

in which the average value is for the temperature distribution present when  $\omega$  was observed. Implicit in expression (7) is the assumption that  $J/l$  is a constant regardless of temperature. Actually the ratio varies because of thermal expansion so that (7) implies a redefinition of the modulus  $G$  at some temperature  $T$  in terms of the dimensions at room temperature rather than at the temperature  $T$ . The matter is of little practical importance here because measurements of the increase in length for the conditions of the experiment showed that the maximum change in  $J/l$  was of the order of one percent. Now let  $G_0/G_T$  in which  $G_0$  and  $G_T$  are the values of the modulus at room temperature and at a temperature  $T$ , respectively, be represented by a polynomial in powers of  $(T-T_0)$ , adequate to represent the data over a restricted interval. We then have

$$G_0/G_T = 1 + A_1(T-T_0) + A_2(T-T_0)^2 + A_3(T-T_0)^3 + \dots, \quad (8)$$

with the coefficients  $A_n$  to be determined from the experimental data. The determination is made by solving a set of simultaneous equations involving the  $A_n$ . A particular equation of the set is obtained by substituting an experimentally observed value for  $(\omega_0/\omega)^2$  in Eq. (7) and by substituting for  $\langle G_0/G(z) \rangle_{AV}$  the reciprocal of the rod length multiplied by the integral of Eq. (8) over the length of the rod. The integration is performed by using the temperature distribution which was observed for the value of  $(\omega_0/\omega)^2$  selected. In actual practice the corrected curve was obtained by determining three coefficients which matched the integrated experimental results for three points at 1500°C and above and by determining three coefficients which matched the results for three points at 1500°C and below. The corrected curve for the high region and that for the low region were observed to agree well in an intermediate range, and the final corrected curve was obtained by a smooth joining of the low temperature and the high temperature curves.

## A Nonsteady Heat Diffusion Problem with Spherical Symmetry\*

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A solution in successive approximations is presented for the heat diffusion across a spherical boundary with radial motion. The approximation procedure converges rapidly provided the temperature variations are appreciable only in a thin layer adjacent to the spherical boundary. An explicit solution for the temperature field is given in the zero order when the temperature at infinity and the temperature gradient at the spherical boundary are specified. The first-order correction for the temperature field may also be found. It may be noted that the requirements for rapid convergence of the approximate solution are satisfied for the particular problem of the growth or collapse of a spherical vapor bubble in a liquid when the translational motion of the bubble is neglected.

### I. INTRODUCTION

A PROBLEM of nonsteady heat diffusion is encountered if one considers the dynamics of a vapor bubble in a liquid. As the size of the bubble changes, heat flows across the moving liquid-vapor interface. The liquid is assumed here to be nonviscous and incompressible, and the thermal conductivity  $k$ , density  $\rho$ , and specific heat  $c$  of the liquid are assumed to have insignificant variation with temperature. The temperature  $T$  in the liquid then satisfies the equation

$$\Delta T = \frac{1}{D} \frac{dT}{dt} - \frac{1}{k} \dot{\eta}. \quad (1)$$

Here  $D = k/\rho c$  is the thermal diffusivity of the liquid,  $\eta = \eta(t)$  is the heat source per unit volume in the liquid which is taken to be a function of time only, and  $dT/dt$

denotes the particle derivative so that

$$dT/dt = \partial T/\partial t + \mathbf{v} \cdot \nabla T,$$

where  $\mathbf{v}$  is the liquid velocity which in general varies with position and time.

### II. FORMULATION OF THE PROBLEM

It is advantageous to transform Eq. (1) from Eulerian to Lagrangian coordinates. It will be assumed that the motion possesses spherical symmetry; i.e., the vapor bubble is spherical, and its radial motion is sufficiently rapid that any translational motion may be neglected. The Eulerian coordinates will be chosen as  $(r, t)$  with origin  $r=0$  at the center of the bubble. If  $R(t)$  is the bubble radius at time  $t$ , Lagrange coordinates may be defined by

$$h = (1/3)[r^3 - R^3(t)], \quad (2a)$$

$$t = t, \quad (2b)$$

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and Eq. (1) can now be written as

$$\frac{\partial}{\partial h} \left( r^4 \frac{\partial T}{\partial h} \right) = \frac{1}{D} \frac{\partial T}{\partial t} - \frac{1}{k} \dot{\eta}. \tag{3}$$

If at  $t=0$ , the temperature at  $r = \infty$  is  $T_0$ , then at a later time  $t$  the temperature at infinity is  $T_\infty = T_0 + (D/k)\eta(t)$ , where  $\eta(0)=0$ . It is convenient to introduce a function  $U(h, t)$  such that

$$\partial U / \partial h = T - T_0, \tag{4}$$

and Eq. (3) may then be written

$$\frac{\partial}{\partial h} \left\{ r^4 \frac{\partial^2 U}{\partial h^2} - \frac{1}{D} \frac{\partial U}{\partial t} \right\} = -\frac{1}{k} \dot{\eta},$$

so that by integration

$$r^4 \frac{\partial^2 U}{\partial h^2} - \frac{1}{D} \frac{\partial U}{\partial t} = -\frac{h}{k} \dot{\eta} + \psi(t), \tag{5}$$

where  $\psi(t)$  is an arbitrary function of time. From Eq. (4),

$$U = \int_0^h (T - T_0) dh + \chi(t),$$

and the function  $\chi(t)$  may be chosen so that  $\psi(t)=0$ , and also so that

$$U(h, 0) = 0, \tag{6}$$

if it is assumed that the temperature at  $t=0$  is  $T_0$  everywhere. Equation (5) now becomes

$$r^4 \frac{\partial^2 U}{\partial h^2} - \frac{1}{D} \frac{\partial U}{\partial t} = -\frac{h}{k} \dot{\eta}. \tag{7}$$

In addition to Eq. (6), one has the boundary condition at infinity

$$(\partial U / \partial h)_{h=\infty} = (D/k)\eta(t). \tag{8}$$

The further boundary condition required to fix the temperature field may be given by the temperature gradient at the bubble wall,  $r=R(t)$ , as a function of time. One then has

$$(\partial^2 U / \partial h^2)_{h=0} = (1/R^2)(\partial T / \partial r)_{r=R(t)} = F(t). \tag{9}$$

The diffusion problem thus defined can be solved by successive approximations, if the assumption is made that  $U$  and  $T$  vary appreciably only in the region

$$h \ll R^3(t). \tag{10}$$

Physically, for the vapor bubble problem, this assumption is made plausible by the fact that not only is the heat capacity much greater in the liquid state than in the vapor state but the thermal diffusivity is also much smaller.

### III. ZERO-ORDER SOLUTION

It is convenient to introduce in Eq. (7) a new time variable  $\tau$  defined by

$$\tau = \int_0^t R^4(t) dt.$$

Equation (7) now becomes, to the first order in  $h/R^3$ ,

$$\frac{\partial^2 U}{\partial h^2} - \frac{1}{D} \frac{\partial U}{\partial \tau} + \frac{h}{k} \frac{d\eta}{d\tau} = -\frac{4h}{R^3} \frac{\partial^2 U}{\partial h^2}, \tag{11}$$

since

$$r^4 \approx R^4(1 + 4h/R^3)$$

to this order. By the usual procedure of successive approximations, one sets

$$U = U^0 + U^1 + \dots, \quad T = T^0 + T^1 + \dots,$$

where the superscript denotes the order of the approximation in powers of the perturbation parameter  $h/R^3$ . The zero-order approximation,  $U^0$ , is thus determined by

$$\frac{\partial^2 U^0}{\partial h^2} - \frac{1}{D} \frac{\partial U^0}{\partial \tau} + \frac{h}{k} \frac{d\eta}{d\tau} = 0, \tag{12}$$

with the boundary conditions (6), (8), and (9).

The appropriate solution of Eq. (12) is readily found by taking the Laplace transform on the variable  $\tau$ . If

$$u(h, s) = \mathcal{L}\{U^0(h, \tau)\} \equiv \int_0^\infty e^{-s\tau} U^0(h, \tau) d\tau,$$

$$\sigma(s) = \mathcal{L}\{\eta(\tau)\},$$

$$f(s) = \mathcal{L}\{F(\tau)\},$$

then

$$\frac{d^2 u}{dh^2} - \frac{s}{D} u = -\frac{hs}{k} \sigma(s), \tag{13}$$

where use has been made of Eq. (6) and the specification that  $\eta(0)=0$ . One has further, from Eqs. (8) and (9),

$$(du/dh)_{h=\infty} = (D/k)\sigma(s), \tag{14}$$

$$(d^2 u / dh^2)_{h=0} = f(s). \tag{15}$$

The required solution of Eq. (13) is

$$u = (D/s)f(s) \exp\{-h(s/D)^{1/2}\} + h(D/k)\sigma(s),$$

so that

$$\frac{du}{dh} = -(D/s)^{1/2} f(s) \exp\{-h(s/D)^{1/2}\} + (D/k)\sigma(s). \tag{16}$$

From Eq. (16), one finds

$$\begin{aligned} \mathcal{L}^{-1}\{du/dh\} &= \partial U^0/\partial h = T^0(h, \tau) - T_0 \\ &= (D/k)\eta(\tau) - (D/\pi)^{\frac{1}{2}} \int_0^\tau \frac{F(\xi)}{(\tau-\xi)^{\frac{1}{2}}} \\ &\quad \times \exp\left\{-\frac{h^2}{4D(\tau-\xi)}\right\} d\xi. \end{aligned} \quad (17)$$

If one sets

$$\theta(h, \tau) = T - T_0 - (D/k)\eta(\tau) = T - T_\infty,$$

then

$$\begin{aligned} \theta^0(h, \tau) &= -(D/\pi)^{\frac{1}{2}} \int_0^\tau \frac{F(\xi)}{(\tau-\xi)^{\frac{1}{2}}} \\ &\quad \times \exp\left\{-\frac{h^2}{4D(\tau-\xi)}\right\} d\xi. \end{aligned} \quad (17')$$

Thus, the difference between the temperature at the spherical boundary,  $T^0(0, \tau)$ , and the temperature at infinity,  $T_\infty$ , is given by

$$\theta^0(0, \tau) = -(D/\pi)^{\frac{1}{2}} \int_0^\tau \frac{F(\xi) d\xi}{(\tau-\xi)^{\frac{1}{2}}}. \quad (18)$$

One also has from Eq. (16) for  $h=0$

$$\theta^0(0, \tau) = \mathcal{L}^{-1}\{(D/s)^{\frac{1}{2}}f(s)\},$$

so that the inverse Laplace transform of Eq. (16) may be written in an alternative form as

$$\begin{aligned} \theta^0(h, \tau) &= [h/(4\pi D)^{\frac{1}{2}}] \int_0^\tau \frac{\theta^0(0, \xi)}{(\tau-\xi)^{\frac{1}{2}}} \\ &\quad \times \exp\left\{-\frac{h^2}{4D(\tau-\xi)}\right\} d\xi. \end{aligned} \quad (19)$$

In terms of the original time variable, one has, for example, from Eq. (18)

$$\begin{aligned} T^0(0, t) - T_0 &= (D/k)\eta(t) \\ &- (D/\pi)^{\frac{1}{2}} \int_0^t \frac{R^2(x)(\partial T/\partial r)_{r=R(x)}}{\left\{\int_x^t R^4(y)dy\right\}^{\frac{1}{2}}} dx. \end{aligned} \quad (20)$$

If the variations in  $R(t)$  are sufficiently small, Eq. (20) simplifies to

$$\begin{aligned} T^0(0, t) - T_0 &= (D/k)\eta(t) \\ &- (D/\pi)^{\frac{1}{2}} \int_0^t \frac{(\partial T/\partial r)_{r=R(x)}}{(t-x)^{\frac{1}{2}}} dx, \end{aligned} \quad (21)$$

which represents the "plane approximation" obtained if the curvature of the boundary  $r=R(t)$  is completely neglected.

It may be noted, when

$$T^0(0, \tau) - T_0 - (D/k)\eta(\tau) = \theta^0(0, \tau)$$

is a monotonic function of  $\tau$ , that one obtains from Eq. (19) the inequality

$$\frac{\theta^0(h, \tau)}{\theta^0(0, \tau)} \leq \text{erfc}\{h/(4D\tau)^{\frac{1}{2}}\}.$$

#### IV. FIRST-ORDER CORRECTION TO THE SOLUTION

If one continues the procedure of successive approximation, the right side of Eq. (11) is now considered as determined from the zero-order solution  $U^0$ , and the first-order correction  $U'$  is determined by

$$\frac{\partial^2 U'}{\partial h^2} - \frac{1}{D} \frac{\partial U'}{\partial \tau} = -\frac{4h}{R^3} \frac{\partial^2 U^0}{\partial h^2} = G(h, \tau). \quad (22)$$

The boundary conditions for  $U'(h, \tau)$  are

$$U'(h, 0) = (\partial U'/\partial h)_{h=\infty} = (\partial^2 U'/\partial h^2)_{h=0} = 0. \quad (23)$$

From Eq. (4), the first-order correction,  $T'$ , to the temperature field is given by

$$T' = \partial U'/\partial h.$$

Denoting the Laplace transforms of  $U'(h, \tau)$ ,  $G(h, \tau)$  by  $v(h, s)$ ,  $g(h, s)$ , respectively, one has

$$\frac{d^2 v}{dh^2} - \frac{s}{D} v = g(h, s), \quad (24)$$

with the boundary conditions

$$(dv/dh)_{h=\infty} = (d^2 v/dh^2)_{h=0} = 0. \quad (25)$$

By Eq. (22),  $G(0, \tau) = 0$ , and the solution to Eqs. (24) and (25) is readily found to be

$$\begin{aligned} v(h, s) &= -\frac{1}{2} \left(\frac{D}{s}\right)^{\frac{1}{2}} \left\{ e^{h(s/D)^{\frac{1}{2}}} \int_h^\infty e^{-x(s/D)^{\frac{1}{2}}} g(x, s) dx \right. \\ &\quad \left. + e^{-h(s/D)^{\frac{1}{2}}} \int_0^h e^{x(s/D)^{\frac{1}{2}}} g(x, s) dx \right. \\ &\quad \left. - e^{-h(s/D)^{\frac{1}{2}}} \int_0^\infty e^{-x(s/D)^{\frac{1}{2}}} g(x, s) dx \right\}, \end{aligned}$$

so that

$$\begin{aligned} \frac{dv}{dh} &= -\frac{1}{2} \left\{ e^{h(s/D)^{\frac{1}{2}}} \int_h^\infty e^{-x(s/D)^{\frac{1}{2}}} g(x, s) dx \right. \\ &\quad \left. - e^{-h(s/D)^{\frac{1}{2}}} \int_0^h e^{x(s/D)^{\frac{1}{2}}} g(x, s) dx \right. \\ &\quad \left. + e^{-h(s/D)^{\frac{1}{2}}} \int_0^\infty e^{-x(s/D)^{\frac{1}{2}}} g(x, s) dx \right\}. \end{aligned} \quad (26)$$

For  $h=0$ , one has

$$\left(\frac{dv}{dh}\right)_{h=0} = - \int_0^\infty e^{-x^{(s/D)^{1/2}}g(x,s)dx = \mathcal{L}\{T'(0,\tau)\}. \quad (27)$$

From Eq. (27), the first-order correction to the temperature at the spherical boundary is

$$T'(0,\tau) = - \int_0^\infty dx \int_0^\tau G(x,\xi) \times \frac{x \exp[-x^2/4D(\tau-\xi)]}{[4\pi D(\tau-\xi)^3]^{1/2}} d\xi. \quad (28)$$

By Eq. (22) and the definition of  $\theta^0$

$$G(h,\tau) = - \frac{4h}{R^3(\tau)} \frac{\partial \theta^0(h,\tau)}{\partial h},$$

so that Eq. (28) becomes

$$T'(0,\tau) = \int_0^\infty dx \int_0^\tau \frac{4x^2}{R^3(\xi)} \frac{\partial \theta^0(x,\xi)}{\partial x} \times \frac{\exp[-x^2/4D(\tau-\xi)]}{[4\pi D(\tau-\xi)^3]^{1/2}} d\xi,$$

and, if it is permissible to interchange the order of the integrations, one gets

$$T'(0,\tau) = 2 \int_0^\tau \frac{d\xi}{R^3(\xi)[\pi D(\tau-\xi)^3]^{1/2}} \int_0^\infty \frac{\partial \theta^0(x,\xi)}{\partial x} \times \exp[-x^2/4D(\tau-\xi)] x^2 dx. \quad (29)$$

Differentiation of Eq. (19) gives

$$\frac{\partial \theta^0(x,\xi)}{\partial x} = \frac{1}{(4\pi D)^{1/2}} \int_0^\xi \frac{\theta^0(0,\zeta)}{(\xi-\zeta)^{3/2}} \frac{\partial}{\partial x} \times \{x \exp[-x^2/4D(\xi-\zeta)]\} d\zeta.$$

Substitution of this relation in Eq. (29) leads to the result

$$T'(0,\tau) = 2 \left(\frac{D}{\pi}\right)^{1/2} \int_0^\tau \frac{d\xi}{R^3(\xi)} \times \int_0^\xi \frac{\theta^0(0,\zeta)}{(\tau-\zeta)^{3/2}} \left[1 - 3 \frac{\tau-\xi}{\tau-\zeta}\right] d\zeta. \quad (30)$$

One may obtain the following inequalities from Eq. (30)

when  $R$  is a monotonic increasing function of time:

$$-\left(\frac{D}{\pi}\right)^{1/2} \frac{1}{R^3(\tau)} \int_0^\tau \frac{\theta^0(0,\zeta)}{(\tau-\zeta)^{3/2}} d\zeta \leq T'(0,\tau) \leq -\frac{4}{3} \left(\frac{D}{\pi}\right)^{1/2} \times \int_0^\tau \frac{\theta^0(0,\zeta)}{R^3(\zeta)(\tau-\zeta)^{3/2}} d\zeta. \quad (31)$$

When  $R$  is a monotonic decreasing function, the sense of the inequalities should be reversed. It may be noted in Eq. (31) that

$$-\left(\frac{D}{\pi}\right)^{1/2} \int_0^\tau \frac{\theta^0(0,\zeta)}{(\tau-\zeta)^{3/2}} d\zeta = D \int_0^\tau F(\zeta) d\zeta,$$

where  $F(\zeta)$  is defined by the boundary condition of Eq. (9). When  $R$  is a monotonic increasing function, Eq. (31) thus gives

$$\frac{D}{R^3} \int_0^\tau F(\zeta) d\zeta \leq T'(0,\tau) \leq \frac{4D}{3R_0^3} \int_0^\tau F(\zeta) d\zeta, \quad (32)$$

where  $R_0 = R(0)$ . The upper bound in Eq. (32) represents, of course, a poorer estimate than that given in Eq. (31).

### V. CONCLUSION

The approximations developed here have been applied by the authors to the problem of the growth of a vapor bubble in a superheated liquid. For this specific problem, one can examine in detail the validity of the assumption of the thin "thermal boundary layer" which has been justified previously only in general physical terms. Such an examination of the predicted temperature field shows that the zero-order approximation, as given by Eq. (17) or (17'), is sufficient. Therefore, an explicit expression for the first-order temperature correction at any point in the liquid has not been given here, although it may be found from Eq. (26). The first-order temperature correction at the boundary  $r=R(t)$  is given by a fairly simple expression, and the given bounds upon it provide a convenient estimate of the rapidity of the convergence of the approximation theory.

The approximation procedure presented here is not limited to heat diffusion across a spherical vapor bubble in a liquid. The theory applies without alteration to diffusion across any spherical boundary with radial motion in a fluid, provided the thin "thermal boundary layer" approximation is valid. It is to be emphasized that effects of any translational motion of the spherical boundary have not been considered. For the case of the vapor bubble in a liquid, the solution is therefore applicable only over time intervals so short that no significant translation can take place.