SYNTHESIS OF ROBUST CONTROLLERS AND FILTERS

John C. Doyle

Honeywell, Inc., Systems and Research Center, Minneapolis, Minnesota
and
Department of Mathematics, University of California, Berkeley

ABSTRACT

This paper outlines a general framework for analysis and synthesis of linear control systems and reports on a new solution to a very general L_∞/H_∞ optimal control problem.

I. Introduction

This paper outlines a general framework for analysis and synthesis of linear control systems that unifies and extends many existing methods. These include covariance, singular value [1], and structured singular value ([2], [3]) analysis. The synthesis methods include the Wiener-Hopf-Kalman (WHK) approaches (e.g., Wiener and Kalman filtering, LQG, etc.) for time-invariant systems and the newer L_∞/H_∞ methods (see [4] for a review of recent work on this subject). The main result reported in this paper is a new solution to a very general L_∞/H_∞ optimal control problem. The practical significance of this result is enhanced by the fact that the H_∞-optimal controller can be computed using standard real matrix operations (i.e., solving Lyapunov and Riccati equations, QR and SV decompositions, etc.) on state-space representations.

This paper will present a rather bare outline of these results. The final sections rely heavily on the theory of Ball and Helton [5]. A tutorial introduction to the key issues in this theory is presented in this proceedings [4].

III. Analysis

Various modeling assumptions will be considered and the impact of these assumptions on analysis and synthesis methods will be explored. Referring to Figure 1a, the nominal model is assumed throughout to be a Linear, Time-Invariant Ordinary Differential Equation (LTIODE). The uncertain inputs are assumed to be either filtered white noise or weighted Lp-norm bounded signals. The plant uncertainty is modelled as perturbations (not necessarily small) to the nominal. Performance is measured in terms of either the weighted error covariance or the weighted Lp-norm of the error.

Figure 1b. Analysis Model

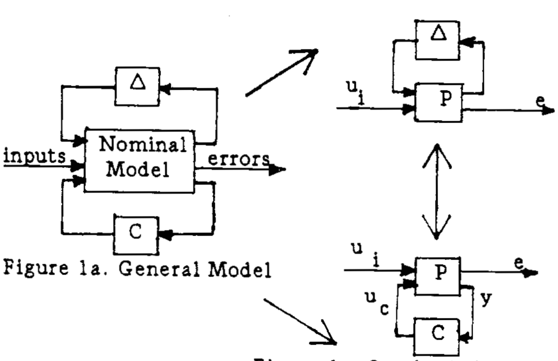

Figure 1a. General Model

Figure 1c. Synthesis Model

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These performance measures are intended to reflect engineering issues such as good command response or small errors in regulation or estimation. Perturbations typically arise in an attempt to model changes and uncertainty in operating conditions and plant characteristics as well as unmodelled dynamics. Uncertain inputs model disturbances, noises and commands. The analysis and synthesis framework used in this paper includes all the standard linear time-invariant filtering and control problems, including the so-called two-degree-of-freedom control problem. This last problem is obtained when commands are modelled in the usual way as uncertain input signals.

Since the focus of this section is on analysis, the controller can be viewed as just another system component. Thus for analysis purposes, Figure 1a may be reduced to Figure 1b. Here P is a 2x2 block transfer function matrix providing connections from external inputs and perturbations to outputs and perturbations. Note than <u>any</u> interconnection of inputs and outputs with components and perturbations may be rearranged into this form. Then the output can be written as

$$e = (P_{22} + P_{21}\Delta(I-P_{11}\Delta)^{-1}P_{12})u. \qquad (3.1)$$

It is assumed that stability is always a performance requirement and thus that P has all its poles in the open left-half plane.

The standard modelling assumptions and the resulting analysis methods are summarized in Table I. The first option is that uncertainty is modelled as white noise and performance is measured in terms of error covariance. It is well-known that the error covariance can be evaluated in terms of the L_2-norm of P_{22}. This model is appealing in that many physical noises and disturbances have existing, accepted models as filtered white noise and that computation of $\|P_{22}\|_2$ is quite easy using Lyapunov equations. Furthermore, synthesis (the so-called Wiener-Hopf-Kalman (WHK) theory) in this context involves linear approximation in a Hilbert space, also computationally appealing. Unfortunately, few physical systems are adequately modelled with additive white noise as the only uncertainty.

Case	Modeling Assumptions: Input	Modeling Assumptions: Perturbation	Performance Requirement	Analysis Test
1	White Noise Unit Covariance	$\Delta=0$	Covariance ≤ 1	$\frac{1}{2\pi}\|P_{22}\|_2 \le 1$
2	$\|u_i\|_2 \le 1$	$\Delta=0$	$\|e\|_2 \le 1$	$\|P_{22}\|_\infty \le 1$
3	-	$\|\Delta\|_\infty < 11$	BIBO Stable	$\|P_{11}\|_\infty \le 1$
4	-	$\Delta=$	BIBO Stable	$\|P_{11}\|_\mu \le 1$
5	$\|u_i\|_2 \le 1$	diag($\Delta 1,\Delta 2, \ldots, \Delta_n$) (structured uncertainty)	$\|e\|_2 \le 1$	$\|P\|_\mu \le 1$

Table 1. Analysis Summary

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Some alternatives to the white noise view of uncertainty are summarized in the remainder of Table I. These options may be thought of as being separate cases of the following general form of a performance/robustness theorem:

Given	Modelling Assumptions	,	
Performance Requirement is satisfied	iff	Analysis Test is satisfied	(3.2)

The first alternative (case 2) involves modelling inputs as unknown-but-bounded (in an L_2-sense) and requiring that the output remain bounded (in L_2) below a specified level for all such inputs. The resulting analysis test involves the L_∞-norm of P_{22}, the same norm which appears in the robust stability test of Case 3. The chief advantage of these assumptions over Case 1 is that both uncertain inputs and plant perturbations are handled with the same $\|\bullet\|_\infty$ test. A less compelling reason is that designs for unknown-but-bounded L_2 inputs and outputs can be given a minimax energy interpretation.

Note that any induced operator norm would provide an analysis test that would handle both uncertain norm-bounded inputs and (induced) norm-bounded perturbations. For example, modeling signals as unknown-but-bounded in magnitude (and using the resulting induced norm for convolution operators on L_∞ for analysis) has obvious advantages in applications where signal magnitude is a more natural notion than energy. On the other hand, there are some important reasons for choosing L_2 signal models, $\|\bullet\|_\infty$ perturbation bounds, and $\|\bullet\|_\infty$ analysis tests over, say, other L_p spaces:

1. The induced convolution-operator norm on L_2 (i.e., $\|\bullet\|_\infty$ on transfer functions) is the only induced norm which yields necessary as well as sufficient robust stability tests.

2. Perturbation models of this type are currently the most easily obtained.

3. An optimal synthesis theory analogous to that of WHK is now available. (i.e., the main result of this paper)

4. Engineers have developed substantial experience with these methods through the use of Bode plots and more recently, their singular value generalizations.

Clearly, these reasons are not entirely independent. The $\|\bullet\|_\infty$ norm on transfer functions is reasonably easily computed, but it does involve a search over one frequency variable.

It should be noted that in practice the use of weights on signals and perturbations is essential, since both vary with direction and frequency. This is true independent of the particular assumptions being made. By absorbing any weightings into the interconnection function P, the weighted case can be reduced to that considered in Figure 1a and Table I. This is one advantage of the framework proposed here over less general ones in that any interconnection of signals, systems and perturbations, including weights, can be rearranged to fit the framework.

While case 2 and 3 provide a single framework in which to analyze performance and robustness (of stability), the $\|\bullet\|_\infty$ norm alone provides no systematic, reliable method for analyzing robust performance. Furthermore, $\|\bullet\|_\infty$ analyzes robustness with respect to purely unstructured uncertainty. A more sophisticated tool that treats robust performance with respect to structured uncertainty involves the structured singular value, μ, and $\|\bullet\|_\mu$ [2], [3]. Although μ is not a norm, we will abuse notation and let $\|P\|_\mu \equiv \mathrm{ess}\sup_\omega \mu(P(j\omega))$.

Cases 4 and 5 of Table I summarize the two basic applications of μ to analysis. Case 4 gives a structured version of Case 3 by characterizing robust stability with respect to block-diagonal perturbations. This is quite general since any interconnection of perturbations can be rearranged to fit the structure of Figure 1b with a block diagonal Δ. Case 5 generalizes cases 2, 3, and 4 by characterizing the performance (in an L_2-bounded sense) for systems with structured uncertainty. This is currently the only available method for systematically analyzing the performance of complex systems with plant perturbations.

We have seen that the standard analysis tools of linear control theory plus a new, more powerful method based on μ can be viewed as special cases of the general framework outlined in this section and summarized in Figure 1b, Table I and (3.2). In this framework analysis of system performance and robustness reduces to computing $\|P\|_\alpha$ for $\alpha=2$, ∞, or μ for some transfer function P. The goal of the remainder of the paper is to develop a similar framework for the synthesis of controllers to meet specifications expressed in terms of these analysis methods.

IV. Synthesis Framework

From the previous section on analysis, we know that evaluating the performance/robustness of the control system in Figure 1a can be reduced to the computation of $\|P(C)\|_\alpha$ for $\alpha=2$, ∞ or μ. Here $P(C) = P_{11} + P_{12}C(I-P_{22}C)^{-1} P_{21}$ as shown in Figure 1c (the P_{ij}'s here are in general different from the P_{ij}'s in Figure 1b). A natural approach to synthesis would be to solve

$$\min_C \|P(C)\|_\alpha \qquad \alpha=2,\ \infty \text{ or } \mu \tag{4.1}$$

The rest of this paper is primarily concerned with a synthesis framework that provides solutions for the $\alpha=2$ and ∞ cases. The $\alpha=\mu$ case will be considered briefly at the end of this paper. Until then, α will be used to denote those cases when either $\alpha=2$ or $\alpha=\infty$ apply equally well.

The first step in the solution is to use the stabilizing controller parameterization [6] to turn $\min\|P(C)\|$ into a problem affine in a stable parameter Q. For simplicity, suppose throughout that P is open-loop stable (the P_{ij} have all their poles in the open left-half plane). Relaxing this assumption complicates the parameterization but does not affect the synthesis methods. If the $\{P_{ij}\}$ are open-loop stable, P(C) is stable iff $Q=C(I-P_{22}C)^{-1}$ is stable. Using this parameterization, the synthesis problem becomes

$$\min_{Q\in RH_\alpha} \|P_{11} + P_{12}QP_{21}\|_\alpha \tag{4.2}$$

where the prefix R denotes real-rational. It greatly simplifies the discussion to drop the requirement that Q be real-rational. It is a fortunate consequence of the theory that the optimal Q is in fact real-rational when the P_{ij} are real-rational.

The two cases $\alpha=2$ and $\alpha=\infty$ can be developed in a parallel fashion. Figure 2 gives a flowchart outlining the steps involved in solving the synthesis problem with labels indicating the technique and section relevant to the step. Note that in each case the general problem is reduced to finding the nearest H_α approximation to a function in L_α. The first step in this reduction involves inner-outer factorization of rational matrices.

V. Inner-Outer and Spectral Factorizations

This section will develop purely state-space methods for performing the factorizations needed in the remaining sections. By using standard algorithms involving only real matrix algebra, these methods should prove computationally reliable. The key idea is to reduce each factorization to solving the standard Algebraic Riccati Equation

$$F^TX + XF - XWX + H=O \tag{5.1}$$

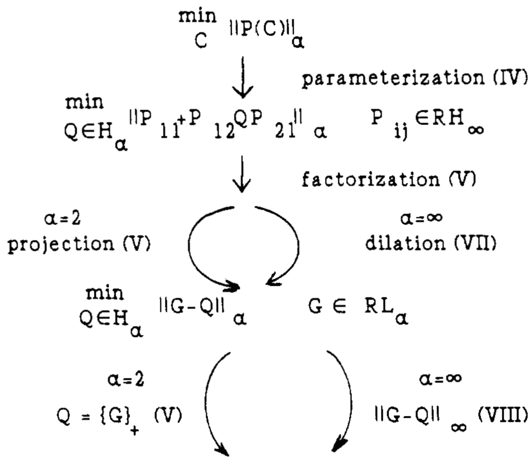

Figure 2. Synthesis Summary

The following theorem gives an algebraic relation between solutions of (5.1) and spectral factorizations (SF) [7], [8], [9].

<u>Theorem</u> (SF) Solving (5.1) for X with $F=A-BR^{-1}S^T$, $W=BR^{-1}B^T$, $H=P-SR^{-1}S^T$, $P=P^T$, $R=R^T>0$, and letting $K=R^{-1}(S^T+B^TX)$ yields the factorization

$$[B^T(-sI-A^T)^{-1}\ \ I]\begin{bmatrix}P & S\\ S^T & R\end{bmatrix}\begin{bmatrix}(sI-A)^{-1}B\\ I\end{bmatrix} = M^* RM$$

where $M = I+K(sI-A)^{-1}B$.

The proof is straightforward algebra and well-known. The different solutions of (5.1) correspond to alternative zero patterns for M. If (A,B) is controllable, there exists a unique solution $X=X^T\geq 0$ so that M has only lhp zeros [9], [10]. This theorem will be required in Section VII to form $(\gamma^2I-G^*G)^{1/2}$ for $\gamma>\|G\|_\infty$. Here $(\ )^{1/2}$ denotes a spectral factor with all poles and zeros in the open left half plane. Note that if G has rhp poles the theorem must be applied twice, the second time to M^{-1} to reflect the rhp poles into the lhp.

Suppose $G(s) = D+C(sI-A)^{-1}B \in RH_\infty^{n\times m}$, $(n\geq m)$ with rank (D) = m, [C,A,B] minimal, and no zeros with zero real part. An inner-outer factorization (IOF) of G is G(s) = Θ(s) M(s) where

1) $M(s)\in RH_\infty^{m\times m}$, $[M(s)]^{-1}\in RH_\infty^{m\times m}$ (outer)

2) $\Theta(s)\in RH_\infty^{n\times m}$ and $\Theta^*\Theta=I$ $(\Theta^*\equiv\Theta^T(-s))$ (inner)

Suppose wolog that D is already factored so that $D^TD=I$. Let $D_\perp$ be such that $[D\ D_\perp]$ is an orthogonal matrix. If n=m, then $D_\perp$ does not exist. Under these conditions, the following theorem holds:

<u>Theorem</u> (IOF) G has an IOF with

$$\Theta(s) = D + (D_\perp D_\perp^TC-DB^TX)(sI-F+BB^TX)^{-1}B$$

$$M(s) = I + (D^TC+B^TX)(sI-A)^{-1}B \text{ where } X=X^T\geq 0 \text{ solves}$$

(5.1) with $F=A-BD^TC$, $G=BB^T$, $H=C^TD_\perp D_\perp^TC$.

Proof: Obtain M(s) directly from SF theorem with

$$\begin{bmatrix}P & S\\ S^T & R\end{bmatrix} = \begin{bmatrix}C^T\\ D^T\end{bmatrix}[C\ D].$$

The Θ(s) is obtained from $\Theta(s) = G(s)[M(s)]^{-1}$ with a little algebra. Note that if G has rhp poles everything applies except that M will also have the same rhp poles.□

The inner-outer factorization is sufficient to solve the H_2 optimization problem, but the H_∞ problem is simplified by the introduction of an additional factorization. Suppose G is as before (except possibly unstable) and n>m. If Θ(s) is n×(n-m) and inner and $\Theta^*G=0$ then Θ will be called a complementary inner factor (CIF) of G. The following theorem and corollary relate particular CIF's with solutions to (5.1). When no confusion should arise, dimension superscripts are omitted..

<u>Theorem</u> (stable CIF) An RH_∞ CIF for G is

$$\Theta = [D_\perp+(DB^T-D_\perp D_\perp^T CX)(sI-F+C^TD_\perp D_\perp^T X)^{-1}C^TD_\perp]$$

where $F = -A^T + C^TDB^T$ and $X=X^T\geq 0$ solves

$$F^TX + XF - XC^TD_\perp D_\perp^TCX + BB^T = 0$$

<u>Corollary</u> (unstable CIF) An $R\bar{H}_\infty$ CIF Θ (all poles in rhp, $\Theta^*\epsilon RH_\infty$) may be obtained by replacing X in the stable CIF with the negative semidefinite solution to the same Riccati equation.

Proof: Write $G(s) = D+C(sI-A)^{-1}B$

$$= [D_\perp D]\left(\begin{bmatrix}0\\ I\end{bmatrix} + \begin{bmatrix}D_\perp^T\\ D^T\end{bmatrix} C(sI-A)^{-1}B\right) = [D_\perp D]\begin{bmatrix}G_1(s)\\ G_2(s)\end{bmatrix}$$

Let $G_3 = [D_\perp D]\begin{bmatrix}I\\ -[G_1G_2^{-1}]^*\end{bmatrix}$. Then $G_3^*G=0$.

The RH_∞ CIF is obtain as the inner factor Θ of G_3 from the IOF Theorem. The RH_∞ CIF is obtained in the same way with s replaced by -s. Note that if $(D_\perp^TC,A)$ is not observable, the Riccati equation may not have a positive semidefinite solution. In this case, simply proceed using a reduced, minimal realization. This will result in a similar formula for Θ but with new state coordinates. □

Combining these results yields a factorization $G(s)=\Theta(s)\begin{bmatrix}M(s)\\ 0\end{bmatrix}$ where $\Theta(s) = [\Theta_1\Theta_2]$ and $\Theta^*\Theta=I$. A similar result may be obtained for cases where n<m simply by taking transposes throughout.

These two factorizations play a central role in both the L_2/H_2 and L_∞/H_∞ solutions because both $\|\bullet\|_2$ and $\|\bullet\|_\infty$ are invariant under unitary (i.e., inner or allpass) transformations. In particular, suppose P_{12} and P_{21} factor as

$$P_{12} = \Theta\begin{bmatrix}M_{12}\\ 0\end{bmatrix} \text{ and } P_{21} = [M_{21}\ 0]\Psi$$

with $\Theta^*\Theta=I$, $\Psi\Psi^*=I$ and the M's and their inverses in RH_∞. Then for both α=2 and ∞

$$\|P_{11}+P_{12}QP_{21}\|_\alpha = \|\Theta^*(P_{11}+P_{12}QP_{21})\Psi^*\|_\alpha$$

$$= \left\|\begin{bmatrix}\Theta_1^*P_{11}\Psi_1^*+M_{12}QM_{21} & \Theta_1^*P_{11}\Psi_2^*\\ \Theta_2^*P_{11}\Psi_1^* & \Theta_2^*P_{11}\Psi_2^*\end{bmatrix}\right\|_\alpha \quad (5.5)$$

Since L_2 is a Hilbert space the optimal Q for α=2 is obtained immediately by orthogonal projection from L_2 onto H_2 as

$$Q_{opt} = -M_{12}^{-1}\{\Theta_1^*P_{11}\Psi_1^*\}_+M_{21}^{-1} \quad (5.6)$$

where $\{\ \}_+$ indicates projection onto H_2 (e.g. by partial fractions expansion).

This is the classical Wiener-Hopf solution. The simplicity of the L_2/H_2 optimal solution is very appealing. The L_∞/H_∞ problem cannot be solved quite so simply because L_∞ is not a Hilbert space and therefore there is no notion of orthogonal projection from L_∞ onto H_∞. Nevertheless, (5.5) can be reduced to a problem of approximating L_∞ function by functions in H_∞. An algorithm for performing this reduction is developed in Section VII and an algorithm for obtaining an optimal $RH_\infty^{m\times n}$ approximation to an $RL_\infty^{m\times n}$ function will be developed in Section VIII. The next section (VI) uses the factorizations of this section and the H_2-optimal Q of (5.6) to provide a simple solution to the standard "LQG" control problem.

Note that the factorization theorems require the D term to be maximal rank. This implies that the synthesis methods in this paper require that P_{12} and P_{21}, though nonsquare, must have maximal rank D terms. This assumption is quite reasonable from an engineering point of view, but does exclude certain singular problems (e.g., no penalty on the control signal or no sensor noise) which may still be of some interest. Most of these problems can be treated by using some simple modifications and will not be studied in detail here.

VI. L_2/H_2 Optimal Controllers and the Standard LQG Problem

In this section, we will briefly digress to look at the special case of the standard LQG problem using the more general L_2/H_2 theory from the last section. It is hoped that this will provide readers well-versed in the LQG problem with a familiar reference point.

Consider the standard problem

$$\dot{x} = Ax + Bu_c + Gd \quad , \quad y = Cx + Nn$$

$$e = \begin{bmatrix} r \\ v \end{bmatrix} = \begin{bmatrix} Hx \\ Ru_c \end{bmatrix}, \; u_i = \begin{bmatrix} d \\ n \end{bmatrix} \tag{6.1}$$

where x is the state, u_c the control, d and n white noise with identity covariance, and y the measured output. The control objective is to design a linear controller L(s) (i.e., $u_c(s)=L(s)y(s)$ that minimizes $E(|e|^2)$, the steady state "error" covariance. All variables are vector quantities of compatible, but otherwise arbitrary finite dimension.

This fits naturally into the synthesis framework of the last section and can be solved using inner-outer factorization and the L_2/H_2 optimal Q of (5.6). The interconnection structure is

$$\begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} r \\ v \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} u_i \\ u_c \end{bmatrix}$$

$$= \begin{bmatrix} H\Phi G & 0 & H\Phi B \\ 0 & 0 & R \\ C\Phi G & N & C\Phi B \end{bmatrix} \begin{bmatrix} d \\ n \\ u_c \end{bmatrix}$$

$$u_c(s) = L(s)y(s) \qquad \Phi = (sI-A)^{-1}$$

The parameterization $Q=L(I-P_{22}L)^{-1}$ leads to the optimization problem

$$\min_{Q\in H_2} \|P_{11} + P_{12}QP_{21}\|_2$$

and the solution is given by (5.6). It simply remains to perform the algebra to obtain a simpler description of Q_{opt} and L_{opt}. The steps are as follows:

1)

Factor $P_{12} = \begin{bmatrix} H\Phi B \\ R \end{bmatrix} = \begin{bmatrix} H\Phi BM_{12}^{-1} \\ RM_{12}^{-1} \end{bmatrix} M_{12} = \Theta_1 M_{12}$

From the IOF theorem (5.3) $M_{12}(s) = (R^TR)^{1/2}\,[I+K\Phi B]$

where $K = (R^TR)^{-1}B^TX$ and $X=X^T \geq 0$ solves

$$A^TX + XA - XB(R^TR)^{-1}B^TX + H^TH = 0$$

This is the standard (state-feedback) controller Riccati equation.

2) $P_{21} = [C\Phi G \;\; N] = M_{21}[M_{21}^{-1}C\Phi G \quad M_{21}^{-1}M] = M_{21}\Psi_1$

where $M_{21} = [I+C\Phi F]\,(NN^T)^{1/2}$, $F = \Sigma C^T(NN^T)^{-1}$,

and $\Sigma = \Sigma^T \geq 0$ solves the standard filter Riccati equation.

$$A\Sigma+\Sigma A^T - \Sigma C^T(NN^T)^{-1}C\Sigma + GG^T = 0.$$

3) From (5.6),

$$Q_{opt} = -M_{12}^{-1}\{\Theta_1^* P_{11}\Psi_1^*\}_+ M_{21}^{-1}$$

$$= -M_{12}^{-1}\{[H\Phi BM_{12}^{-1}]^* H\Phi G\,[M_{21}^{-1}C\Phi G]^*\}_+ M_{21}^{-1}$$

A little algebra yields:

$$Q_{opt} = -[I+K(sI-A)^{-1}B]^{-1}K(sI-A)^{-1}F[I+C(sI-A)^{-1}F]^{-1}$$

4) $L_{opt} = Q(I+C\Phi BQ)^{-1}$ which after a little more algebra reduces to $L_{opt} = -K(sI-A+BK+FC)^{-1}F$.

This is the well-known formula for the optimal LQG controller for the special case considered here.

VII. L_∞/H_∞: Reduction to $\min\|G-Q\|$

In this section we will develop an algorithm for reducing the general L_∞/H_∞ synthesis problem in (4.2) (for $\alpha=\infty$) to the problem of approximating an L_∞ function with one in H_∞. This is similar to the L_2/H_2 case and involves similar factorizations, but L_∞ lacks the Hilbert space structure so the algorithm will be more complicated. The approach taken here is closely related to that of Davis, et al [11], who characterize the contraction dilations of a contraction. The interested reader should compare the algorithm (7.4) in this section with (5.5) (which isn't formed explicitly) and with the central problem treated in [11].

Two simple facts are needed before proceeding. Suppose

$$X\in L_\infty^{m\times n},\; Y\in L_\infty^{k\times n},\; W\in L_\infty^{m\times p}, \text{ and } \|X\| < \gamma.$$

For this section $\|\bullet\|$ will denote $\|\bullet\|_\infty$. Then the following two facts are easily verified.

$$1) \quad \left\| \begin{bmatrix} Y \\ X \end{bmatrix} \right\| \leq \gamma \quad \text{iff} \quad \|Y(\gamma^2 I-X^*X)^{-1/2}\| \leq 1 \tag{7.1}$$

$$2) \quad \|[W \;\; X]\| \leq \gamma \quad \text{iff} \quad \|(\gamma^2 I-XX^*)^{-1/2}W\| \leq 1 \tag{7.2}$$

The main result of this section will use these two facts and the factorizations from Section V to reduce the general L_∞/H_∞ problem to a simple approximation problem.

This reduction will be expressed as a series of equivalent statements starting with the most general L_∞/H_∞ problem in (4.2) and ending with a $\|G-Q\|$ problem, which is solved in Section VIII. Recall that the most general L_∞/H_∞ synthesis problem involves solving

$$\min_{Q_o\in H_\infty^{m\times n}} \|G_o - H_oQ_oK_o\|_\infty \tag{7.3}$$

$$G_o\in RH_\infty^{k\times p},\; H_o\in RH_\infty^{k\times m},\; K_o\in RH_\infty^{n\times p}.$$

or nontriviality, suppose $k \geq m$ and $p \geq n$ and further suppose that both H_o and K_o have maximal rank D terms. Previous results in this area have required k=m and p=n. With these assumptions and $\gamma \in R$, $\gamma > o$ the following holds.

$$\|G_o - H_o Q_o K_o\| \leq \gamma \quad (7.4a)$$

$$\text{iff} \quad \|G_o - H_o Q_o M_1 \Psi_1\| \leq \gamma \quad (7.4b)$$

$$\text{where } K_o = M_1 \Psi_1 = [M_1 \; 0] \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$$

$$\text{iff} \quad \|G_o \Psi^* - H_o [Q_1 \; 0]\| \leq \gamma \text{ where } Q_1 = Q_o M_1 \quad (7.4c)$$

$$\text{iff} \quad \|[G_o \Psi_1^* - H_o Q_1 \qquad G_o \Psi_2^*]\| \leq \gamma \quad (7.4d)$$

$$\text{iff} \quad \|M_2 (G_o \Psi_1^* - H_o Q_1)\| \leq 1 \quad (7.4e)$$

$$\text{by (7.2) with } M_2 = (\gamma^2 I - G_o \Psi_2^* \Psi_2 G_o^*)^{-1/2}$$

$$\text{iff} \quad \|G_1 - H_1 Q_1\| \leq 1 \quad (7.4f)$$

$$\text{where } G_1 = M_2 G_o \Psi_1^* \text{ and } H_1 = M_2 H_o$$

$$\text{iff} \quad \|G_1 - \Theta_1 M_3 Q_1\| \leq 1 \quad (7.4g)$$

$$\text{where } H_1 = \Theta_1 M_3 = [\Theta_1 \Theta_2] \begin{bmatrix} M_3 \\ 0 \end{bmatrix}$$

$$\text{iff} \quad \|\Theta^* G_1 - \begin{bmatrix} Q_2 \\ 0 \end{bmatrix}\| \leq 1 \text{ where } Q_2 = M_3 Q_1 \quad (7.4h)$$

$$\text{iff} \quad \left\| \begin{matrix} \Theta_1^* G_1 - Q_2 \\ \Theta_2^* G_2 \end{matrix} \right\| \leq 1 \quad (7.4i)$$

$$\text{iff} \quad \|(\Theta_1^* G_1 - Q_2) M_4\| \leq 1 \quad (7.4j)$$

$$\text{by (7.1) with } M_4 = (I - G_1^* \Theta_2 \Theta_2^* G_1)^{-1/2}$$

$$\text{iff} \quad \|G_2 - Q_3\| \leq 1 \quad (7.4k)$$

$$\text{where } G_2 = \Theta_1^* G_1 M_4 \text{ and } Q_3 = Q_2 M_4$$

If there exists a Q_3 such that $\|G_2 - Q_3\| \leq 1$, then letting $Q_o = M_3^{-1} Q_3 M_4^{-1} M_1^{-1}$ yields a Q_o such that $\|G_o - H_o Q_o K_o\| \leq \gamma$. The computation of the optimal Q_3 will be treated in the next section.

The equations in (7.4) can be turned into an algorithm for solving (7.3) by guessing a γ and computing (7.4b)-(7.4k) successively. If γ is too small, either (7.4e) or (7.4j) will fail or $\min\|G_2 - Q_3\| > 1$. If γ is too large then $\min\|G_2 - Q_3\| < 1$. Note that, just as in the case of analysis using $\|\bullet\|_\infty$, the synthesis problem involves a one-parameter search (over γ) to find the optimal norm. A solution arbitrarily close to the optimal can be found in a finite number of iterations of (7.4). By using the factorizations from Section V each iteration involves fairly routine computations involving real matrix operations on the state-space representations.

Although not required for this paper, it is possible to relax the conditions on G_o, H_o and K_o to RL_∞ without altering (7.4). An extra factorization must be performed in steps c and g to insure the M_1 and M_3 are in H_∞. Note also that the steps in (7.4) apply equally well when considering $Q_o \in H_{k,\infty}^{m \times n}$, in which case $Q_3 \in H_{k,\infty}^{m \times n}$ ($H_{k,\infty}$ allows k rhp poles).

VIII. Solution of $\min \|G-Q\|_\infty$

This section will outline a method to solve

$$\min_{Q \in H_\infty^{m \times n}} \|G-Q\|_\infty, \; G \in R\bar{H}_\infty^{m \times n}. \quad (8.1)$$

In the last section, the more general problem (7.3) was reduced to (8.1) with $G \in RL_\infty^{m \times n}$. By partial fractions expansion, the $H_\infty^{m \times n}$ part of G may be absorbed into Q, leaving (8.1). The solution to (8.1) will exploit Silverman and Bettayeb's [12] treatment of Hankel operators using Moore's balanced realization to provide a computational scheme based on the theory of Ball and Helton ([4], [5]).

To use these results, we must transform the right half-plane onto the unit disc, by say, taking $s=(\lambda+z)/(\lambda-z)$ $s=(\lambda+z)/(\lambda>o$ and $\lambda \notin$ spectrum (A). Suppose, then, that G(z) is given as

$$G(z) = C(zI-A)^{-1}B = \sum_{k=1}^{\infty} G_k z^{-k}, \; G_k = CA^{k-1}B \quad (8.2)$$

with the associated infinite Hankel matrix

$$H = \begin{bmatrix} G_1 & G_2 & G_3 & \cdots \\ G_2 & G_3 & \cdots & \\ G_3 & & & \\ & & & \end{bmatrix} \begin{bmatrix} C \\ CA \\ CA^2 \\ \end{bmatrix} [B \; AB \; A^2B \; \dots \;] = OR. \quad (8.3)$$

The controllability and observability grammians $W=RR^T$ and $M=O^TO$ can be computed as the unique solutions of the Lyapunov equations

$$\begin{aligned} AWA^T - W &= -BB^T \\ A^TMA - M &= -C^TC. \end{aligned} \quad (8.4)$$

Assume that (A,B,C) is a balanced realization so that $M=W=\Sigma=\text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$. Then the following hold

1) $H^T H x = \lambda x \rightarrow \Sigma^2 (Gx) = \lambda (Gx)$

2) $H^T H (R^T e_i) = \sigma_i^2 (R^T e_i)$ (8.5)

3) $HH^T (Oe_i) = \sigma_i^2 (Oe_i)$

This yields the singular value decomposition of H as

$$H = U_H \Sigma V_H^T = (O\Sigma^{-1/2}) \Sigma (\Sigma^{-1/2} G) \quad (8.6)$$

A corresponding SVD of the Hankel operator in terms of its symbol G is

$$\{G(z)U(z)\}_- = V(z)\Sigma \quad (8.7)$$

where $V=C(zI-A)^{-1}\Sigma^{-1/2}$ and $U=B^T(I-zA^T)^{-1}\Sigma^{-1/2}$. It is well known that the minimum in (8.1) is equal to σ_1. Suppose that G has been normalized by a constant so that $\sigma_1=1$.

Let <F,G> for $F \in L_\infty^{m \times j}$, $G \in L_\infty^{m \times k}$ denote the unique matrix in $C^{n \times k}$ that solves <Fx,Gy>=x <F,G>y for every $x \in C^j$ and $y \in C^k$. Then <V,V>=I and <U,U>=I and it is easily verified that

$$\langle U, zU \rangle = \Sigma^{-1/2} A \Sigma^{1/2} \text{ and } \langle V, zV \rangle = \Sigma^{1/2} \; A \Sigma^{-1/2} \quad (8.8)$$

The Ball-Helton theory requires the introduction of several spaces. Let $U=UH_2^r$ and $V=VH_2^r$ and let the ambient Krein space be

$$X = \begin{bmatrix} V \\ 0 \end{bmatrix} + \begin{bmatrix} H_2^m \\ H_2^n \end{bmatrix} \quad (8.9)$$

with the usual indefinite inner product denoted by [,]. Define the subspace $M \subset X$ as

$$M \triangleq \begin{bmatrix} G \\ I \end{bmatrix} H_2^n + \begin{bmatrix} I \\ 0 \end{bmatrix} H_2^m$$

$$= \begin{bmatrix} V\Sigma \\ U \end{bmatrix} C^r + \begin{bmatrix} 0 \\ I \end{bmatrix} U_\perp + \begin{bmatrix} I \\ 0 \end{bmatrix} H_2^n. \qquad (8.10)$$

As in ([4], [5]), we want to find a maximal negative, shift-invariant subspace of M and the operator whose graph is this subspace. This will solve (8.1). The key step is to obtain an explicit representation for the wandering subspace $L = M \ominus (SM)'$, where S denotes shift (multiplication by z). As a first step, note that

$$M' = \begin{bmatrix} V \\ U\Sigma \end{bmatrix} C^r \qquad (8.11)$$

$$SM = \begin{bmatrix} zV\Sigma \\ zU \end{bmatrix} C^r + \begin{bmatrix} 0 \\ zI \end{bmatrix} U_\perp + \begin{bmatrix} zI \\ 0 \end{bmatrix} H_2^m$$

$$(SM)' = \begin{bmatrix} zY & 0 & C_\perp \\ zU\Sigma & I & 0 \end{bmatrix} \begin{bmatrix} C^r \\ C^n \\ C^p \end{bmatrix} = \Theta(z)\, C^{r+n+p}$$

where $C_\perp$ spans the orthogonal complement of the range of C. Usually, C is onto and p=0.

Let [F,G] for $F \in L_\infty^{(m+n)xj}$ and $G \in L_\infty^{(m+n)xk}$ denote the matrix in C^{jxk} that solves $[Fx, Gy] = x^* [F,G] y$ for every $x \in C^j$, $y \in C^k$. Then a basis for $L = M \cap (SM)'$ can be found from (8.11) by letting $P \epsilon C^{(r+n+p)xq}$ span the kernel of

$$\Gamma = \left[\left[\begin{smallmatrix} V \\ U\Sigma \end{smallmatrix}\right], \Theta(z) \right]$$

$$= [\ \Sigma^{1/2} A \Sigma^{-1/2} - \Sigma^{1/2} A \Sigma^{3/2} \qquad -\Sigma^{1/2} B \qquad 0] \qquad (8.12)$$

Then $L = \Theta(z) P C^q$. Equation (8.12) is obtained by taking the Krein inner product of the representations for M' and (SM)' from (8.11) using (8.8).

The next step is to extract a maximal negative subspace of L. To this end let

$$M_1 = [\Theta, \Theta] = \begin{bmatrix} I-\Sigma^2 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & C_\perp^T C_\perp \end{bmatrix}, \qquad M_2 = P^T M_1 P,$$

and find Y such that $M_2 = Y^T \Lambda Y$ with

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{a signature matrix.}$$

Partition $Y = [Y_0 Y_- Y_+]$ to match Λ and let $L_-(z) = \Theta(z) P [Y_0 Y_-]$. This gives a Krein-orthogonal basis for a maximal negative subspace $L_- C^{q-}$. There is, in general, an infinite family of maximal negative subspaces but this is a natural choice as it includes no part of the positive subspace from the Krein-orthogonal decomposition of L.

By shift variance, $L_- H_2^{q-}$ is a maximal negative shift-invariant subspace of M. Partition

$L_- = \begin{matrix} L_1(z) \\ L_2(z) \end{matrix}$ $\quad L_1 \in RL_\infty^{mxq-}$ and $L_2 \in RL_\infty^{nxq-}$. By the unique correspondence between maximal negative sift-invariant subspaces and graphs of contractions, there is a unique solution $Q_{opt} \in RH_\infty^{mxn}$ to

$$(G - Q_{opt}) L_2 = L_1. \qquad (8.13)$$

The simplest way to solve (8.13) is to transform the unit disk back into the rhp. Generically, this will result in full rank D terms for the transfer functions in (8.13). Then simple formulas will yield a nonminimal realization for Q_{opt} which can be reduced to obtain the optimal controller. Note that all the operations performed to obtain the optimal Q can be done using standard computations on real matrices.

The algorithm for finding the optimal $Q \in H_\infty^{mxn}$ is easily generalized to handle $Q \in H_{k,\infty}^{mxn}$ by simply dividing through by σ_{k+1} initially instead of σ_1. The rest of the solution goes through unchanged. This is a further consequence of the Ball-Helton theory [5]. The $Q \in H_{k,\infty}^{mxn}$ problem arises in the optimal Hankel-norm model reduction problem, and the computational scheme outlined above may prove useful there.

IX. Synthesis for Plants With Structured Uncertainty

We have seen that L_∞/H_∞ optimal control theory can be generalized to handle as rich a class of problems as the L_2/H_2 theory. The advantage of the L_∞/H_∞ framework is it is potentially more relevant to practical engineering problems since it handles both uncertain inputs and uncertain plants. The price is increased conceptual and computational complexity.

While the results reported in this paper are encouraging, they are just one more step towards a truly practical, systematic synthesis method for linear systems. The next important step would be to synthesize optimal controllers for performance/robustness expressed in terms of μ by solving

$$\min_{Q \in RH_\infty} \|P_{11} + P_{12} Q P_{21}\|_\mu \qquad (9.1)$$

An appealing approach to this problem is to "solve"

$$\min_{Q,D} \|D(P_{11} + P_{12} Q P_{21}) D^{-1}\|_\infty \qquad (9.2)$$

by iteratively minimizing over Q and D. Here $D \epsilon RH_\infty$ is taken to be of the form appropriate for the uncertainty structure of the problem ([2], [3]). Each minimization over D or Q with the other fixed is a convex problem and a global solution can be found. Unfortunately, (9.2) is not convex jointly in D and Q. It is a reasonable conjecture, however, that the global solution to (9.2) is approached with such a scheme. Lacking a proof of this conjecture, this scheme remains ad hoc.

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