Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control

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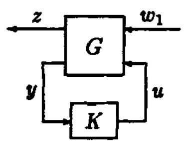
Abstract

Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm analysis and synthesis problems are considered in this paper. It is shown that the mixed norm analysis combined with structured uncertainty can be used to give bounds on robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance. It is also shown that the mixed norm controller shares a separation property similar to those of pure $\mathcal{H}_2$ or $\mathcal{H}_\infty$ controllers. The obvious advantage for a mixed norm is that it gives a natural trade-off between $\mathcal{H}_2$ performance and $\mathcal{H}_\infty$ performance, and provides a potential framework for extending the μ-synthesis framework to address robust $\mathcal{H}_2$ performance. A simple example is used to motivate the possible advantages such a framework might have over a pure $\mathcal{H}_\infty$ theory.

1 Introduction

Standard $\mathcal{H}_2$ or $\mathcal{H}_\infty$ control begins with the following setup

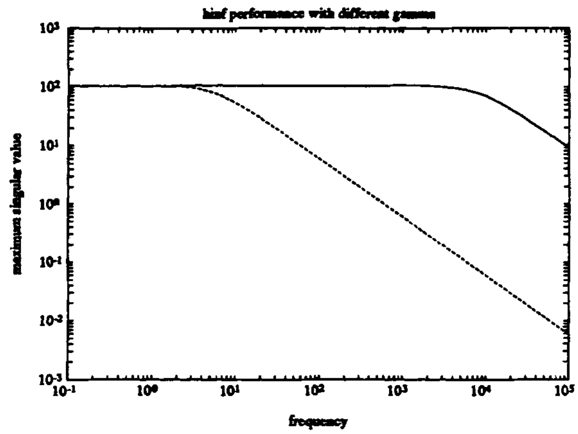


The objective is to design a controller K such that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the closed loop transfer function from w_1 to z is minimized. The $\mathcal{H}_\infty$ design methodology has become very popular in recent years. The primary significance of $\mathcal{H}_\infty$ theory is that it can be combined with certain analysis methods, for example, structured singular value or μ analysis, to give a robust controller synthesis technique for systems with structured uncertainty. This particular combination of μ analysis and $\mathcal{H}_\infty$ synthesis is referred to as μ synthesis. Even though this is a somewhat ad hoc approach, there is no comparable method yet for robust $\mathcal{H}_2$ synthesis. It should be emphasized that $\mathcal{H}_\infty$ by itself does not have much advantage over the more conventional $\mathcal{H}_2$ theory.

Since $\mathcal{H}_\infty$ theory involves an optimality criteria a question that arises when applying it is should we design an *optimal controller* (or as close to optimal as possible if not optimal)? The answer is generally no. There are several reasons for this

- computing the exact optimal value (to within limits imposed by the computer hardware) is numerically difficult and expensive.
- optimal controllers tend to have some undesirable characteristics when compared with slightly suboptimal controllers.

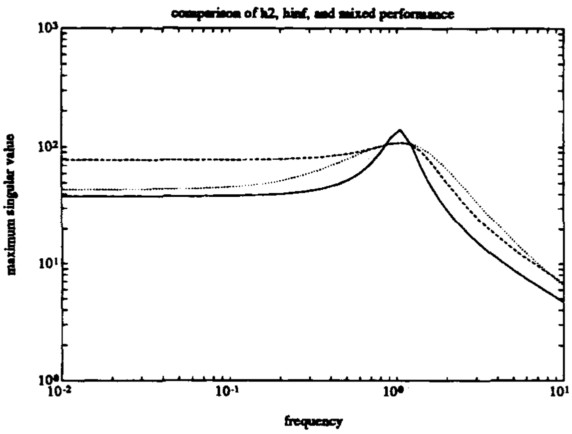
This latter point may be illustrated by a simple example. Shown below are the maximum singular value plots of 2 closed loop systems with different suboptimal $\mathcal{H}_\infty$ controllers. The solid line corresponds to a suboptimal $\mathcal{H}_\infty$ controller whose norm is less than .0001% above the optimal, while the dashed line has a performance level about 0.2% above the optimal. The optimal controller would have a horizontal line indistinguishable at low frequencies from the two shown. It would be very hard to argue that the "more optimal" was better in any meaningful way. The more optimal controllers reduce the peak level by a negligible amount while degrading the performance substantially over large frequencies.



Although this is not an example with any specific engineering motivation, we believe it is likely that we would normally prefer the 0.2% suboptimal controller. It clearly has lower $\mathcal{H}_2$ norm and it might have, for example, more desirable time-domain characteristics. We should add that this example is typical, and not an anomaly. Our experience with $\mathcal{H}_\infty$ control designs on practical problems suggest that slightly suboptimal controllers are both easier to compute and better when analyzed with respect to all the additional considerations that arise in a practical problem, but which are only approximately treated by the $\mathcal{H}_\infty$ theory. A full exposition of this issue is well beyond the scope of this paper, but we will consider one obvious point to motivate the mixed norm theory discussed in the remainder of the paper.

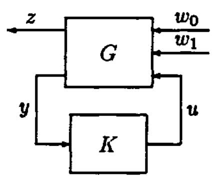
Recall that suboptimal and optimal $\mathcal{H}_\infty$ controllers are generally nonunique, so there are infinitely many controller choices for a given $\mathcal{H}_\infty$ performance level. This freedom can of course be used to further shape the close loop frequency response. However, the current $\mathcal{H}_\infty$ theory gives no direct method for doing so. Hence one almost always ends up in picking the so-called central or maximum entropy controller [Glover and Mustafa, 1989], as was done in this case above.

It is also interesting to compare a pure $\mathcal{H}_2$ controller design on the same problem as above, i.e., the generalized plant G is fixed. Then typically a $\mathcal{H}_\infty$ (central) controller design will give a lower, flatter closed loop frequency response than the $\mathcal{H}_2$ controller. This is shown in the figure below: the solid line corresponds to the $\mathcal{H}_2$ design, the dashed line corresponds to an $\mathcal{H}_\infty$ design that is 5% suboptimal, and the dotted line is a mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design. Note that this plot has a different frequency range than the previous one.



These observations suggest that it would be nice to have a theory that directly handles both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives at the same time. This motivates us to consider a more general problem which achieves this goal naturally and also gives a unified approach to solve both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. Of course, the real motivation for the mixed problem is that $\mathcal{H}_2$ usually makes more sense for performance, but $\mathcal{H}_\infty$ is better for robustness to plant perturbations. Thus naturally we want a theory that handles both.

Consider the following diagram:



where signal w_0 is assumed to be fixed and white, and w_1 is assumed to be bounded in power; the design performance objective is to minimize the power of the output error signal z. It will be seen that if only w_0 is present, then the problem reduces to the standard $\mathcal{H}_2$ problem. Similarly, if only w_1 is present we obtain the standard $\mathcal{H}_\infty$ problem. This setup can arise from a practical problem naturally. For instance, it is often desirable to keep the power or RMS value of the system output small. Additionally, some inputs to the system may have certain spectral characteristics which can be modeled as the output of a filter with white input while other inputs are more naturally modeled as signals of bounded power.

This paper presents both mixed norm analysis and synthesis results. Some earlier versions were presented in [Doyle, Zhou, and Bodenheimer,1989] (in short [DZB]). The first part presents some new analysis results and the second part presents the synthesis results with all the conjectures in [DZB] resolve positively.

2 Preliminaries

This section reviews some elementary mathematical and system theoretic results. The notation used in this paper is fairly standard, and only deterministic signals and systems are considered. The development of the signal sets here is somewhat peripheral to the main theme of this paper, and will be quite sketchy. The objective is to motivate certain induced norms, which are mixtures of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. These mixed norms could also be motivated in a stochastic framework.

Bounded Power Signals

Given a signal $u(t)$, we define its autocorrelation matrix as

$$R_{uu}(\tau) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t+\tau)u'(t)dt,$$

if the limit exists and finite for all τ. It can be shown that $R_{uu}(\tau) = R'_{uu}(-\tau) \geq 0$.

For the purpose of this paper, we further assume the Fourier transform of the signal's autocorrelation matrix function exists (but may contain impulses). This Fourier transform is called the *spectral density* of u, denoted $S_{uu}(j\omega)$:

$$S_{uu}(j\omega) := \int_{-\infty}^{\infty} R_{uu}(\tau)e^{-j\omega\tau}d\tau.$$

Then $R_{uu}(\tau)$ can be obtained from $S_{uu}(j\omega)$ by inverse Fourier transform as

$$R_{uu}(\tau) := \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{uu}(j\omega)e^{j\omega\tau}d\omega.$$

Note that spectral density matrices are Hermitian ($S_{uu}(j\omega) = S'_{uu}(j\omega)$) and positive semidefinite ($S_{uu}(j\omega) \geq 0$).

We will call a signal $u(t)$ a *bounded power signal* if $u(t)$ satisfies the following conditions:

(BP1) $u(t) \in \mathcal{L}_\infty$;

(BP2) the autocorrelation matrix $R_{uu}(\tau)$ exists and is finite for all τ;

(BP3) the power spectral density function $S_{uu}(j\omega)$ exists (it need not be bounded and may include impulses).

The set of all signals having bounded power is denoted by

$$\mathcal{P} := \{u(t) : u(t) \text{ satisfies (BP1 - BP3)}\}$$

A semi-norm can be defined on the space of signals of bounded power, i.e.,

$$\|u\|_P = \left(\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|u(t)\|^2\, dt\right)^{1/2} = (\mathrm{Trace}\,[R_{uu}(0)])^{1/2}.$$

The capital "P" is used to differentiate this power semi-norm from the usual Lebesgue $\mathcal{L}_p$ norm. The power norm of a signal can also be computed from its spectral density function

$$\|u\|_P^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}[S_{uu}(j\omega)]d\omega$$

This expression implies S_{uu} is strictly proper if it is rational. We note that if $u \in \mathcal{P}$ and $\|u(t)\|_\infty < \infty$, then $\|u\|_P \leq \sqrt{m}\|u\|_\infty$, where m is the dimension of u. However, not every $\mathcal{L}_\infty$ signal is in $\mathcal{P}$, because the limit in the definition of the autocorrelation matrix may not exist. Note also that signals of bounded power may be persistent signals in time such as sines or cosines. Clearly an $\mathcal{L}_2$ signal has zero power so $\|\cdot\|_P$ is only a semi-norm, not a norm.

The cross-correlation between two signals u and v is defined as

$$R_{uv}(\tau) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t+\tau)v'(t)dt$$

if it exists and is finite for all τ. It is easy to show that the cross-correlation has the following property

- $R_{uv}(\tau) = R'_{vu}(-\tau)$;

The Fourier transform of $R_{uv}(\tau)$ is called the cross-spectral density and is denoted as $S_{uv}(j\omega)$.

output\input	$\|u\|_2$	spectrum $\|u\|_S$	power $\|u\|_P$
$\|z\|_2$	$\|G\|_\infty$	∞	∞
$\|z\|_S$	0	$\|G\|_\infty$	∞
$\|z\|_P$	0	$\|G\|_2$	$\|G\|_\infty$

Table 1: Systems input/output gains

Bounded Spectrum Signals

A signal $u(t)$ is said to have bounded spectrum if $u \in \mathcal{P}$ and $\|S_{uu}(j\omega)\|_\infty < \infty$, and the set of such signals is denoted as

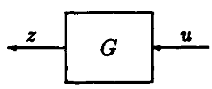
$$\mathcal{S} := \{u(t) \in \mathbf{R}^m : \|S_{uu}(j\omega)\|_\infty < \infty\} \subset \mathcal{P}.$$

The quantity $\|u\|_s := \|S_{uu}(j\omega)\|_\infty^{1/2}$ is a seminorm on $\mathcal{S}$.

The engineering relevance of the set $\mathcal{S}$ is that it can be used to model signals with fixed or bounded spectral characteristics. Similarly, $\mathcal{P}$ could be used to model signals whose spectrum is not bounded but which are bounded in power. In both cases, these signals can be passed through weighting filters to produce signals with desired frequency content. We will primarily view the signals in $\mathcal{S}$ and $\mathcal{P}$ directly in the frequency domain in terms of their spectrums.

Note that, strictly speaking, white noise is not in $\mathcal{S}$, but can be thought of as the limit of a sequence of signals in $\mathcal{S}$ whose spectra in the limit approaches a constant matrix. Thus in the rest of this paper, we will act as if $\mathcal{S}$ includes white noise, recognizing that a suitable limiting argument would be required to make the development rigorous. The term white noise will then be used to describe the case where $S_{uu} = I$. We will assume that the white signal here is such that when it is applied to a strictly proper system the output will be bounded, i.e., in $\mathcal{L}_\infty[0,\infty)$.

We now list some useful spectral analysis facts for a linear system G with convolution kernel $g(t)$, input u, and output z



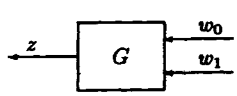
The following properties are standard:

- $R_{zu}(\tau) = g(\tau) * R_{uu}(\tau)$;
- $R_{zz}(\tau) = g(\tau) * R_{uu}(\tau) * g'(-\tau)$;
- $S_{zu}(j\omega) = G(j\omega)S_{uu}(j\omega)$;
- $S_{zz}(j\omega) = G(j\omega)S_{uu}(j\omega)G'(j\omega)$.

These properties are useful in establishing some input and output relationships, in particular we have the relationship listed in Table 1. Note that the induced norms from energy ($\mathcal{L}_2$) to energy, power to power, and spectrum to spectrum are all ∞-norms. While the the induced norm from spectrum to power is the 2-norm, in particular, if the input signal is white with unit spectral density, then the power of the output equals the 2-norm of the transfer matrix.

3 Mixed Norm Performance Analysis

We will examine the norms induced on G with inputs $w_0(t)$ and $w_1(t)$ from different sets.



The performance of system is measured by the power of the output $z(t)$. Thus our objective is to compute

$$\sup_{w_0 \in W_0,\ w_1 \in W_1} \|z\|_P^2 \tag{1}$$

where W_0 is either white or $\mathcal{BS}$ and $W_1 = \mathcal{BP}$. This problem is referred to as the "mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$" problem because, from the earlier tables, if we ignore w_1 then the norm induced on G from w_0 to z is the $\mathcal{H}_2$ norm; similarly, if we ignore w_0 then the norm induced on G from w_1 to z is the $\mathcal{H}_\infty$ norm. This mixed problem has an important motivation from the robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance analysis which will be discussed later on. It turns out that there are several different interpretations that can be given to (1), with somewhat different answers. These different interpretations will be discussed in the remainder of this section.

Denote the cross spectral of w_0 and w_1 by $S_{w_0w_1}(j\omega)$. Now assume G is stable and partition G compatibly with w_0 and w_1 as $[G_0\ G_1]$, where G_0 is assumed strictly proper (otherwise the output signal can have unbounded power). In terms of the state-space matrices, this can be represented as

$$G(s) = \left[\begin{array}{c|cc} A & B_0 & B_1 \\ \hline C & 0 & D_1 \end{array}\right] =: \left[\begin{array}{cc} G_0 & G_1 \end{array}\right].$$

Now we can compute the power spectral of the output z. To do that let

$$w := \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

Then the spectral density matrix of w can be computed as

$$S_{ww} = \begin{bmatrix} S_{w_0w_0} & S_{w_0w_1} \\ S_{w_0w_1}^* & S_{w_1w_1} \end{bmatrix}$$

Using this formula and the formula shown before, we get

$$S_{zz} = \begin{bmatrix} G_0(j\omega)^* \\ G_1(j\omega)^* \end{bmatrix}^* \begin{bmatrix} S_{w_0w_0} & S_{w_0w_1} \\ S_{w_0w_1}^* & S_{w_1w_1} \end{bmatrix} \begin{bmatrix} G_0(j\omega)^* \\ G_1(j\omega)^* \end{bmatrix} \tag{2}$$

and

$$\|z\|_P^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}[S_{zz}(j\omega)]d\omega \tag{3}$$

These relations form the basis for our study below.

3.1 Orthogonal Case

If we assume that w_0 and w_1 are orthogonal, i.e.,

$$S_{w_0w_1} = 0.$$

then we have

$$\sup_{w_0 \in \mathcal{BS},\ w_1 \in \mathcal{BP}} \|z\|_P^2 = \|G_0\|_2^2 + \|G_1\|_\infty^2$$

and the worst signal w_0 is white noise with unit spectral density matrix, $S_{w_0w_0} = I$. This case is the simplest and also the least interesting.

3.2 White and Causal Case

In this case $w_0(t)$ is *assumed* to be white with unit spectral density, i.e., $S_{w_0w_0} = I$, and $w_1(t) \in \mathcal{P}$. We will further assume that $w_1(t)$ can be generated from $w_0(t)$ through a strictly causal filter, i.e., $S_{w_1w_0} = W(s)$ and $W(s) \in \mathcal{RH}_2$ (i.e., W is strictly proper). This is the next simplest case, and the one that will be the main focus of this paper.

With x denoting the system states, the system equation can be written as

$$\begin{aligned} \dot{x} &= Ax + B_0 w_0(t) + B_1 w_1(t), \quad \|x(-\infty)\| < \infty \\ z &= Cx + D_1 w_1(t) \end{aligned}$$

Suppose $\gamma > \|G_1\|_\infty$ so $\gamma > \bar{\sigma}(D_1)$. Denote $R := \gamma^2 I - D_1' D_1$ and

$$X = Ric \begin{bmatrix} A + B_1 R^{-1} D_1' C & B_1 R^{-1} B_1' \\ -C'(I + D_1 R^{-1} D_1')C & -(A + B_1 R^{-1} D_1' C)' \end{bmatrix} \geq 0.$$

Theorem 1. *Suppose $\gamma > \|G_1\|_\infty$. Then*

$$\sup_{w_1 \in \mathcal{P}} \left\{ \|z\|_P^2 - \gamma^2 \|w_1\|_P^2 \right\} = \mathrm{Trace}(B_0' X B_0)$$

with a worst-case signal $\tilde{w}_1 = R^{-1}(D_1' C + B_1' X)x$.

This follows from differentiating $x'Xx$ along solutions of the differential equation, then completing the square, and finally taking the time average integral using the relation $R_{xw_0}(0) = \frac{1}{2} B_0$.

Finally, to compute (1), we have to find a suitable γ such that $\tilde{w}_1 \in \mathcal{BP}$, this is given in the following theorem.

Theorem 2. *Let γ_0 be such that $\|R(\gamma_0)^{-1}(D_1'C + B_1' X(\gamma_0))x\|_P = 1$. Then*

$$\sup_{w_1 \in BP} \|z\|_P^2 = \mathrm{Trace}(B_0' X(\gamma_0) B_0) + \gamma_0^2.$$

3.3 Non-white and Non-causal Case

For the purpose of comparison, we now examine the case when w_0 is not restricted to be white and w_1 is not restricted to be a causal function of w_0. This case is more complicated. The following study will show that in this case the worst case signal w_0 is actually white, but the worst case signal w_1 is not, in general, a causal function of w_0.

Let $\gamma > 0$ be such that $\|G_1\|_\infty < \gamma$ and without loss of generality assume that the spectral density of w_0 and w_1 have the following decompositions:

$$\begin{aligned} S_{w_0 w_0} &= S_{00} S_{00}^* \\ S_{w_0 w_1} &= S_{00} S_{01}^* \\ S_{w_1 w_1} &= S_{01} S_{01}^* + S_{11} S_{11}^* \end{aligned}$$

where S_{00} can be restricted to stable and minimum phase transfer matrix, in fact, w_0 can be thought as the output of the stable system S_{00} with an unit density white input.

Introduce the following maximization problem:

$$\sup_{w_0 \in BS,\, w_1 \in \mathcal{P}} \left\{ \|z\|_P^2 - \gamma^2 \|w_1\|_P^2 \right\} = \sup_{w_0 \in BS,\, w_1 \in \mathcal{P}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Trace}[J] d\omega \qquad (4)$$

where

$$J := \begin{bmatrix} G_0(j\omega)^* \\ G_1(j\omega)^* \end{bmatrix}^* \begin{bmatrix} S_{00} S_{00}^* & S_{00} S_{01}^* \\ S_{01} S_{00}^* & S_{01} S_{01}^* + S_{11} S_{11}^* \end{bmatrix} \begin{bmatrix} G_0(j\omega)^* \\ G_1(j\omega)^* \end{bmatrix} - \gamma^2 (S_{01} S_{01}^* + S_{11} S_{11}^*)$$

It can be shown that Trace$[J]$ is maximized by $S_{11} = 0$ and

$$S_{01} = (\gamma^2 I - G_1^* G_1)^{-1} G_1^* G_0 S_{00} \qquad (5)$$

Hence we have

$$\mathrm{Trace}\, J = \mathrm{Trace} \left\{ \gamma^2 G_0 S_{00} S_{00}^* G_0^* (\gamma^2 I - G_1 G_1^*)^{-1} \right\}$$

Thus

$$\sup_{w_0 \in BS,\, w_1 \in \mathcal{P}} \left\{ \|z\|_P^2 - \gamma^2 \|w_1\|_P^2 \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Trace} \left\{ \gamma^2 G_0 G_0^* (\gamma^2 I - G_1 G_1^*)^{-1} \right\} d\omega$$

Again, the worst case signal w_0 is white with unit spectral density $S_{00} = I$.

Theorem 3. *Let γ be such that*

$$\|w_1\|_P = \left\| (\gamma^2 I - G_1^* G_1)^{-1} G_1^* G_0 \right\|_2 = 1. \qquad (6)$$

Then

$$\sup_{w_0 \in BS,\, w_1 \in BP} \|z\|_P = \left\| \gamma^2 (\gamma^2 I - G_1 G_1^*)^{-1} G_0 \right\|_2$$

with the worst signal w_0 satisfying $S_{w_0 w_0} = I$ and w_1 having spectral density $S_{w_1 w_1} = S_{01} S_{01}^*$ where S_{01} is given by (5).

Remark 1. Note that from equation (5), it is seen that the worst signal w_1 can be generated from passing w_0 through the acausal linear system $(\gamma^2 I - G_1^* G_1)^{-1} G_1^* G_0$.

Remark 2. The case where w_1 is assumed to be causal function of w_0, but w_0 is not restricted to being white, is not solved. Examples exist which show that in this case, the worst w_0 is not white.

3.4 Comparison of the induced norms

Note that for any of the assumptions above

$$\|z\|_P = \|G_0 w_0 + G_1 w_1\|_P \leq \|G_0 w_0\|_P + \|G_1 w_1\|_P$$

so we have that

$$\begin{aligned} \sup_{w_0 \in W_0, w_1 \in W_1} \|z\|_P &\leq \|G_0\|_2 + \|G_1\|_\infty \\ &\leq \sqrt{2(\|G_0\|_2^2 + \|G_1\|_\infty^2)} \end{aligned}$$

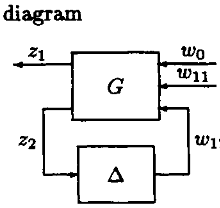
where W_0 is white or bounded spectrum and W_1 is bounded power (causal or noncausal). Then the relationships among the costs of $\|z\|_P$ in different cases can be summarized as

$$\begin{aligned} \text{orthogonal} &\leq \text{white and causal} \\ &\leq \text{nonwhite and causal} \\ &\leq \text{nonwhite and noncausal} \\ &= \text{white and noncausal} \\ &\leq \sqrt{2}\ \text{orthogonal} \end{aligned}$$

Note that the different assumptions make very little difference in that actual induced norm. We will focus on the white and causal case, because it is the simplest case that we can give reasonable meaning to, and it naturally leads to a tractable synthesis problem.

4 Robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance

In this section we consider system performance analysis when the system model has structured norm-bounded perturbations, as in the following diagram



where G is partitioned according to the inputs and outputs as

$$G = \begin{bmatrix} G_{00} & G_{01} & G_{02} \\ G_{10} & G_{11} & G_{12} \end{bmatrix} =: \begin{bmatrix} G_0 & G_1 \end{bmatrix}$$

and $G_0 = \begin{bmatrix} G_{00} \\ G_{10} \end{bmatrix}$ is strictly proper. The uncertainty is structured such that $\Delta \in \boldsymbol{\Delta}$ where

$$\boldsymbol{\Delta} = \left\{ \text{diag}\{\Delta_1, \Delta_2, \ldots, \Delta_m\} : \ \Delta_i \in (\mathcal{H}_\infty)^{t_i \times t_i}, \ \|\Delta_i\|_\infty \le 1 \right\}$$

Again we assume $w_0 \in \mathcal{S}$ and $w_{11} \in \mathcal{P}$. The robust performance problem in this setting concerns the following question: when does

$$\frac{\|z_1\|_{\mathcal{P}}^2}{\|w_{11}\|_{\mathcal{P}}^2 + \|w_0\|_{\mathcal{S}}^2} \le 1, \ \ \forall \Delta \in \boldsymbol{\Delta} \tag{7}$$

hold?

Unlike the pure $\mathcal{H}_\infty$ case, exact analysis of this problem is difficult, even theoretically. A sufficient condition for this problem can be obtained using the mixed norm analysis results in the previous section. Define a set of scaling transfer matrices

$$\mathcal{D} = \left\{ \text{diag}\{d_1(s)I_{t_1}, \ldots, d_m(s)I_{t_m}\} : \ d_i(s), d_i^{-1}(s) \in \mathcal{H}_\infty, \right\}$$

Then $D\Delta D^{-1} = \Delta$ for all $\Delta \in \boldsymbol{\Delta}$ and $D \in \mathcal{D}$.

Let $D(s) \in \mathcal{D}$ and

$$z := \begin{bmatrix} z_1 \\ D(s)z_2 \end{bmatrix}, \quad w_1 := \begin{bmatrix} w_{11} \\ D(s)w_{12} \end{bmatrix}$$

Then we have

$$\begin{aligned} z &= \begin{bmatrix} G_{00} & G_{01} & G_{02}D^{-1} \\ DG_{10} & DG_{11} & DG_{12}D^{-1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \\ &=: \begin{bmatrix} \hat{G}_0 & \hat{G}_1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \end{aligned}$$

Now consider a mixed norm analysis problem

$$J_m = \inf_{D \in \mathcal{D}} \sup_{w_1 \in \mathcal{P}, \ w_0 \in \mathcal{S}} \frac{\|z\|_{\mathcal{P}}^2}{\|w_1\|_{\mathcal{P}}^2 + \|w_0\|_{\mathcal{S}}^2} \tag{8}$$

Theorem 4. *The system satisfies robust performance, i.e., (7) holds, if $J_m \le 1$.*

Now in particular let w_0 be such that $\|w_0\|_{\mathcal{S}} \le 1$. Then the test $J_m \le 1$ for a given $D \in \mathcal{D}$ is equivalent to

$$\sup_{w_1 \in \mathcal{P}, \ w_0 \in \mathcal{S}} \left\{ \|z\|_{\mathcal{P}}^2 - \|w_1\|_{\mathcal{P}}^2 \right\} \le 1. \tag{9}$$

This can be tested using the results obtained in previous sections. To get the least conservative test possible, a search on D is required.

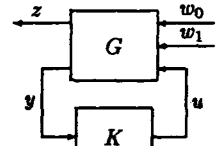
Remark 3. The robust performance (7) has two special cases:

- $w_0 = 0$. This is so called robust $\mathcal{H}_\infty$ performance problem. In this case the analysis problem is reduced to structured singular value problem, see for example, [Doyle, Wall, and Stein, 1982]. For 1 or 2 blocks, the condition $J_m \le 1$ is necessary as well as sufficient for robust performance.
- $w_{11} = 0$. We shall call this problem *robust $\mathcal{H}_2$ performance problem.*

Remark 4. The problem of selecting the best D scalings for the mixed problem is not as simple as for the $\mathcal{H}_\infty$ case, where the problem can be reduced to constant matrices at each frequency. Of the various methods available to compute (9), the least conservative consistent with the assumptions on Δ would be the white and causal case.

5 Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Synthesis

In this section, we consider the synthesis problem when the system is subjected to mixed disturbance signals. Specifically, consider the system described by the block diagram



where again the plant G and controller K are assumed to be real-rational and proper. We consider only the white and causal case.

> **Problem (G)** *Given the plant G, a constant γ, exogenous signals w_0, with $S_{w_0w_0} = I$ and $w_1 \in \mathcal{P}$ depending causally on w_0. The mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problem is to find a controller K such that*
>
> $$\min_K \sup_{w_1 \in \mathcal{P}} \left\{ \|z\|_{\mathcal{P}}^2 - \gamma^2 \|w_1\|_{\mathcal{P}}^2 \right\}$$
>
> *is solved, where the minimization is constrained to those K providing internal stability.*

The phrase "problem (G)" means the minimization problem corresponding to the plant "G". In this paper, a controller K is said to be an *admissible controller* if it internally stabilizes G. As mentioned earlier, when $w_0 = 0$ or $w_1 = 0$, the induced norm becomes the the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm, respectively. Thus, Problem (G) is solvable only if the corresponding pure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems are solvable. In this paper, we do not usually address the issue of the optimal mixed controller and only discuss optimality in terms of a given γ, restricting γ to be greater than the corresponding $\mathcal{H}_\infty$ optimal level, γ_∞. Thus, optimal controller means optimal for a given γ level. Clearly, any mixed optimal controller is a sub-optimal pure $\mathcal{H}_\infty$ controller, but the converse need not be true.

Lemma 1. *Problem (G) is solvable only if there exists a K such that $\|T_{zw_1}\|_\infty < \gamma$, i.e., the corresponding $\mathcal{H}_\infty$ problem ($w_0 = 0$) is solvable.*

The results in this paper state that the condition in the lemma is not only necessary, but also sufficient. If a sub-optimal pure $\mathcal{H}_\infty$ controller exists, so $\gamma > \gamma_\infty$, then an (sub-) optimal mixed controller also exists.

Assumptions on the Plant G

The system has the following realization

$$G(s) = \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_1 & 0 & 0 & D_{12} \\ C_2 & D_{20} & D_{21} & 0 \end{array} \right]$$

The following assumptions are made:

(i) (A, B_2) is stabilizable and (C_2, A) is detectable.

(ii) D_{12} has full column rank with $\begin{bmatrix} D_{12} & D_{12}^{\perp} \end{bmatrix}$ unitary.

(iii) $R_0 = D_{20}D_{20}' > 0$ and $R_1 = D_{21}D_{21}' \geq 0$.

(iv) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbf{R}$.

(v) $\begin{bmatrix} A - j\omega I & B_0 & B_1 \\ C_2 & D_{20} & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbf{R}$.

Assumption (i) is clearly necessary for internal stability. The essential assumption in (ii) is that D_{12} has full column rank, while the second part of the assumption is only made to simplify the formulas in our solution. There is no loss of generality, since scaling can be applied first to bring it to this standard form. One should note that unlike the $\mathcal{H}_\infty$ problem there is no explicit assumption on D_{21}; instead we require condition on D_{20}, assumption (iii). The significance of assumption (iii) is that it insures that the corresponding $\mathcal{H}_2$ problem is nonsingular.

Assumption (iv) are made for the same reason as in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem to guarantee that the Riccati equation associated with the pure $\mathcal{H}_2$ problem has a stabilizing solution. As for the assumption (v), it is weaker than the dual to (iv), and is a necessary condition for a filtering problem in the mixed setting to be solvable. The assumptions are not surprising as we should have expected that some conditions with combined $\mathcal{H}_2$ and $\mathcal{H}_\infty$ features are required.

There is no loss of generality in assuming $D_{22} = 0$, since the controller for the $D_{22} \neq 0$ case can be found from the controller for $D_{22} = 0$ case by a linear fractional transformation [see Glover and Doyle, 1988]. On the other hand, the solution for $D_{11} \neq 0$ case is much more complicated as can be seen from Glover and Doyle (1988) for $\mathcal{H}_\infty$ problem. The formulas in this paper should generalize in the same way.

Theorem 5. *Given $\gamma > 0$ and plant G, there exists a controller $K(s)$ which solves Problem (G) if and only if the following conditions hold:*

(i) $H_\infty \in \mathrm{dom}(Ric)$ and $X_\infty := Ric(H_\infty) \geq 0$ where

$$H_\infty := \begin{bmatrix} A - B_2D_{12}'C_1 & \gamma^{-2}B_1B_1' - B_2B_2' \\ -C_1'D_{12}^{\perp}(D_{12}^{\perp})'C_1 & -(A - B_2D_{12}'C_1)' \end{bmatrix}.$$

(ii) There exist L, Y, and P which satisfy

$$LR_0 + B_0D_{20}' + PC_2' + \gamma^{-2}PYLR_1 + \gamma^{-2}PYB_1D_{21}' = 0$$

$$Y(A_{tmp} + LC_2) + (A_{tmp} + LC_2)'Y + Y\tilde{R}Y + F_\infty'F_\infty = 0$$

$$Y \geq 0 \text{ and } A_{tmp} + LC_2 + \tilde{R}Y \text{ is stable}$$

$$\{A_{tmp} + LC_2 + \tilde{R}Y\}P + P\{A_{tmp} + LC_2 + \tilde{R}Y\}' + (B_0 + LD_{20})(B_0 + LD_{20})' = 0$$

Moreover, when these conditions hold, one such controller is

$$K(s) := \left[\begin{array}{c|c} A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + LC_2 & -L \\ \hline F_\infty & 0 \end{array}\right]$$

where $\tilde{R} = \gamma^{-2}(B_1+LD_{21})(B_1+LD_{21})'$, $A_{tmp} = A+\gamma^{-2}B_1B_1'X_\infty$, and $F_\infty = -(D_{12}'C_1 + B_2'X_\infty)$.

This theorem can be proven in two steps:

- using the separation principle in [DZB] to reduce the original problem to a mixed full control problem;
- solve the mixed full control problem as in [DZB]. This involves proving all the conjectures in [DZB].

The first step has been presented in [DZB], but the proof for the second step is very long and will not be included here due to space limitations. The key ideas in the proof will be discussed in the talk, time permitting.

Remark 5. It would be useful to compare the results in this paper with those of Bernstein and Haddad (1989), which have a superficial similarity that hints at deeper connections. Unfortunately, there isn't space to fully explore the connections. Briefly, our results were definitely inspired by their work, and could be viewed as an attempt to improve on it in certain respects. In particular, we have given an induced norm interpretation to our performance objective, instead of an ad hoc upper bound. We remove a certain nagging technicality that is roughly equivalent to assuming a priori that Y in Theorem 5 exists and is positive definite, which it need not be. Finally, we prove the necessity of the conditions in Theorem 5.

References

Bernstein, D.S. and W.M. Haddad (1989). "LQG control with an $\mathcal{H}_\infty$ performance bound: A Riccati equation approach," *IEEE Trans. Auto. Control*, vol. AC-34, pp. 293-305.

Doyle, J.C., K. Glover, P.P. Khargonekar, B.A. Francis (1989). " State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Auto. Control*, vol. AC-34, No. 8, pp. 831-847. Also see 1988 American Control Conference, Atlanta, June, 1988.

Doyle, J.C., K. Zhou, and B. Bodenheimer (1989). "Optimal Control with Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance Objectives," 1989 American Control Conference, pp. 2065-2070.

Doyle, J.C., J. Wall and G. Stein (1982). "Performance and robustness analysis for structured uncertainty," in *Proc. 21st IEEE Conf. Decision Contr.*, pp. 629-636.

Francis, B.A. (1987). *A course in $\mathcal{H}_\infty$ control theory*, Lecture Notes in Control and Information Sciences, vol. 88, Springer-Verlag, Berlin.

Glover, K. and J. Doyle (1988). "State-space formulae for all stabilizing controllers that satisfy an $\mathcal{H}_\infty$ norm bound and relations to risk sensitivity," *Systems and Control Letters*, vol. 11, pp. 167-172.

Glover, K. and D. Mustafa (1989). "Derivation of the maximum entropy $\mathcal{H}_\infty$ controller and a state space formula for its entropy," *Int. J. Control.*

Mariton, M. and R. Bertrand (1985). "A homotopy algorithm for solving coupled Riccati equations," *Optimal. Contr. Appl. Meth.*, vol. 6, pp. 351-357.

Rotea, M. A. and P. P. Khargonekar (1989). " $\mathcal{H}_2$ optimal control with an $\mathcal{H}_\infty$ constraint: the state feedback case ", University of Michigan.