

ON INNER-OUTER AND SPECTRAL FACTORIZATIONS

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Abstract

This paper outlines methods for computing the key factorizations necessary to solve general H_2 and H_∞ linear optimal control problems.

Notation

$L_a \triangleq$ { Lebesgue space }

$H_a \triangleq$ { Hardy space }

$R \triangleq$ { Proper, real-rational }

$RP^{m \times m} \triangleq$ { $p \times m$ matrices in R } (similarly for H and L)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \triangleq D + C(sI - A)^{-1}B$$

rcf \triangleq right coprime factorization over RH_∞

Throughout this paper, α will be used whenever either $\alpha=2$ or $\alpha=\infty$ would apply equally. The term *unit* in RH_∞ refers to any $M \in RH_\infty$ such that $M^{-1} \in RH_\infty$. When R is used as a prefix, it denotes real-rational.

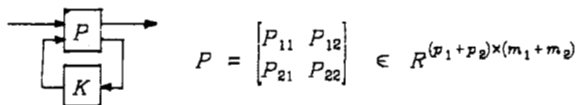
1. Introduction :

The importance of inner-outer (IOF), spectral and coprime factorizations in obtaining solutions to certain H_2 and H_∞ optimal control problems has been known for some time. The solution to the general H_a ($\alpha=2, \infty$) optimal control problems [1],[2] uses these factorizations and, in addition, the "complementary inner factor" (CIF), to reduce the general problem to that of approximating an L_a rational matrix by one in H_a .

This paper focuses on the factorizations used in [2] and, in particular, on explicit formulas and methods for computation. We show that all the factorizations needed in the H_2 and H_∞ optimal synthesis problem can be obtained using standard real matrix operations on state-space representations. The Algebraic Riccati Equation (ARE) plays a central role in computing the desired factorizations. Because of space limitations, the "proofs" of the results in this paper are extremely sketchy.

2. Background :

The general H_a optimal control problem is shown in the following figure.



The objective is to find a stabilizing $K \in R^{m_2 \times p_2}$ which solves $\min_K \|F_1(P;K)\|_\alpha$ where $F_1(P;K) \triangleq P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$. For nontriviality, assume that $p_1 > m_2$ and $m_1 > p_2$.

The first step is to find $K_0 = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \in R^{(m_2+p_2) \times (p_2+m_2)}$ such that $F_1(P;F_1(K_0;Q)) = F_1(T;Q) = T_{11} + NQ\tilde{N} \in RH_a^{p_1 \times m_1}$ is

stable and affine for any $Q \in RH_\infty^{m_2 \times p_2}$. This is the Youla parametrization of all stabilizing controllers and is obtained by finding coprime factorizations of P over the ring of stable rationals and solving a double Bezout identity to obtain the coefficients of K_0 .

We are interested in a particular K_0 which results in both N and \tilde{N} being inner. That is, $N^*N=I$ and $\tilde{N}\tilde{N}^*=I$. This requires a coprime factorization with inner numerator. In addition, we require N_\perp and \tilde{N}_\perp inner so that $\begin{bmatrix} N & N_\perp \end{bmatrix}$ and $\begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix}$ are square and inner. N_\perp and \tilde{N}_\perp are called complementary inner factors (CIF). With these we have that

$$\begin{aligned} \left\| \begin{bmatrix} T_{11} + NQ\tilde{N} \end{bmatrix} \right\|_\alpha &= \left\| \begin{bmatrix} N & N_\perp \end{bmatrix}^* [T_{11} + NQ\tilde{N}] \begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix} \right\|_\alpha \\ &= \left\| \begin{bmatrix} N & N_\perp \end{bmatrix}^* [T_{11}] \begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right\|_\alpha \end{aligned} \tag{2.1}$$

since both the $\alpha = 2$ and ∞ norms are unitary invariant.

The H_2 case immediately reduces to a best approximation problem with $Q_{opt} = P_{H_2}(N^* [T_{11}] \tilde{N}^*)$, where P_{H_2} denotes projection onto H_2 . The H_∞ case is somewhat more complicated and requires an additional spectral factorization. To see how this arises, consider the special case when $T_{11}\tilde{N}_\perp^* = 0$ and (2.1)

reduces to $\left\| \begin{bmatrix} R+Q \\ G \end{bmatrix} \right\|_\infty$ with $R = N^* [T_{11}] \tilde{N}^*$ and $G = N_\perp^* [T_{11}] \tilde{N}^*$.

It is easily verified that for any $\gamma > \|G\|_\infty$

$$\left\| \begin{bmatrix} R+Q \\ G \end{bmatrix} \right\|_\infty \leq \gamma \text{ iff } \left\| (\gamma^2 I - G^*G)^{-\frac{1}{2}} (R+Q) \right\|_\infty \leq 1$$

where $(H)^{\frac{1}{2}}$ denotes the unit spectral factor of the para-Hermitian matrix H . Thus, the H_∞ problem also reduces to a best approximation problem since the $(\gamma^2 I - G^*G)^{-\frac{1}{2}}$ is a unit and can be absorbed into Q . The general case similarly involves both inner-outer and spectral factorizations [2]. The remainder of the paper outlines methods for computing these factorizations.

3. Algebraic Riccati Equation :

Consider the Algebraic Riccati Equation,

$$F^T X + X F - X W X + Q = 0 \tag{ARE}$$

where $F, W, Q \in \mathbb{R}^{n \times n}$, $W = W^T \geq 0$ and $Q = Q^T$

with the associated Hamiltonian matrix

$$A_H = \begin{bmatrix} F & -W \\ -Q & -F^T \end{bmatrix}$$

Our main interest is to find the unique real symmetric stabilizing solution such that the matrix $(F - WX)$ is asymptotically stable. For simplicity we will use "solution" of the ARE to mean a real symmetric one. The ARE considered here is more general than the ARE which arises in linear quadratic optimal control and Kalman-Bucy filtering theory in that there is no assumption on the definiteness of the matrix Q .

The following theorem gives the necessary and sufficient conditions for the existence of a unique stabilizing solution of (ARE). Without loss of generality, we will assume that $W = GG^T$. This is a slight generalization of a theorem of Kucera [3].

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Theorem 3-1 :

The stabilizability of (F, G) and $\text{Re}[\lambda_i(A_H)] \neq 0$ ($\forall i = 1, 2, \dots, 2n$) are necessary as well as sufficient for the existence of a unique stabilizing solution of (ARE).

Remarks :

- (1) The unique stabilizing solution of Theorem 3-1 will be denoted by $\text{Ric}(A_H)$.
- (2) If $Q \geq 0$, then the stabilizing solution $X \geq 0$.

Theorem 3-2 :

If $Q = H^T H \geq 0$ in (ARE) and X is its solution, then $\text{Ker}(X) \subset \text{Ker}(H)$.

4. Inner-Outer Factorization (IOF) :

Assume $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in R^{p \times m}$ ($p \geq m$) and the realization is minimal. We use the notation " D_\perp " for the orthogonal complement of D and $R^{\#}$ ($R \geq 0$) for any square matrix V s.t. $V^T V = R$.

Theorem 4-1 :

G has a rcf $G = NM^{-1}$ with N inner if and only if G has no (transmission) zeros on the $j\omega$ -axis, including ∞ .

A particular realization for the factorization is

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} A-BF & BR^{-\#} \\ -F & R^{-\#} \\ C-DF & DR^{-\#} \end{bmatrix} \in RH_{\infty}^{(m+p) \times m}$$

where $R = D^T D > 0$, $F = R^{-1}(B^T X - D^T C)$,

$$\text{and } X = \text{Ric} \begin{bmatrix} A-BR^{-1}D^T C & -BR^{-1}B^T \\ -C^T D_\perp D_\perp^T C & -(A-BR^{-1}D^T C)^T \end{bmatrix} \geq 0.$$

[Proof] :

(only if) :

Suppose $G = NM^{-1}$ is a rcf and $N^* N = I$. Then $G^* G = (M^{-1})^* M^{-1} > 0$ on the $j\omega$ -axis since M is stable. Thus G has no $j\omega$ -axis zeros.

(if) :

The if part can be proven by directly verifying that the above realization is a rcf of G and that $N^* N = I$. The conditions on G insure, by Theorem 3-1, the existence of X .

Theorem 4-2 :

If $p > m$ in Theorem 4-1, then there exists a CIF $N_\perp \in RH_{\infty}^{p \times (p-m)}$ such that the matrix $[N \ N_\perp]$ is square and inner. A particular realization is

$$N_\perp = \begin{bmatrix} A-BF & -X^T C^T D_\perp \\ C-DF & D_\perp \end{bmatrix}$$

where X^\dagger is the pseudo-inverse of the stabilizing Riccati solution X from Theorem 4-1.

[Proof] :

The proof consists of verifying directly that $[N \ N_\perp]$ is inner using the above realization for N_\perp and the realization for N from Theorem 4-1. A key step requires that $\text{Ker}(X) \subset \text{Ker}(D_\perp^T C)$, which follows from Theorem 3-2.

Theorem 4-3 :

There exists a rcf $G = NM^{-1}$ such that $M \in RH_{\infty}^{m \times m}$ is inner if and only if G has no poles on the $j\omega$ -axis. A particular realization is

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} A-BF & B \\ -F & I \\ C-DF & D \end{bmatrix} \in RH_{\infty}^{(m+p) \times m}$$

$$\text{where } F = B^T X \text{ and } X = \text{Ric} \begin{bmatrix} A & -BB^T \\ 0 & -A^T \end{bmatrix} \geq 0.$$

[Proof] :

Similar to Theorem 4-1.

Remarks :

- (1) The minimality condition in Theorem 4-3 can be weakened to (A, B) stabilizable and $\text{Re}[\lambda_i(A)] \neq 0$ and the theorem still holds.
- (2) If $G \in RH_{\infty}^{m \times m}$ in Theorem 4-1, then M is a unit in RH_{∞} . In this case, $G = N(M^{-1})$ is called "inner-outer factorization" (IOF).
- (3) Dual results for all factorizations can be obtained when $p \leq m$. In these factorizations, output injection using the dual Riccati solution replaces state feedback to obtain corresponding left factorizations.

5. Spectral Factorization :

Recall from Section 2 that the H_{∞} solution requires the computation of the spectral factor which is a unit in RH_{∞} . The following theorem characterizes this spectral factor. The same assumptions on G apply here as did in Section 4 except that $G \in RL_{\infty}^{p \times m}$.

Theorem 5-1 :

If $\gamma > \|G(s)\|_{\infty}$, there exists a unit $M \in RH_{\infty}^{m \times m}$ such that $M^* M = \gamma^2 I - G^* G$. A particular realization is given by

$$M = \begin{bmatrix} A-K_F K_C & B-K_F \\ R^{\#} K_C & R^{\#} \end{bmatrix}$$

where $R = \gamma^2 I - D^T D > 0$, $K_C = R^{-1}(B^T X - D^T C)$, $K_F = Y K_C^T R$,

$$X = \text{Ric} \begin{bmatrix} A+BR^{-1}D^T C & -BR^{-1}B^T \\ C^T(I+DR^{-1}D^T)C & -(A+BR^{-1}D^T C)^T \end{bmatrix}$$

and

$$Y = \text{Ric} \begin{bmatrix} A^T & -K_C^T R K_C \\ 0 & -A \end{bmatrix}$$

[Proof] :

The proof consists of verifying from the above realizations that M and M^{-1} are stable and that $M^* M = \gamma^2 I - G^* G$.

Remark :

If G is stable, then $Y = 0$ and $K_F = 0$.

6. Implementation :

All the factorizations in the theorems have explicit state-space representations. Thus, the implementation of algorithms can be done easily using real matrix operations. Two key steps in the algorithms are:

- (1) Solution of ARE : the ARE can be solved using the Schur method which reduces the Hamiltonian matrix to real Schur form by orthogonal similarity transformation and then uses the invariant subspace associated with n stable eigenvalues to find the stabilizing solution. See [4] for a complete treatment.
- (2) $(D^T D)^{\#}$ and D_\perp : using the QR factorization, we find $D = \begin{bmatrix} Q_1 & Q_2 \\ 0 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$ where $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ is orthogonal. Then $R = (D^T D)^{\#}$ and $D_\perp = Q_2$.

Using the well-known software packages EISPACK, LINPACK and some of their extensions, experimental versions of these algorithms have been implemented successfully. Experience to date indicates that these algorithms are very reliable.

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