WHEN IS CLASSICAL LOOP SHAPING H^∞-OPTIMAL?

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ABSTRACT

This paper examines conditions under which a given SISO LTI control system is H^∞-optimal with respect to weighted combinations of its sensitivity function and its complementary sensitivity function. The specific weighting functions considered are defined in terms of the sensitivity and complementary sensitivity functions. We show that a large class of practical controllers are in fact H^∞-optimal, including typical stable controllers.

1. INTRODUCTION.

This paper examines conditions under which a given SISO control system is H^∞-optimal with respect to weighted combinations of its sensitivity function, $S(s)=[1+GK(s)]^{-1}$, and its complementary sensitivity function, $T(s)=GK(s)[1+GK(s)]^{-1}$. We are not interested in arbitrary weighting functions, however, but in weighting functions whose magnitudes are the inverses of $|S(s)|$ and $|T(s)|$, respectively. Control systems which are H^∞-optimal with respect to these special weights will be called self-optimal.

The concept of self-optimality is motivated by our desire to determine whether the sensitivity and complementary sensitivity functions of a given controller K_o obtained from, say, classical SISO loop shaping techniques, could possibly be improved upon, uniformly across frequency, by H^∞ optimization. Self-optimal controllers cannot be uniformly improved, while others can be uniformly improved. It turns out that a large class of practical controllers are in the self-optimal category. Thus, they cannot be improved at some frequencies without incurring deteriorations elsewhere.

Our study also provides insights into certain properties of H^∞ designs. It has been conjectured that H^∞ optimal control designs generically produce unstable controllers. This is true, for example, for plants with two or more RHP zeros optimized for weighted sensitivity [ref.(7)]. In practical feedback design, of course, one picks stable controllers whenever possible. Hence, the conjecture suggests that such designs are suboptimal, though perhaps otherwise adequate. We show with the concept of self-optimality that a large class of practical controllers are in fact H^∞ optimal, including typical stable controllers.

We note that the self-optimal concept also applies to MIMO systems, but only for a very restrictive class of design problems. In general, the trade offs between sensitivity and complementary sensitivity in MIMO systems are not transparent, see ref. (2) for example. In addition, not all important MIMO control design issues for MIMO systems can be captured by simple specifications on these two functions, ref. (5).

The paper is organized into four sections. Section 2 provides a formal definition of the self-optimality property and presents the main results of the paper. These take the form of simple pole-zero counting conditions (winding numbers) applied to G and K_o. Section 3 shows several examples which illustrate these results. Section 4 presents a proof of a special case of a theorem, due to Helton ref. (4), which we use to prove our main results.

2. MAIN RESULTS.

We begin by stating assumptions made on the plant G and controller K_o which simplify our exposition.

[1] G is a given SISO LTI plant and K_o is a given SISO LTI controller which internally stabilizes G under negative feedback.

[2] G and K_o have no poles or zeros on the $j\omega$-axis including ∞. With minor modifications to given G and K one can meet this requirement.

[3] For technical reasons which will emerge as we prove our results, assume that $GK_o(j\omega) \neq 1$ for all ω in $R\cup\{\infty\}$ and that if GK_o has constant magnitude across the $j\omega$-axis, ie. $GK_o(j\omega) = \alpha U(j\omega)$ where $U^*U = 1$, then G has at least one RHP pole or zero. This restriction should not be serious since a desirable loop shape for classical design rarely has constant magnitude.

Define weights

$$W_1(s) := outer\left[1 + GK_o\right], \quad W_2(s) := outer\left[\frac{1 + GK_o}{GK_o}\right].$$

Here the *outer* part of a proper rational function F with no poles or zeros on the $j\omega$-axis is defined to be the unique rational function F_o with no poles or zeros in the closed RHP such that $|F| = |F_o|$ on the $j\omega$-axis.

Define Ψ, Λ, and Θ to be the weighted mixed sensitivity, the weighted sensitivity, and the weighted complementary sensitivity transfer functions respectively as shown in Figures 1, 2, and 3 and given by

$$\Psi(K) := \begin{bmatrix} W_1\frac{1}{1+GK} \\ W_2\frac{GK}{1+GK} \end{bmatrix}, \qquad (1)$$

$$\Lambda(K) := \left[W_1 \frac{1}{1+GK}\right], \quad \Theta(K) := \left[W_2 \frac{GK}{1+GK}\right].$$

The controller K_o is called **self optimal** for the mixed sensitivity problem, or **S&T-optimal,** if and only if K_o is a solution to

$$\inf_K \|\Psi\|_\infty := \inf_K \sup_{j\omega} \bar{\sigma} \begin{bmatrix} W_1 \dfrac{1}{1+GK} \\ W_2 \dfrac{GK}{1+GK} \end{bmatrix}, \tag{2}$$

where the infimum is taken over all internally stabilizing controllers. In other words, $\inf_K \|\Psi\|_\infty = \|\Psi(K_o)\|_\infty$. Define K_o to be **S-optimal** or **T-optimal** analogously.

Denote the number of RHP poles of K_o by p_k, the number of RHP zeros of K_o by z_k, the number of RHP poles of G by p_g, and number of RPH zeros of G by z_g. Define the quantity n to be the number of RHP roots of the polynomial $num(GK_o) - den(GK_o)$, where $num(GK_o)$ and $den(GK_o)$ are the numerator and denominator polynomials of GK_o (assumed to have no common factors).

Our main results are the following.

Theorem 1. For GK_o with assumptions [1], [2], and [3], K_o is S&T-optimal if and only if

$$\mathbf{n} > \mathbf{z_k} + \mathbf{p_k}. \tag{3}$$

Theorem 2. For GK_o with assumptions [1] and [2], K_o is S-optimal if and only if

$$\mathbf{z_g} > \mathbf{p_k}. \tag{4}$$

Theorem 3. For GK_o with assumptions [1] and [2], K_o is T-optimal if and only if

$$\mathbf{p_g} > \mathbf{z_k}. \tag{5}$$

We use Helton's result [ref(4), Theorem 4.1] to prove Theorem's 1, 2, and 3. The key idea is that determination of **S&T-optimality, S-optimality** or **T-optimality** for a controller K_o can be reduced to determination of the winding number of a real rational function.

We now briefly describe Helton's result.

Let $\Gamma := \Gamma(\omega,\Phi)$ be a positive real valued function of a real variable ω and a complex variable Φ. Let s_i and a_{ij} for $i = 1,2,...r$ and $j = 1,2,...m$ be points in the open RHP. Define the interpolation set E to be the set of all functions f analytic and bounded in the RHP and satisfying given interpolation conditions of the form

$$f(s_i) = a_{i1}, \ \frac{df}{ds}(s_i) = a_{i2}, \ \dots \ \frac{d^m f}{ds^m}(s_i) = a_{im}$$

for $i = 1,2,...r$. Suppose the objective is to find

$$\gamma := \inf_{f\in E} \sup_{\omega} \Gamma(\omega, f(j\omega)). \tag{6}$$

Assume that the performance measure Γ has sublevel sets which are disks and (6) has the property that an optimum f_o must make $\Gamma(\omega, f_o(\omega))$ constant in ω. Then

Theorem 4. [ref (4), Theorem 4.1] Suppose $\Gamma(\omega,\Phi)$ is continuously differentiable in Φ for all $\omega \in [-\infty,\infty]$. Suppose $\frac{\partial\Gamma}{\partial\Phi}|_{\Phi=f_o(j\omega)} \neq 0$ for all ω. Then f_o is a local optimum for (6) over E if and only if

(i) $\Gamma(\omega, f_o(j\omega))$ is constant in ω.

(ii) The winding number of $\frac{\partial\Gamma}{\partial\Phi}(\omega, f_o(j\omega))$ is strictly greater than minus the number of interpolating constraints defining E (counting multiplicity).

We now give a proof of Theorem 1. The proofs of Theorem 2 and Theorem 3 are similar, and so omitted.

A controller K solving (2) will also minimize the magnitude squared of Ψ given by

$$\inf_K \|\Gamma(\omega,K)\|_\infty := \tag{7}$$

$$\inf_K \sup_{j\omega} \left\{ W_1^* \left[\frac{1}{1+GK}\right]^* W_1 \frac{1}{1+GK} + W_2^* \left[\frac{GK}{1+GK}\right]^* W_2 \frac{GK}{1+GK} \right\}.$$

As discussed in ref. (8), K internally stabilizes G if and only if the sensitivity function $S := \frac{1}{1+GK}$ belongs to RH^∞ and satisfies the interpolation constraints

$$(S-1) = NU, \quad \text{and} \quad S = MV,$$

where N, U, M, V, are in RH^∞ and NM^{-1} is a coprime factorization of G.

The optimization of Γ over all stabilizing controllers is thus equivalent to the optimization of Γ over the set E of all $S \in RH^\infty$ which satisfy the interpolation constraints

$$S(z_i) = 1 \qquad \textit{for every } z_i \textit{ a RHP zero of } G$$

$$S(p_j) = 0 \qquad \textit{for every } p_j \textit{ a RHP pole of } G.$$

As is well known, in the case of repeated roots one also has interpolation conditions on derivatives of S.

Rewriting (7) as a function of S we obtain

$$\inf_{S\in E} \|\Gamma(\omega,S)\|_\infty := \inf_{S\in E} \sup_{j\omega} \left\{ W_1^* S^* W_1 S + W_2^* (1-S)^* W_2 (1-S) \right\}. \tag{8}$$

We will apply Theorem 4 to the problem as expressed in (8).

First we verify that $\Gamma(\omega,S)$ has the following properties:

1. Γ is continuous in ω. Γ is a quadratic function of S with no explicit dependence on ω.

2. Γ is real valued and non-negative on the $j\omega$-axis. This can be seen by noting that it is the square of the magnitude of $\Psi(j\omega)$.

3. $\Gamma(\omega,S_o(j\omega))$ is constant across frequencies, where $S_o := \frac{1}{1+GK_o}$. One may check that W_1S_o and $W_2(1-S_o)$ are allpass, so that $\Gamma(\omega,S_o(j\omega))$ has the value 2 across the $j\omega$-axis.

4. Γ is continuously differentiable in S for all $\omega \in [-\infty,+\infty]$. Γ is a quadratic, non-analytic, real valued function of the complex variables S and S^*. The partial derivative of Γ with respect to S is

$$\frac{\partial\Gamma}{\partial S}(\omega,S) = W_1^*W_1S^* + W_2^*W_2(S^* - 1). \qquad (9)$$

5. $\frac{\partial\Gamma}{\partial S}(\omega,S_o)$ is nonvanishing, real rational, and given by

$$\frac{\partial\Gamma}{\partial S}(\omega,S_o) = \frac{1 - 2S_o}{S_o(1 - S_o)}. \qquad (10)$$

To derive (10) from (9) we note that

$$W_1^*W_1 = \frac{1}{S_o^*S_o}, \text{ and } W_2^*W_2 = \frac{1}{(1 - S_o^*)(1 - S_o)}. \qquad (11)$$

6. Γ is constant across frequencies at its optimum. To see this we reformulate the mixed sensitivity optimization problem given in (2) as a general distance problem following ref. (1), (3), and (6). First we note that the set of all internally stabilizing proper controllers is given by

$$\left\{K \mid K = (U_o-MQ)(V_o+NQ)^{-1} \quad Q\in RH^\infty \text{ and } V_o+NQ \neq 0 \text{ at } \infty\right\}.$$

where $U_oV_o^{-1}$ is a coprime factorization of K_o and NM^{-1} is a coprime factorization of G with $N, M, V_o, U_o \in RH^\infty$. Given $K \in RL^\infty$, one can find RL^∞ functions R_1 and R_2 such that at each frequency ω

$$\Psi(K)^*\Psi(K) = \begin{bmatrix}(R_1 - Q)^* & R_2^*\end{bmatrix}\begin{bmatrix}R_1-Q \\ R_2\end{bmatrix}. \qquad (12)$$

Thus

$$\gamma := \inf_K \|\Psi(K)\|_\infty = \inf_{Q\in RH^\infty}\left\|\begin{matrix}R_1-Q \\ R_2\end{matrix}\right\|_\infty. \qquad (13)$$

In ref. (1), (3), and (6) it is shown that if

$$\|R_2\|_\infty < \gamma, \qquad (14)$$

then Q minimizing (13) solves

$$\min_{Q\in RH^\infty} \|(R_1-Q)(\gamma^2-R_2^*R_2)^{-1/2}\|_\infty = \min_{\hat{Q}\in RH^\infty} \|(R_1(\gamma^2-R_2^*R_2)^{-1/2}-\hat{Q}\|_\infty = 1.$$

A standard result is that the minimizing $\hat{Q}$ makes $[R_1(\gamma^2-R_2^*R_2)^{-1/2} - \hat{Q}]$ allpass. At each frequency then

$$(R_1 - Q)^*(R_1 - Q)(\gamma^2 - R_2^*R_2)^{-1} = 1, \qquad (15)$$

and so

$$(R_1 - Q)^*(R_1 - Q) + R_2^*R_2 = \gamma^2. \qquad (16)$$

Therefore at its optimum, Γ has the value γ^2 at every frequency if (14) holds.

We observe that

$$\|R_2\|_\infty = \min_{X\in RL^\infty}\left\|\begin{matrix}R_1 - X \\ R_2\end{matrix}\right\|_\infty = \left\|\begin{matrix}0 \\ R_2\end{matrix}\right\|_\infty. \qquad (17)$$

Therefore, at each frequence $\|R_2\|_\infty$ is the minimum value of Ψ over all $K \in RL^\infty$, and so

$$\|R_2\|_\infty^2 = \min_{S\in RL^\infty}\|\Gamma\|_\infty. \qquad (18)$$

Computation of (18) amounts to a frequency by frequency minimization over $S \in C$. The optimal S can be found by solving

$$\frac{\partial\Gamma}{\partial S} = \frac{\partial\Gamma}{\partial S^*} = 0 \qquad \text{for } S \in C.$$

One may verify that the minimum value for Γ is given by $S = \hat{S}$, where

$$\hat{S} = \hat{S}^* = \frac{W_2^*W_2}{W_1^*W_1 + W_2^*W_2} = \frac{1}{G^*K_o^*GK_o + 1}. \qquad (19)$$

Therefore the minimization of (18) is achieved at

$$\Gamma(\hat{S}) = \frac{W_1^*W_1W_2^*W_2}{W_1^*W_1 + W_2^*W_2} = 1 + \frac{G^*K_o^* + GK_o}{G^*K_o^*GK_o + 1}.$$

To show that (14) holds we verify that $\Gamma(\hat{S}) < \gamma^2$ at all frequencies. We note from (19) that $\hat{S}$ is a real valued function of $j\omega$. Hence if $\hat{S} \in RH^\infty$, then $\hat{S}$ must be a constant. Hence $GK_o = \alpha U$ where $U^*U = 1$. Thus $\hat{S} = \frac{1}{\alpha^2 + 1}$. For $\hat{S}$ to be in E, it must satisfy interpolation conditions since by assumption [3] G has at least one RHP pole or zero. Therefore $\hat{S}$ must be identically 1 or 0. This implies that $\alpha = 0$ or ∞. This is a contradiction. Therefore $\hat{S}$ is not in the set E. Because $\hat{S}$ is unique, the optimal $S \in E$ for (8) must result in a strictly larger norm for Γ.

7. The sublevel sets of Γ are disks. One may verify that the set of all $S \in C$ such that

$$\Gamma(\omega,S) = W1^*S^*W_1S+W_2^*(1-S)^*W_2(1-S) < \beta \quad \text{for } \beta > \gamma$$

are in a disk centered at $\hat{S}$.

By Theorem 4, S_o is an optimum if and only if

$$\textit{winding \#}\left[\frac{\partial\Gamma}{\partial S}(\omega,S_o)\right] > -p_g - z_g. \qquad (20)$$

For computing the winding number of $\frac{\partial\Gamma}{\partial S}(\omega,S_o)$ we rewrite (10) in terms of the plant and controller

$$\frac{\partial\Gamma}{\partial S}(\omega,S_o) = \qquad (21)$$

$$\frac{(den(GK_o) + num(GK_o))(num(GK_o) - den(GK_o))}{den(GK_o)num(GK_o)}$$

The winding number of $\frac{\partial\Gamma}{\partial S}(\omega,S_o)$ can now be assessed by counting the RHP poles and zeros of (21). We know that the polynomial $den(GK_o) + num(GK_o)$ has no RHP roots because of our closed loop stability assumption. Let the number of RHP roots of $num(GK_o) - den(GK_o)$ be n as before. The number of RHP roots of $den(GK_o)num(GK_o)$ is the count of

RHP poles and zeros of GK_o. Then

$$winding\ \#\left[\frac{\partial\Gamma}{\partial S}(\omega,S_o)\right] = n - (p_g + z_g + z_k + p_k).$$

Therefore S_o is optimal for Γ if and only if (3) holds.

3. EXAMPLES.

We present three examples illustrating our results.

Example 1.

$$GK_o = \frac{1}{100}\frac{(s+100)}{(s+.01)} \quad where\ G = \frac{s+100}{s+.01},\ K_o = \frac{1}{100}.$$

By Theorems 2 and 3, K_o is not S-optimal or T-optimal because $z_g = p_g = 0$. This conclusion agrees with the fact that the H^∞ optimal values of Λ and Θ are 0 for stable, minimum phase, LTI plants. However, $n = 1$, so K_o is S&T-optimal. Therefore no K exists for which the mixed sensitivity, Ψ, would be improved at every frequency.

Example 2.

$$GK_o = \frac{1}{100}\frac{(s-100)(s+.2)(s-50)}{(s-.01)(s+50)(s-.2)}$$

$$where\ G = \frac{1}{100}\frac{(s-100)(s+.2)}{(s-.01)(s+50)},\quad K_o = \frac{(s-50)}{(s-.2)}.$$

Here $z_g = p_g = z_k = p_k = 1$, so K_o is neither S-optimal nor T-optimal. K_o is not S&T-optimal either since $n = 1$. Performing H^∞ optimization for the weighted mixed sensitivity design Ψ, we found the suboptimal controller

$$K = \frac{-.5s - 38}{s + .05}.$$

With this controller, the norm of Ψ is approximately 1.223, whereas with K_o, the norm of Ψ is approximately 1.414. The H^∞ optimal norm for Ψ is about 1.18. In figure 4 we show a comparison of the unweighted sensitivity and complementary sensitivity functions for the nominal and improved design.

Example 3.

$$GK_o = \frac{1}{100}\frac{(s-100)(s-50)(s+.2)}{(s-.01)(s-.2)(s+50)}$$

$$where\ G = \frac{1}{100}\frac{(s-100)(s-50)}{(s-.01)(s-.2)}\quad K_o = \frac{(s+.2)}{(s+50)}.$$

Although we have the same loop shape as in example 2, in example 3 $z_g = p_g = 2$ and $z_k = p_k = 0$. Therefore K_o is S-optimal and T-optimal. K_o is also S&T-optimal since $n = 1$. The loop shape GK_o crosses over at 1 rad/sec with a $\frac{1}{s}$ slope. The gain margins are −13.45 dB and 30.27 dB. The phase margin is 63.94 degrees. The closed loop poles have damping > .88 .

In examples 1 and 3 we have weighted mixed sensitivity problems whose H^∞ optimal controllers are low order, stable, and minimum phase. In example 3 it is interresting to note that one obtains the same controller, namely K_o, from performing weighted sensitivity, weighted complementary sensitivity, or mixed sensitivity optimization.

4. PROOF OF A SPECIAL CASE OF THEOREM 4.

We prove the following special case of Helton's result [ref. (4), Theorem 4.1] which applies directly to our problem. Assume that Γ is as given in (8) and our assumptions [1], [2], and [3] hold.

Lemma 1. ***S_o* is optimal for Γ if and only if**

$$\textbf{winding}\ \#\left[\frac{\partial\Gamma}{\partial S}(\omega,S_o)\right] > -(z_g + p_g).$$

Proof of Lemma 1:

Γ has a unique minimum (local and global) over all $S \in E$. Therefore, consider the variable $S = S_o + ez$ where $e \in R^+$ and $z \in RH^\infty$. The Taylor Series for Γ about S_o is

$$\Gamma(S) = \Gamma(S_o) + \frac{\partial\Gamma}{\partial S}(\omega,S_o)(ez) + \frac{\partial\Gamma}{\partial S^*}(\omega,S)(\overline{ez}) + (\ higher\ terms\ ).$$

For $\Gamma(\bar{S})$ to be optimal, where $\bar{S} \neq S_o$, it is necessary that $\Gamma(\bar{S}) < \Gamma(S_o)$ at every frequency $j\omega$ and that $\Gamma(\bar{S})$ be flat across frequencies as was discussed in section 2. We note that

$$\frac{\partial\Gamma}{\partial S}(\omega,S_o)(ez) + \frac{\partial\Gamma}{\partial S^*}(\omega,S)(\overline{ez}) = 2\ real\left[\frac{\partial\Gamma}{\partial S}(\omega,S_o)ez\right]$$

Hence, if a direction z can be found so that $-\frac{\partial\Gamma}{\partial S}(\omega,S_o)z$ stays withing the RHP, ie. is positive definite for all $j\omega$, we can find e small enough so that

$$\Gamma(S_o + ez) < \Gamma(S_o)$$

for all $j\omega$. If no such z can be found, $\Gamma(S_o)$ is the global minimum. To be positive definite for all $j\omega$, $-\frac{\partial\Gamma}{\partial S}(\omega,S_o)\,z$ must have a winding number of zero. $-\frac{\partial\Gamma}{\partial S}(\omega,S_o)\,z$ is a real rational function and so its winding number is the number of its RHP zeros minus the number of its RHP poles.

Because the sensitivity function $S_o + ez$ must be in E, the parameter z must be zero at all the interpolating points given by the locations of the right half plane poles and zeros of the plant G. Rewrite z as $z = \phi h$, where $h \in RH^\infty$ and $\phi \in RH^\infty$ is allpass with zeros at all the interpolating points and nowhere else. If $\frac{\partial\Gamma}{\partial S}(\omega,S_o)\phi$ has at least as many RHP poles as RHP zeros, an h can be found to cancel the excess RHP poles and remove enough phase at each frequency to keep $-\frac{\partial\Gamma}{\partial S}(\omega,S_o)\,\phi h$ in the RHP. One may refer back to section 2 for the derivation of the winding number of $\frac{\partial\Gamma}{\partial S}(\omega,S_o)$. For $-\frac{\partial\Gamma}{\partial S}(\omega,S_o)z$ to be positive definite, we must have

$$winding\ \#\left[\frac{\partial\Gamma}{\partial S}(\omega,S_o)\right] = n-(p_g+z_g+z_k+p_k) > -(p_g+z_g).$$

We note that the result of Lemma 1 can be shown for the norm squared of Λ, and for the norm squared of Θ, in the same manner that we have shown it for Γ.

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NOTATION.

RHP = the open right half plane := $\{s \in C : \mathrm{Re}(s) > 0\}$.
L^∞ = the essentially bounded functions on jR.
H^∞ = the functions in L^∞ which admit an analytic extension to the RHP.
RL^∞ = the real rational functions in L^∞.
RH^∞ = the real rational functions in H^∞.
SISO = Single Input Single Output.
MIMO = Multiple Input Multiple Output.
LTI = Linear Time Invariant.
$\bar{\sigma}(A)$ = the maximum singular value of the matrix A

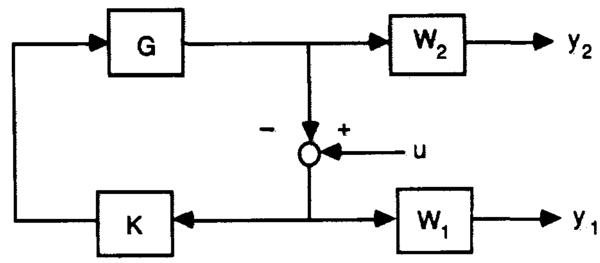

Figure 1 The Weighted Mixed Sensitivity Problem

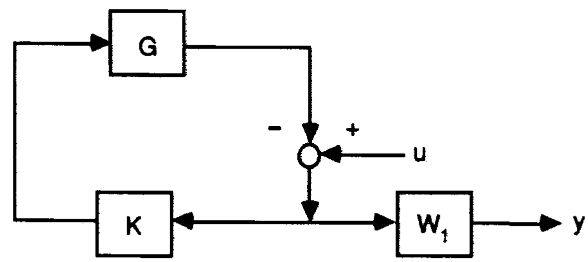

Figure 2 The Weighted Sensitivity Problem

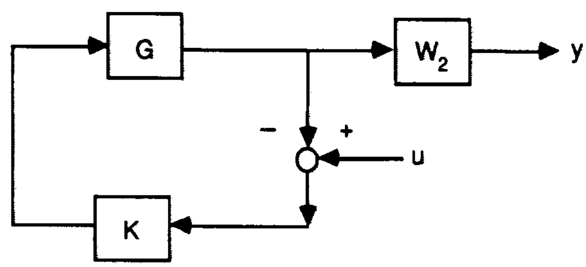

Figure 3 The Weighted Complementary Sensitivity Problem

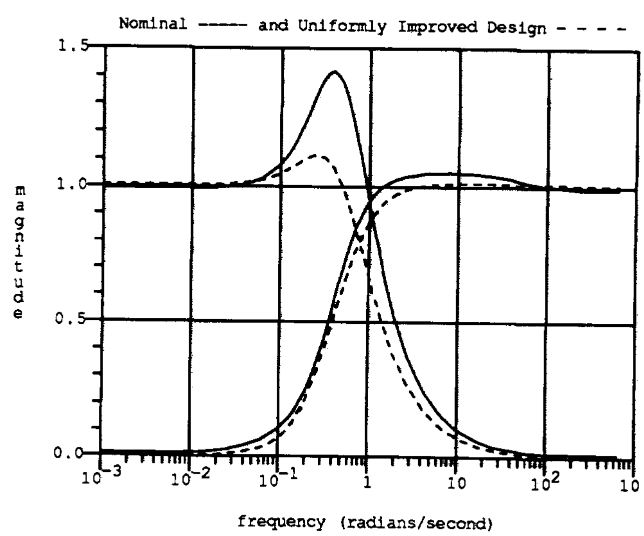

Figure 4 Sensitivity and Complementary Sensitivity for the Designs in Example 2