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THE GENERAL DISTANCE PROBLEM IN H_∞ SYNTHESIS

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Abstract

The general distance problem which arises in the general H_∞ optimal control problem is considered. The existence of an optimal solution is proved and the expression of the optimal norm γ_o is obtained from a somewhat abstract operator point of view. An iterative scheme, called γ-iteration, is introduced which reduces the general distance problem to a standard best approximation problem. Bounds for γ_o are also derived. The γ-iteration is viewed as a problem of finding the zero crossing of a function. This function is shown to be continuous, monotonically decreasing, convex and be bounded by some very simple functions. These properties make it possible to obtain very rapid convergence of the iterative process. The issue of model-reduction in H_∞-synthesis will also be addressed.

Notation

$L_2\ (L_\infty) \triangleq$ { Lebesque space }

$H_2\ (H_\infty) \triangleq$ { Hardy space }

$R \triangleq$ { Proper, real-rational }

$R^{p\times m} \triangleq$ {$p\times m$ matrices in R} (similarly for H and L)

$$\|A\|_\infty \triangleq \begin{cases} L_\infty\text{-norm if } A \in L_\infty \\ \bar{\sigma}(A) \text{ if } A \text{ is a constant matrix} \end{cases}$$

$\mathbf{P}_{H_2}\ (\mathbf{P}_{H_2^\perp}) \triangleq$ the orthogonal projection from L_2 onto H_2 $(H_2^\perp)$.

$\mathbf{H}_G \triangleq$ the *Hankel* operator (matrix) generated by $G \in L_\infty$.

$\mathbf{T}_G \triangleq$ the *Toeplitz* operator (matrix) generated by $G \in L_\infty$.

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq D+C(sI-A)^{-1}B$$

The term *unit* in RH_∞ refers to any $M \in RH_\infty$ such that $M^{-1} \in RH_\infty$. When R is used as a prefix, it denotes real-rational.

1. Introduction

In this paper, we shall consider the "General Distance Problem" (GDP) [6,11,13,14] which can be stated as follows :

Given $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in L_\infty$, find the optimal $Q \in H_\infty$ such that

$$\left\| \begin{bmatrix} R_{11}-Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty$$

is minimized. Note that the minimum norm, denoted as γ_o, is the distance

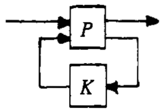
$$dist\,(R, \begin{bmatrix} H_\infty & 0 \\ 0 & 0 \end{bmatrix})$$

from R to the set of (matrix) functions of the form

$$\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix},\ Q \in H_\infty.$$

This class of problems will be called the "4-block problem" in this paper to distinguish from the special case where $\begin{bmatrix} R_{21} & R_{22} \end{bmatrix}$ or $\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}$ is identically zero. The latter will be referred as the "2-block problem". Note that if both $\begin{bmatrix} R_{21} & R_{22} \end{bmatrix}$ and $\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}$ are zero, this is known as the "best (or Hankel) approximation" problem [1,2,3,17]. The general distance problem can also be regarded as a matrix dilation problem with the constraint of the causality (i.e., $Q \in H_\infty$ required).

The **GDP** arises in the solution of the general H_∞ optimal control problem [5,6,8,10,11,13,14,15,16,23,24]. The basic framework for the general H_∞ optimal control problem is shown in the following figure



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in R^{(p_1+p_2)\times(m_1+m_2)}$$

The objective is to find a stabilizing $K \in R^{m_2\times p_2}$ which solves $\min_K \|\mathbf{F}_l(P;K)\|_\infty$ where $\mathbf{F}_l(P;K) \triangleq P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}$. For nontriviality, assume that $p_1 > m_2$ and $m_1 > p_2$.

The first step is to find $K_o = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \in R^{(m_2+p_2)\times(p_2+m_2)}$ such that $\mathbf{F}_l(P;\mathbf{F}_l(K_o;Q)) = \mathbf{F}_l(T;Q) = T_{11}-NQ\tilde{N} \in RH_\infty^{p_1\times m_1}$ is stable and affine for any $Q \in RH_\infty^{m_2\times p_2}$. This is the Youla parametrization of all stabilizing controllers [8,11,21,22] and is obtained by finding coprime factorizations of P over the ring of stable rationals and solving a double Bezout identity to obtain the coefficients of K_o.

We are interested in a particular K_o which results in N and $\tilde{N}$ being inner and co-inner respectively. That is, $N^*N=I$ and $\tilde{N}\tilde{N}^*=I$. This requires a coprime factorization with inner numerator [5,11]. In addition, we require $N_\perp$ and $\tilde{N}_\perp$ inner so that $\begin{bmatrix} N & N_\perp \end{bmatrix}$ and $\begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix}$ are *square* and inner. $N_\perp$ and $\tilde{N}_\perp$ are called complementary inner factors (CIF). With these we have that

$$\left\|T_{11}-NQ\tilde{N}\right\|_\infty = \left\| \begin{bmatrix} N & N_\perp \end{bmatrix}^* [T_{11}-NQ\tilde{N}] \begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix}^* \right\|_\infty$$

$$= \left\| \begin{bmatrix} N & N_\perp \end{bmatrix}^* [T_{11}] \begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix}^* - \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} R_{11}-Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \quad (1.1)$$

since ∞ norm is unitary invariant.

The solution to this problem requires an additional spectral factorization. To see how this arises, consider the special case when $T_{11}\tilde{N}_\perp^* = 0$ and (1.1) reduces to

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$$\left\| \begin{bmatrix} R_1 - Q \\ R_2 \end{bmatrix} \right\|_\infty \tag{1.2}$$

with $R_1 = N^* [T_{11}] \tilde{N}^*$ and $R_2 = N_\perp^* [T_{11}] \tilde{N}^*$.

It is easily verified that for any $\gamma > \|R_2\|_\infty$

$$\left\| \begin{bmatrix} R_1 - Q \\ R_2 \end{bmatrix} \right\|_\infty \le \gamma \quad \Leftrightarrow \quad \left\| (\gamma^2 I - R_2^* R_2)^{-1/2} (R_1 - Q) \right\|_\infty \le 1 \tag{1.3}$$

where $(H)^{1/2}$ denotes the unit spectral factor of the para-Hermitian matrix H. Thus, the H_∞ problem also reduces to a best approximation problem since the $(\gamma^2 I - R_2^* R_2)^{-1/2}$ is a unit and can be absorbed into Q. The general 4-block case similarly involves both inner-outer and spectral factorizations. Algorithms for obtaining these factorizations using standard real matrix operations on state-space representations were presented in [5,11].

This paper will focus on the solution to (1.1) and (1.2) and algorithms for obtaining the optimal Q. In Section 2, a somewhat abstract operator point of view is adopted and existence of an optimal Q and expressions for the optimal norm are obtained. These expressions are in terms of an operator norm or an equivalent generalized eigenvalue problem. Unfortunately, the operator and generalized eigenvalue problem are infinite rank and these results do not yield computable formulas for either the optimal norm or Q. In Section 3, an alternative approach, called γ-iteration, is introduced. It essentially involves guessing a γ and then reducing the problem of finding all Q that give norm less than γ to a standard finite rank Hankel norm approximation problem as in (1.3) above. The guess for γ is iterated on until it converges to the optimal norm, γ_o, and the optimal Q is thus obtained. Section 3 gives some fairly tight bounds for the optimal γ_o in terms of easily computable quantities, which immediately allows for reasonable estimates of γ_o.

In order for the γ-iteration scheme to converge rapidly, it is necessary to exploit some properties of the process. In Section 4, the γ-iteration is viewed as the problem of finding the zero crossing of a function. This function is shown to be continuous, monotonically decreasing, convex, and bounded by some very simple functions. These properties make it possible to obtain very rapid convergence of the γ-iteration. In Section 5, we address the issue of model reduction in general distance problems. The error bound will be given.

Most of proofs in this paper are omitted. The details can be found in [6].

2. Optimal Solutions of General Distance Problems

In this section, we will discuss the existence of the optimal solution of the general distance problem. The proof [6,13,14] is essentially a generalization of that for best approximation problems in [11] where the Parrott/Davis-Kahan-Weinberger theorem [7] on norm-preserving dilation is used. In the following, the 2-block **GDP** will be studied first. The results are then generalized to the 4-block **GDP**. It is more convenient in this section to consider H_∞ of the disc instead of the half-plane. This does not loss of generality since there is a well-known isometric isomorphism between the half-plane and the unit disc (see [11]).

Consider the following 2-block **GDP**:

$$\gamma_o = \min_{Q \in H_\infty} \left\| R - \begin{bmatrix} Q \\ 0 \end{bmatrix} \right\|_\infty \quad \text{where } R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \in L_\infty. \tag{2gdp}$$

Define the operator Γ_R from H_2 to $H_2^\perp \oplus L_2$ as follows :

$$\Gamma_R : H_2 \to H_2^\perp \oplus L_2$$

$$\Gamma_R f = \begin{bmatrix} \mathbf{P}_{H_2^\perp} R_1 f \\ R_2 f \end{bmatrix} \quad , f \in H_2 \tag{2.1}$$

Theorem 2.1

$$\gamma_o = \min_{Q \in H_\infty} \left\| \begin{bmatrix} R_1 - Q \\ R_2 \end{bmatrix} \right\|_\infty = \| \Gamma_R \| \tag{2.2}$$

γ_o can also be expressed in terms of the following eigenvalue problem.

Corollary 2.2

$$\| \Gamma_R \| = \lambda_{\max}^{1/2} \left[\mathbf{H}_{R_1}^* \mathbf{H}_{R_1} + \mathbf{T}_{R_2^* R_2} \right] \tag{2.3}$$

where $\mathbf{H}_{R_1}$ is the *Hankel* matrix generated by R_1 and $\mathbf{T}_{R_2^* R_2}$ is the *Toeplitz* matrix generated by $R_2^* R_2$.

Remark

This corollary can be proved using a lemma by Sarason [20]. Recognizing that

$$\| \Gamma_R \| = \left\| \begin{bmatrix} \mathbf{H}_{R_1} \\ \mathbf{T}_{R_2} \\ \mathbf{H}_{R_2} \end{bmatrix} \right\|_\infty = \lambda_{\max}^{1/2} \left[\mathbf{H}_{R_1}^* \mathbf{H}_{R_1} + \mathbf{T}_{R_2}^* \mathbf{T}_{R_2} + \mathbf{H}_{R_2}^* \mathbf{H}_{R_2} \right]$$

The result is immediate from Sarason's lemma, since

$$\mathbf{T}_{R_2^* R_2} - \mathbf{T}_{R_2}^* \mathbf{T}_{R_2} = \mathbf{T}_{R_2^* R_2} - \mathbf{T}_{R_2^*} \mathbf{T}_{R_2} = \mathbf{H}_{(R_2^*)^*}^* \mathbf{H}_{R_2} = \mathbf{H}_{R_2}^* \mathbf{H}_{R_2}$$

Although Corollary 2.2 gives an explicit formula for the smallest achievable norm, unfortunately, it is an infinite-dimensional eigenvalue problem and is difficult to solve. A Hermitian *Toeplitz* operator has no point spectrum (i.e., no eigenvalues). This is known as Hartman-Winter theorem [9]. Therefore, in (2.3), $\mathbf{T}_{R_2^* R_2}$ has infinite rank. This is quite different from the best-approximation problem [17]. In the real-rational case, the corresponding *Hankel* matrix has only finite rank which is equal to the McMillan degree of the given transfer matrix. Another difficulty is the following: although the proof [6,12,13] using dilation theory is conceptually elegant, the construction of optimal Q using norm-preserving dilation is not a trivial problem. There does not currently exist a computational attractive procedure to obtain these coefficients currently. Nevertheless, Theorem 2.1 shows that the optimal solution for the 2-block **GDP** exists. Furthermore, the *Hankel* ⊕ *Toeplitz* structure appearing in Corollary 2.2 is of particular interest and provides a lot of insights for the problem.

To avoid these difficulties which arise in the direct approach, an iterative scheme, called γ-iteration, will be proposed in the next section. First, we consider the generalization of Theorem 2.1 to the 4-block problem.

For the 4-block **GDP**, i.e.,

$$\gamma_o = \min_{Q \in H_\infty} \left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \tag{4gdp}$$

let's define the operator Γ_R as follows:

$$\Gamma_R : H_2 \oplus L_2 \to H_2^\perp \oplus L_2$$

$$\Gamma_R \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{H_2^\perp} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}.$$

Theorem 2.3 ([6,13,14])

$$\gamma_o = \min_{Q \in H_\infty} \left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty = \| \Gamma_R \| .$$

3. γ - Iteration and Bounds

In this section, we propose an iterative scheme, called γ-iteration, to solve the general distance problem. The idea is that by guessing a value for the minimal norm, the distance problem can be simplified to an equivalent best approximation problem which can be solved by existing algorithms. This guess can be iterated to obtain convergence to the optimal norm and optimal Q.

The γ-iteration will converge more rapidly if good initial guesses are obtained for the optimal norm. Some upper and lower bounds are derived in this section which provide accurate bounds on the minimal norm. The general iterative procedure is also described. The 2-block and 4-block cases are considered separately. The proofs are omitted and can be found in [6]. The following theorem lies at the heart of the γ-iteration scheme.

Theorem 3.1 ([10,11])

Assume $Q \in H_\infty$, then

$$\left\|\begin{bmatrix} R_1 - Q \\ R_2 \end{bmatrix}\right\|_\infty \le \gamma \qquad (\gamma > \gamma_o) \tag{3.1}$$

if and only if

$$\| (R_1 - Q) M^{-1} \|_\infty \le 1 \tag{3.2}$$

where M is the spectral factor of the para-Hermitian matrix $(\gamma^2 I - R_2^* R_2)$.

The theorem says that if $\gamma > \gamma_o$, there exists a $\bar{Q} \in H_\infty$ such that $\|R_1 M^{-1} - \bar{Q}\|_\infty \le 1$ and hence, $Q = \bar{Q}M$ satisfies (3.1). This implies that $\|\mathbf{H}_{R_1 M^{-1}}\| \le 1$. Therefore, a solution to (3.2) can be obtained by considering the following best approximation problem

$$\bar{\gamma}_o = \min_{\bar{Q} \in H_\infty} \| R_1 M^{-1} - \bar{Q} \|_\infty. \tag{3.3}$$

In case that the function is real-rational, the algorithms in [3,17] can then be applied to solve the optimal $Q \in RH_\infty$ corresponding to the given γ. A practical state-space solution to the **GDP** can be found in [6].

It is well-known that $(\bar{\gamma}_o)^2$ is the largest eigenvalue of the following standard eigenvalue problem

$$(\mathbf{H}_{R_1 M^{-1}})^* (\mathbf{H}_{R_1 M^{-1}}) u = \lambda u \tag{3.4}$$

Since $M^{-1} \in H_\infty$, $\mathbf{H}_{R_1 M^{-1}}$ is equal to $\mathbf{H}_{R_1} \mathbf{T}_{M^{-1}}$. Hence,

$$\begin{aligned}
(3.4) \quad &\Leftrightarrow \quad \mathbf{T}_{M^{-1}}^* \mathbf{H}_{R_1}^* \mathbf{H}_{R_1} \mathbf{T}_{M^{-1}} u = \lambda u \\
&\Leftrightarrow \quad \mathbf{H}_{R_1}^* \mathbf{H}_{R_1} v = \lambda (\mathbf{T}_{M^{-1}}^*)^{-1} (\mathbf{T}_{M^{-1}})^{-1} v \\
&\Leftrightarrow \quad \mathbf{H}_{R_1}^* \mathbf{H}_{R_1} v = \lambda \mathbf{T}_M^* \mathbf{T}_M v \\
&\Leftrightarrow \quad \mathbf{H}_{R_1}^* \mathbf{H}_{R_1} v = \lambda \mathbf{T}_{M^*} \mathbf{T}_M v \\
&\Leftrightarrow \quad \mathbf{H}_{R_1}^* \mathbf{H}_{R_1} v = \lambda \mathbf{T}_{M^* M} v \\
&\Leftrightarrow \quad \mathbf{H}_{R_1}^* \mathbf{H}_{R_1} v = \lambda (\mathbf{T}_{\gamma^2 I - R_2^* R_2}) v \\
&\Leftrightarrow \quad \mathbf{H}_{R_1}^* \mathbf{H}_{R_1} v = \lambda (\gamma^2 I - \mathbf{T}_{R_2^* R_2}) v \qquad (3.5)
\end{aligned}$$

Eq.(3.5) is a generalized eigenvalue problem and the dependency of (generalized) eigenvalues on γ is clear. This formulation of the problem will be very useful later.

Since the approach proposed is an iterative one, it will be helpful if the upper and lower bounds can be provided in advance. Some results are summarized in the following theorem.

Theorem 3.2

Assume γ_o is the minimal achievable norm as in (2gdp),

$$\gamma_u = \left\| \begin{bmatrix} \|\mathbf{H}_{R_1}\| \\ \|R_2\|_\infty \end{bmatrix} \right\|_2$$

$$\gamma_{l_1} = \max \left\{ \|\mathbf{H}_R\| , \ \|R_2\|_\infty \right\}$$

$$\gamma_{l_2} = \max \left\{ \|\mathbf{H}_{R_1}\| , \ \|R_2\|_\infty \right\}$$

Then

$$\frac{1}{\sqrt{2}} \gamma_o \le \gamma_{l_2} \le \gamma_{l_1} \le \gamma_o \le \gamma_u \le \sqrt{2}\, \gamma_o .$$

The following two theorems are the generalization of Theorem 3.1 and 3.2 for the 4-block GDP.

Theorem 3.3

$$\left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \le \gamma \qquad (\gamma > \gamma_o)$$

if and only if

$$\left\| (I - LL^*)^{-\frac{1}{2}} \left\{ \mathbf{F}_l(\tfrac{1}{\gamma}R, \tfrac{1}{\gamma}R_{22}^*) - \frac{1}{\gamma} Q \right\} (I - \bar{L}^* \bar{L})^{-\frac{1}{2}} \right\|_\infty \le 1 \tag{3.6}$$

where

$$\begin{aligned}
(I - LL^*)^{\frac{1}{2}} &= \text{spectral factor of } (I - LL^*) \\
(I - \bar{L}^* \bar{L})^{\frac{1}{2}} &= \text{spectral factor of } (I - \bar{L}^* \bar{L}) \\
L &= R_{12} S^{-1} \\
\bar{L} &= \bar{S}^{-1} R_{21} \\
S &= (\gamma^2 I - R_{22}^* R_{22})^{\frac{1}{2}} \\
\bar{S} &= (\gamma^2 I - R_{22} R_{22}^*)^{\frac{1}{2}}
\end{aligned}$$

and

$$\mathbf{F}_l(\tfrac{1}{\gamma}R, \tfrac{1}{\gamma}R_{22}^*) = \frac{1}{\gamma} \left\{ R_{11} + R_{12} (\gamma^2 I - R_{22}^* R_{22})^{-1} R_{22}^* R_{21} \right\}$$

Remarks

(i) S and $\bar{S}$ need not to be spectral factors. S and $\bar{S}$ can be any square transfer matrices such that $S^* S = (\gamma^2 I - R_{22}^* R_{22})$ and $\bar{S}\bar{S}^* = (\gamma^2 I - R_{22} R_{22}^*)$.

(ii) Chang and Pearson have derived a similar formula indepedently [4]. However, the fractional transformation $\mathbf{F}_l(\frac{1}{\gamma}R, \frac{1}{\gamma}R_{22}^*)$ in (3.6) was not recognized.

The following theorem provides the upper and lower bounds for the γ-iteration of the 4-block GDP.

Theorem 3.4

Let γ_o be the minimal achievable norm in (4gdp),

$$\gamma_{u_1} = \min \left\{ \left\| \begin{bmatrix} \|\mathbf{H}_{R_{11}}\| \\ \|R_{12}\|_\infty \\ \|[R_{21} \ R_{22}]\|_\infty \end{bmatrix} \right\|_2 , \ \left\| \begin{bmatrix} \|\mathbf{H}_{R_{11}}\| \\ \|R_{21}\|_\infty \\ \left\| \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix} \right\|_\infty \end{bmatrix} \right\|_2 \right\}$$

$$\gamma_{u_2} = \left\| \begin{bmatrix} \|\mathbf{H}_{R_{11}}\| & \|R_{12}\|_\infty \\ \|R_{21}\|_\infty & \|R_{22}\|_\infty \end{bmatrix} \right\|_F$$

$$\gamma_{l_1} = \max \left\{ \|\mathbf{H}_R\| , \ \|[R_{21} \ R_{22}]\|_\infty , \ \left\| \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix} \right\|_\infty \right\}$$

$$\gamma_{l_2} = \max \left\{ \left\| \mathbf{H}_{\begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix}} \right\| , \ \|\mathbf{H}_{[R_{11} \ R_{12}]}\| , \ \|[R_{21} \ R_{22}]\|_\infty , \ \left\| \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix} \right\|_\infty \right\}$$

and

$$\gamma_{l_3} = \max \left\{ \|\mathbf{H}_{R_{11}}\| , \|R_{21}\|_\infty , \|R_{12}\|_\infty , \|R_{22}\|_\infty \right\}$$

Then

$$\frac{1}{2} \gamma_o \le \gamma_{l_3} \le \gamma_{l_2} \le \gamma_{l_1} \le \gamma_o \le \gamma_{u_1} \le \gamma_{u_2} \le 2\, \gamma_o$$

Remark

Note that $\|R\|_\infty = \left\| \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty$ is also an upper bound.

We end the discussion of this section with a general description of the γ-iteration procedure for the 2-block problem. The 4-block case can be stated similarly.

(i) Compute the lower bound $||R_2||_\infty$.

(ii) Choose γ such that $||R_2||_\infty \le \gamma$.

(iii) Find the spectral factor $M = (\gamma^2 I - R_2{}^* R_2)^{1/2}$.

(iv) Let $\bar{\gamma} = ||\mathbf{H}_{R_1 M^{-1}}||$,

  (a) if $\bar{\gamma}>1$, go to (ii) and choose a larger γ.

  (b) if $\bar{\gamma}<1$ and $\gamma>||R_2||_\infty$, go to (ii) and choose a smaller γ.

  (c) if $\bar{\gamma}=1$ and $\gamma \ge ||R_2||_\infty$, go to (v).

  (d) if $\bar{\gamma} \le 1$ and $\gamma = ||R_2||_\infty$, go to (v).

(v) The value of γ is the minimal achievable norm. Find the best approximation of $R_1 M^{-1}$, denoted by $\tilde{Q}_o$.

(vi) The optimal solution $Q_{opt} = \tilde{Q}_{opt} M$.

This algorithm is not complete without some method for selecting the next guess for γ in step (iv). The guaranteed convergence rate for the algorithm will depend on this method and what can be proven about the relationship between γ and $\bar{\gamma}$. This relationship is the focus of the next section.

4. Properties of γ-iteration

In this section, we will illustrate some interesting properties of γ-iteration in the 2-block **GDP**. Based on these properties and some easily obtainable bounds, we will discuss briefly the fundamental principle of the iterative algorithm.

Recall that for a given $\gamma > \gamma_o$, the problem can be solved in terms of an equivalent best (Hankel) approximation problem. It is also known from Section 3 (Eq.(3.5)) that the Hankel norm of this equivalent problem equals the square root of the maximum eigenvalue of the following generalized eigenvalue problem:

$$\mathbf{H}_{R_1}{}^*\mathbf{H}_{R_1}\, v \;=\; \lambda(\gamma^2 I - \mathbf{T}_{R_2{}^* R_2})\, v \qquad \text{(GEP)}$$

The eigenvalues of (GEP) are functions of γ and are nonnegative for all $\gamma > ||T_{R_2{}^*R_2}||^{1/2} = ||R_2||_\infty$. We shall prove that $\lambda_{\max}(\gamma)$ (and its square root) is continuous, strictly monotonically decreasing and convex in γ where, for a given $\gamma > ||R_2||_\infty$, $\lambda_{\max}(\gamma)$ is defined as the maximum eigenvalue of (GEP). The final result of this section is Theorem 4.2, which bounds $\lambda_{\max}(\gamma)$ by simple functions. This can be used along with the the other properties to quickly converge to the optimal norm.

A key observation of (GEP) is that it can be regarded as a "perturbed" generalized eigenvalue problem. Therefore, the perturbation theory of generalized eigenvalue problems for a special case will be considered first. The results can then be used to prove the properties mentioned above.

Consider the following generalized eigenvalue problem,

$$Av(t) \;=\; \lambda(t)B(t)v(t)\,, \qquad t \in (-\varepsilon,\varepsilon)\ ,\ \varepsilon > 0 \qquad \text{(GEP1)}$$

where A is positive semi-definite and independent of t, and $B(t)$ is bounded, positive-definite and analytic in the neighborhood of t=0. Since A and $B(t)$ are Hermitian and $B(t)$>0, it is well-known [19] that by appropriate ordering of the eigenvalues $\{\lambda_i\}$ and selection of eigenvectors $\{v_i\}$, it is possible to pair eigenvalues and eigenvectors $\{\lambda_i(t),v_i(t)\}$ such that

$$Av_i(t) = \lambda_i(t)B(t)v_i(t)$$

for all t, i and $\{\lambda_i(t)\},\{v_i(t)\}$ are analytic for all $t \in (-\varepsilon,\varepsilon)$. At values of t where (GEP1) has simple eigenvalues, this is trivial. At degenerate points, it requires the selection of $\lambda_i(t)$, $v_i(t)$ such that the analyticity is retained through (isolated) point where eigenvalues coalesce.

Define $\lambda_{\max}(\gamma) =$ the maximum eigenvalue of (GEP) at a given γ and $\sigma_{\max}(\gamma) = [\lambda_{\max}(\gamma)]^{1/2}$ for $\gamma > ||R_2||_\infty$. The following two theorems are the main results of this section. It gives some useful properties of γ iteration. The proofs are quite lengthy and will not be presented here.

Theorem 4.1

(i) $\lambda_{\max}(\gamma)$ is continuous, monotonically decreasing, and convex in γ.

(ii) $\sigma_{\max}(\gamma)$ is continuous, monotonically decreasing, and convex in γ.

Note that the generalized eigenvalue formulation used here is similar to that in Helton's broadband matching problem [18], however the motivation here is completely different.

Although the function $\sigma_{\max}$ is unknown, the properties shown in Theorem 4.1 have provided us some useful information about $\sigma_{\max}$ that can be used to obtain fast convergence of the γ-iteration. A detailed study of convergence rates is beyond the scope of this paper, but we will indicate how the properties of $\sigma_{\max}$ can be used to find a next guess for γ. One additional property of $\sigma_{\max}$ is useful in this regard and will be presented in the next theorem.

Define

$$\sigma_u(\bar{\gamma},\gamma) = \frac{c}{(\gamma^2-\beta^2)^{1/2}}$$

where $c = \sigma_{\max}(\bar{\gamma})(\bar{\gamma}^2-\beta^2)^{1/2}$ for some $\bar{\gamma} > \beta$ ($=||R_2||_\infty$).

Theorem 4.2

(i) $\sigma_{\max}(\gamma) \;<\; \sigma_u(\bar{\gamma},\gamma)$ if $\gamma < \bar{\gamma}$.

(ii) $\sigma_{\max}(\gamma) \;=\; \sigma_u(\bar{\gamma},\gamma)$ if $\gamma = \bar{\gamma}$.

(iii) $\sigma_{\max}(\gamma) \;>\; \sigma_u(\bar{\gamma},\gamma)$ if $\gamma > \bar{\gamma}$.

The importance of Theorem 4.2 can be seen from Figure 1. Suppose that at one step in the γ-iteration, we have evaluated $\sigma_{\max}$ at γ_l and γ_u from previous iterations, and want to make a new guess for γ. Without loss of generality, assume that $\beta < \gamma_l < \gamma_u$ such that $\sigma_{\max}(\gamma_l) > 1$ and $\sigma_{\max}(\gamma_u) < 1$. From Theorem 4.2, we know immediately that $\gamma_l < \gamma_o < \gamma_u$. Since $\sigma_{\max}$ is a convex function in γ, $\sigma_{\max}$ must lie below the line segment (denoted by $F_1(\gamma)$) connecting the points $(\gamma_l,\sigma_{\max}(\gamma_l))$ and $(\gamma_u,\sigma_{\max}(\gamma_u))$. In addition, by Theorem 4.2, $\sigma_{\max}$ will lie above the function $\sigma_u(\gamma_l,\gamma) = \dfrac{\sigma_{\max}(\gamma_l)(\gamma_l^2-\beta^2)^{1/2}}{(\gamma^2-\beta^2)^{1/2}}$ when $\gamma > \gamma_l$.

Suppose that γ_u' and γ_l' are the points where $F_1(\gamma_u') = 1$ and $\sigma_u(\gamma_l,\gamma_l') = 1$. We can conclude immediately that $\gamma_l' \le \gamma_o \le \gamma_u'$. The next guess for γ is narrowed considerably over what would be known on the basis of continuity, convexity, and monotonicity alone. Thus it is clearly possible to obtain a scheme for picking the next guess for γ that will provide rapid convergence to the optimal. Further consideration of convergence rates is beyond the scope of this paper.

Remark

Thea γ-iteration can be viewed as the problem of finding the zero crossing of the function $(\sigma_{\max}(\gamma)-1)$.

5. Model Reduction in H_∞ Synthesis

The importance of model reduction in control system design has long been recognized. For practical implementation, it is desired that the order of the controller can be reduced in a way such that the controlled system still satisfies the performance requirements. Typically, there are two ways to obtain a lower order controller: reducing the complexity of the plant model and using model reduction in the design process [12,17]. This section considers the latter issue.

High-order optimal controllers are usually derived when using H_∞ optimization. Therefore, model reduction is inevitable from a practical point of view. The following analysis shows how the model reduction can be performed in the **GDP** with simple L_∞-norm bounds on the resulting loss of performance.

Assume that Q_{opt} is the optimal solution of the **GDP**:

$$\gamma_o \;=\; \min_{Q \in RH_\infty} \left|\left| R - \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right|\right|_\infty$$

Then for model reduction, one has the following two results.

(i) Model reduction on R:

Suppose $\tilde{R}$ is a reduced-order model of R, and $\tilde{Q}_{opt}$ is the optimal solution of

$$\min_{\tilde{Q} \in RH_\infty} \left|\left| \tilde{R} - \begin{bmatrix} \tilde{Q} & 0 \\ 0 & 0 \end{bmatrix} \right|\right|_\infty$$

Define

$$\tilde{\gamma}_o = \left\| R - \begin{bmatrix} \tilde{Q}_{opt} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty .$$

The question is how much error $(\tilde{\gamma}_o - \gamma_o)$ is incurred if the reduced order model $\tilde{R}$ is used in the GDP. This is found as follows:

$$\begin{aligned}
\gamma_o \le \left\| R - \begin{bmatrix} \tilde{Q}_{opt} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty &= \left\| \tilde{R} - \begin{bmatrix} \tilde{Q}_{opt} & 0 \\ 0 & 0 \end{bmatrix} + (R-\tilde{R}) \right\|_\infty \\
&\le \left\| \tilde{R} - \begin{bmatrix} \tilde{Q}_{opt} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty + \|R-\tilde{R}\|_\infty \\
&\le \left\| \tilde{R} - \begin{bmatrix} Q_{opt} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty + \|R-\tilde{R}\|_\infty \\
&= \left\| R - \begin{bmatrix} Q_{opt} & 0 \\ 0 & 0 \end{bmatrix} + (\tilde{R}-R) \right\|_\infty + \|R-\tilde{R}\|_\infty \\
&\le \left\| R - \begin{bmatrix} Q_{opt} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty + 2\|R-\tilde{R}\|_\infty \\
&= \gamma_o + 2\|R-R\|_\infty
\end{aligned}$$

$$\therefore \qquad \tilde{\gamma}_o - \gamma_o \le 2\|R-\tilde{R}\|_\infty \tag{5.1}$$

This inequality shows that the error is no more than $2\|R-\tilde{R}\|_\infty$.

(ii) Model reduction on Q_{opt}:

Suppose that Q_{app} is a reduced order model of the optimal solution Q_{opt}. Then

$$\begin{aligned}
\gamma_o &\le \left\| R - \begin{bmatrix} Q_{app} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty \\
&= \left\| R - \begin{bmatrix} Q_{opt} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Q_{opt}-Q_{app} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty \\
&\le \left\| R - \begin{bmatrix} Q_{opt} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty + \|Q_{opt}-Q_{app}\|_\infty \\
&\le \gamma_o + \|Q_{opt}-Q_{app}\|_\infty
\end{aligned} \tag{5.2}$$

Therefore, model reduction on Q_{opt} will introduce an error of no more than $\|Q_{opt}-Q_{app}\|_\infty$.

Suppose that model reduction in the H_∞ synthesis is done by the two steps:

(a) find the reduced-order model $\tilde{R}$ and the solution, $\tilde{Q}_{opt}$ of the corresponding GDP,

(b) find the reduced-order model, $\tilde{Q}_{app}$, of $\tilde{Q}_{opt}$.

Then,

$$\left\| R - \begin{bmatrix} \tilde{Q}_{app} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty \le 2\|R-\tilde{R}\|_\infty + \|\tilde{Q}_{opt}-\tilde{Q}_{app}\|_\infty \tag{5.3}$$

This error bound can be derived easily by combining the results of (i) and (ii) above.

The above result is very encouraging since if the error bound in 5.3 is guaranteed to be small in model reduction, it will not affect the performance too much. Using either the method of truncation of the balanced realization [12,17] or the method of Hankel-norm approximation [17], the reduced-order model can be found using reliable algorithms. Furthermore, both methods give the error bounds in terms of the L_∞-norm which are computable from the the second order modes of the given system. A more detailed treatment on this subject can be found elsewhere [12,17].

Experience to date has shown that in many practical problems, both the order of $\tilde{R}$ and $\tilde{Q}_{opt}$ can be reduced significantly without incurring too much error. Hence, a (suboptimal) controller with reasonable number of states seems always possible. The discussion in this section addressed just one aspect of the model-reduction issue. More research is needed in this area.

6. Example

In this section, a simple example with a single parameter is constructed to illustrate various properties in the γ-iteration. An exact optimal solution will be derived.

Consider the following 2-block problem:

$$\gamma_o = \min_{Q \in RH_\infty} \left\| \begin{bmatrix} \frac{1}{s-1} - Q \\ \frac{1}{s-a} \end{bmatrix} \right\|_\infty \qquad (a > 0)\,. \tag{6.1}$$

Let Q_o be the optimal solution which achieves the minimum norm. Using the formula in Section 3,

$$\left\| \begin{bmatrix} \frac{1}{s-1} - Q \\ \frac{1}{s-a} \end{bmatrix} \right\|_\infty \le \gamma \iff \|(\frac{1}{s-1} - Q)M^{-1}\|_\infty \le 1\,. \tag{6.2}$$

where $\quad M^{-1} = (\gamma^2 - \frac{1}{-s-a}\,\frac{1}{s-a})^{-1/2} = \frac{s+a}{\gamma s + \sqrt{\gamma^2 a^2 - 1}}$

Let $\tilde{G} = (\frac{1}{s-1}M^{-1})_{unstable} = \dfrac{\frac{1+a}{\gamma + \sqrt{\gamma^2 a^2 - 1}}}{s-1}$ and consider the following best approximation problem :

$$\min_{\tilde{Q} \in RH_\infty} \|\tilde{G} - \tilde{Q}\|_\infty$$

It is not difficult to solve this problem. The minimum norm is

$$\min_{\tilde{Q} \in RH_\infty} \|\tilde{G} - \tilde{Q}\|_\infty = \frac{1}{2}(\frac{1+a}{\gamma + \sqrt{\gamma^2 a^2 - 1}}) \tag{6.3}$$

and the optimal solution is

$$\tilde{Q}_o = -\frac{1}{2}(\frac{1+a}{\gamma + \sqrt{\gamma^2 a^2 - 1}}) \tag{6.4}$$

Of course, in order to have (6.2) make sense, the right-hand side of (6.3) must be less than or equal to 1.

We summarize the solution to (6.1) as follows:

<u>(A) Minimal norm γ_o</u>

(i) $\quad a < 1 \;:\; \gamma_o = \frac{1}{a}$,

(ii) $\quad a = 1 \;:\; \gamma_o = 1$,

(iii) $\quad a > 1 \;:\; \gamma_o = \frac{1}{2(a-1)}\left[-1+\sqrt{a^2+\frac{4(a-1)}{a+1}}\right]$.

<u>(B) Optimal Q</u>

(i) $\quad a < 1 \;:\; Q_o = -\frac{(1+a)s+2a}{2(s+a)}$,

(ii) $\quad a = 1 \;:\; Q_o = -1$,

(iii) $\quad a > 1 \;:\; Q_o = -\dfrac{\gamma_o s + \gamma_o + \frac{a-1}{2}}{s+a}$,

where γ_o is the same as in (A)-(iii).

Remarks

The significance of this example can be stated as follows :

(a) If $0 < a \le 1$, then $\gamma_o = \|R_2\|_\infty$. This tells us that the lower bound in Theorem 3.2 is tight.

(b) From Eq. (6.3), the Hankel norm of $\tilde{G}$ is a convex function of γ. This can be verified by computing its second derivative with respect to γ and show it is always greater than zero.

7. Conclusions

The results in this paper, combined with those of [5,6,10,11,17], greatly reduce the practical computational burden associated with the optimal H_∞ synthesis. Simple bounds and properties of the γ-iteration insure rapid convergence and the iterative process itself can often be avoided. Model reduction can be used to reduce the order of the transfer function matrices at the intermediate stages of the computation. Furthermore, all the computations discussed in this paper can be accomplished using standard real matrix operations on state-space realizations.

The primary contribution of this paper, beyond that in [11], is the set of results in Sections 3-5 on the γ-iteration scheme and model reduction. These results clearly make it possible to obtain algorithms that will converge very rapidly to the optimal solution. This rapid convergence is important since each step in the γ-iteration is potentially computationally intensive. The model reduction results should significantly reduce the computational burden by allowing for order reduction of transfer function matrices in intermediate stages. Although the resulting controller will be suboptimal, the results in Section 5 show that the degree of suboptimality is bounded by the L_∞ norm of the error in the original approximation.

An additional opportunity for obtaining suboptimal controllers with substantial computational savings comes as a direct consequence of the bounds in Theorems 3.2 and 3.4. Note that if Q is chosen to best approximate R_{11} in (4gdp), the bound γ_{u_2} guarantees that the resulting controller will yield a solution with norm no worse than $2\gamma_o$. This suboptimal solution is often satisfactory and avoids the γ-iteration entirely. The corresponding bound in the 2-block GDP is $\sqrt{2}\,\gamma_o$.

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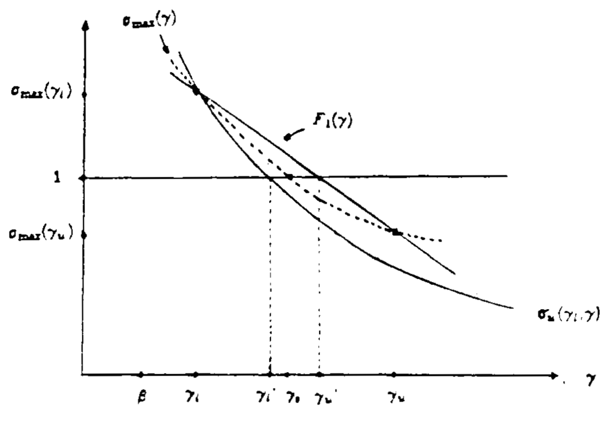

Figure 1. γ-Iteration

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