift from the isothermal limit to the adiabatic limit as the quantity $k_t a$ increases. As k_t is actually the inverse of the diffusion length, this result certainly agrees with what one would expect from physical arguments. Figure 1, 2, 3 and 4 show some numerical results for the case of an air bubble in water. It can be seen that in the range of frequencies shown the effect of thermal absorption is appreciably larger than the effect of viscosity.

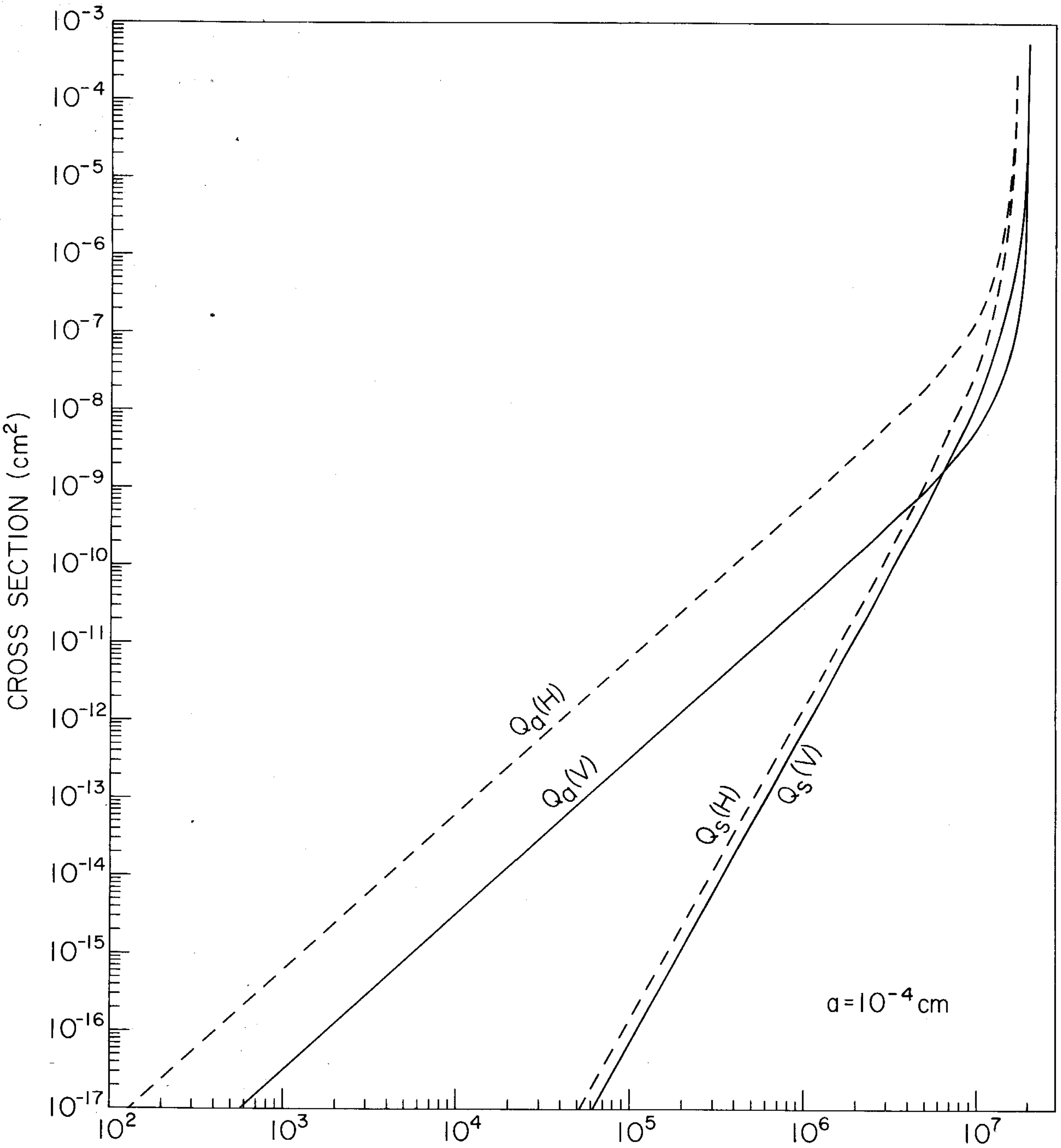

Figure 1. The scattering cross section, $Q_s(H)$, and the absorption cross section, $Q_a(H)$, take the bubble heat conduction only into account. The scattering cross section, $Q_s(V)$, and the absorption cross section, $Q_a(V)$ take bubble viscosity only into account. The bubble radius is 10^{-4} cm.

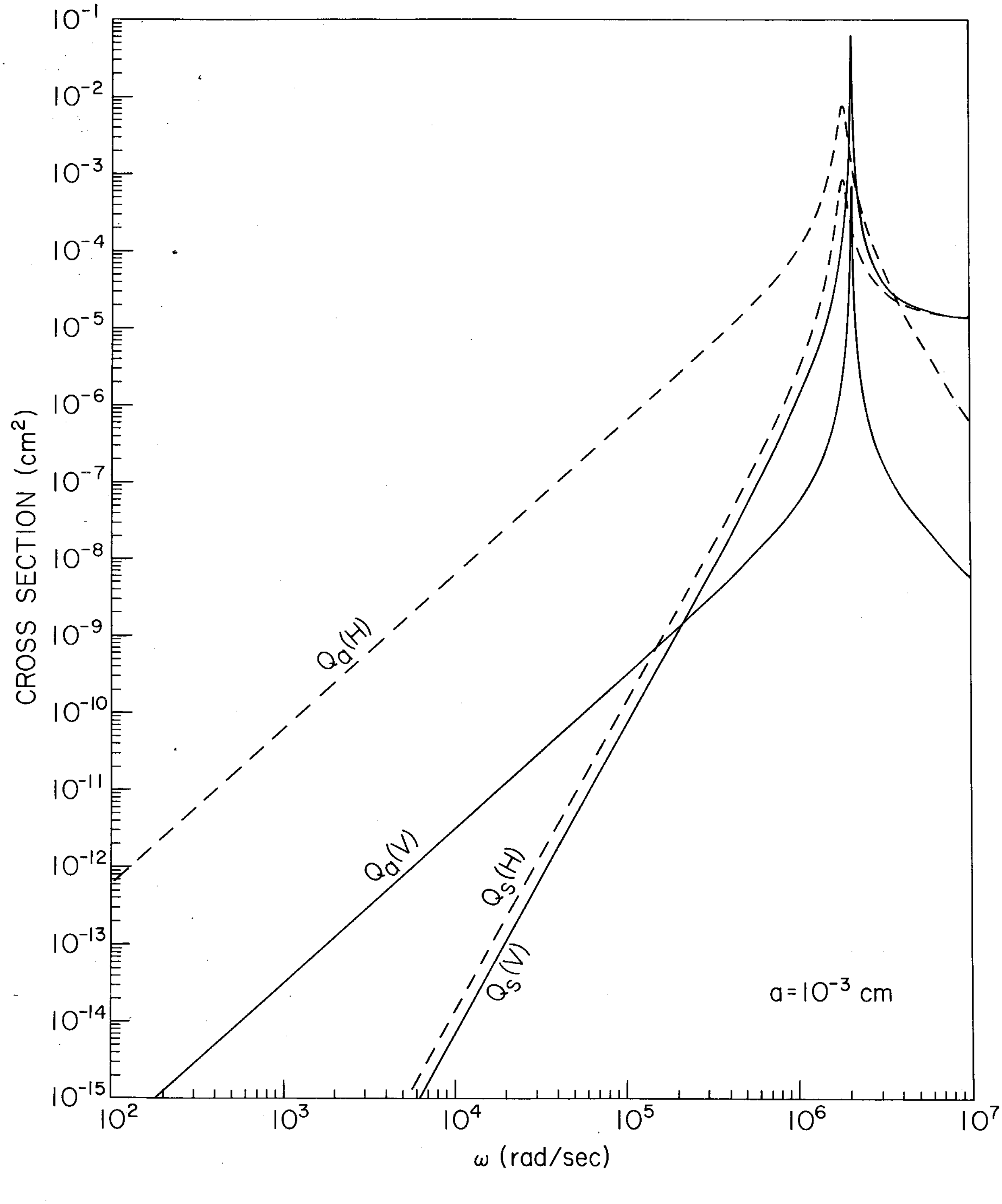

Figure 2. The same cross sections shown in Figure 1 are shown for the bubble radius of 10^{-3} cm.

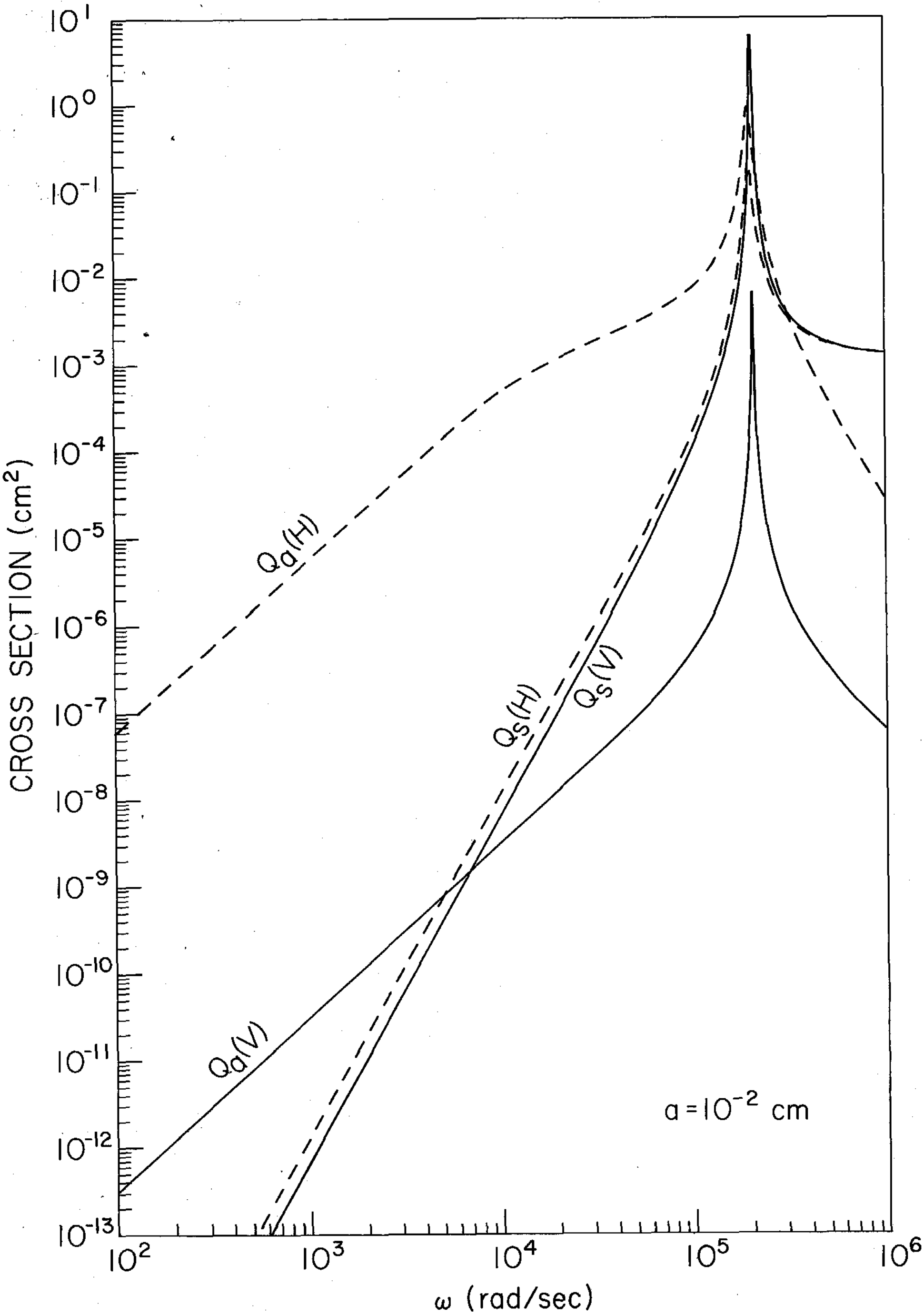

Figure 3. The cross sections of Figure 1 are shown for the bubble radius of 10^{-2} cm.

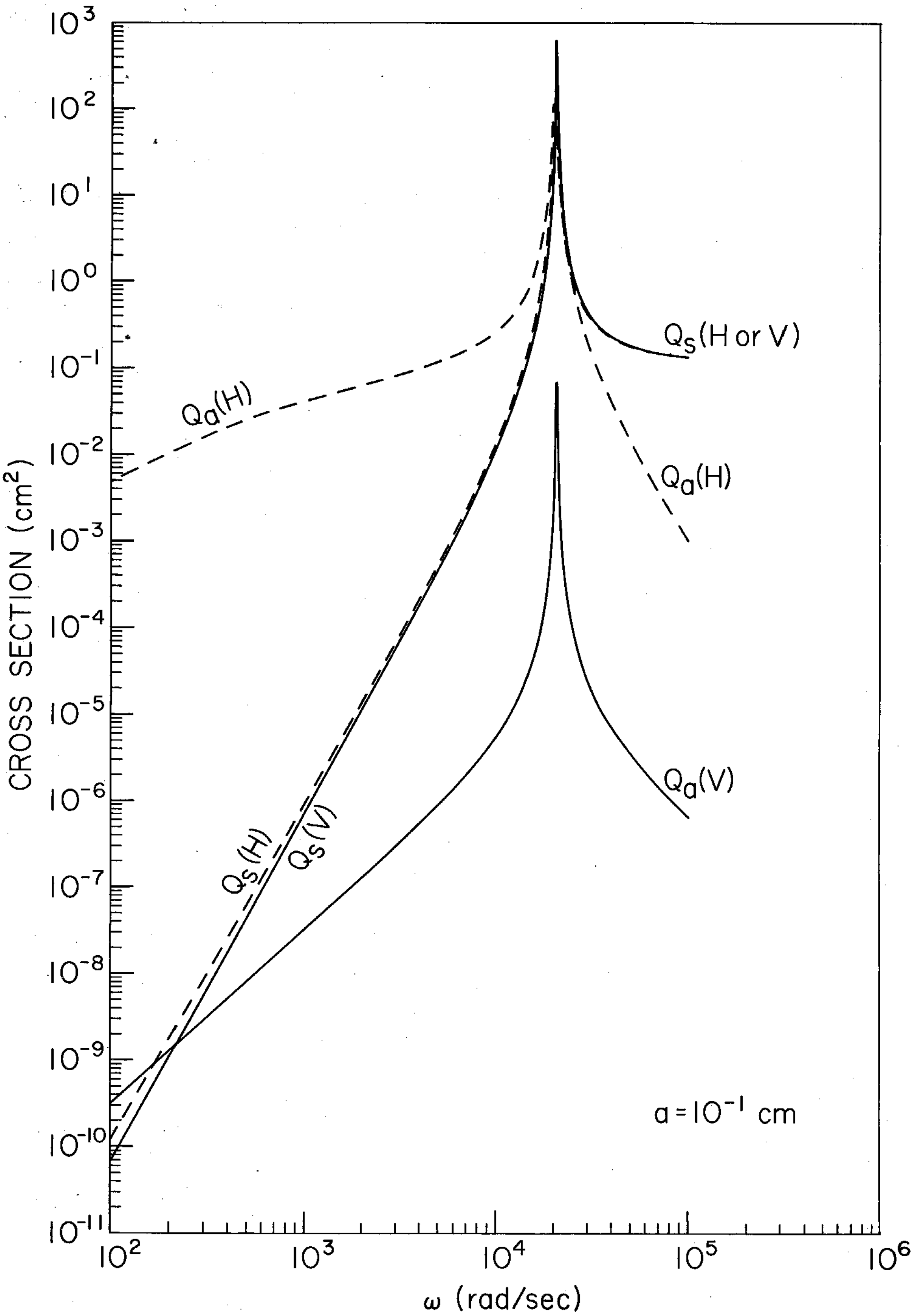

Figure 4. The cross sections of Figure 1 are shown for the bubble radius of 10^{-1} cm.

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