FUNCTIONS ORTHOGONAL IN THE HERMITIAN SENSE. A NEW APPLICATION OF BASIC NUMBERS

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§1. To find a particular set of functions $H_n(u)$ satisfying the Hermitian relation

$$I_{m,n} \equiv \int_{-\infty}^{\infty} e^{-1/2x^2} H_m(ix)H_n(-ix)dx = 0 \qquad (1)$$

in which the exponential factor is exp $(-x^2/2)$ as also in (14) we may put $z = e^{iax}$, where a is an arbitary positive constant and assume that $H_n(ix)$ is a polynomial of the nth degree in z with real coefficients. If $q = \exp(-a^2)$ and use is made of Euler's generalized binomial coefficient which Jensen[1] denotes by the symbol (n, r) and R. Tambs Lyche[2] by the symbol $\begin{bmatrix} n \\ r \end{bmatrix}_q$, or $\begin{bmatrix} n \\ r \end{bmatrix}$, we have, with F. H. Jackson's notation $[n]$ for a basic number[3]

$$[n](1-q) = (1-q^n), \quad (n,r)[1][2]\dots[r] = [n][n-1]\dots [n-r+1] \quad (n,0) = 1. \qquad (2)$$

If, now

$$S_m^{(n)} = \sum_{r=0}^{n} (-)^r (n,r) q^{r(m-n) + \frac{1}{2}r(r+1)} \qquad (3)$$

we readily find that

$$S_{m+1}^{(n)} - S_m^{(n)} = (1-q^n)q^{m-n+1} S_m^{(n-1)}. \qquad (4)$$

Hence

$$S_m^{(n)} = 0 \qquad m < n$$

$$S_n^{(n)} = (1-q^n)\, S_{n-1}^{(n-1)} = (1-q)(1-q^2)\dots(1-q^n). \qquad (5)$$

On account of these relations we may write

$$H_n(ix) = C_n \left\{z^n - (n,1)q^{1/2}z^{n-1} + (n,2)qz^{n-2}\dots + (-)^n(m,n)q^{1/2n}\right\}, \qquad (6)$$

where C_n is a constant which will be chosen so that $I_{nn} = 1$. The appropriate value of C_n is given by the equation

$$C_n^2(2\pi)^{1/2}(1-q)(1-q)^2\dots(1-q^n) = 1. \qquad (7)$$

To expand z^m in a series of functions $H_n(ix)$ we make use of the inversion formula

$$X_n = \sum_{r=0}^{n} (-)^r (n, r) q^{1/2r} Y_{n-r}$$
$$Y_m = \sum_{s=0}^{m} (m, s) q^{1/2s^2} X_{m-s} \tag{8}$$

which is easily verified with the aid of the relations (5). The relation to be established is, in fact

$$X_n = \sum_{r=0}^{n} (-)^r (n, r) q^{1/2r} \sum_{k=0}^{n-r} (n - r, n - r - k) q^{1/2(n-r-k)^2} X_k. \tag{9}$$

Changing the order of summation we have to prove that

$$0 = \sum_{r=0}^{n-k} (-)^r (n\ r)(n - r, n - r - k) q^{kr + 1/2r(r+1)} \quad k \neq n. \tag{10}$$

Now

$$(n, r)(n - r, n - r - k) = (n, k)(n - k, r) \tag{11}$$

and so the relation to be established reduces to a particular case of (5). We obtain, then, the relation

$$z^m = \sum_{r=0}^{m} (m, r) q^{1/2r^2} C_{m-r}^{-1} H_{m-r}(ix) \tag{12}$$

which may be used to transform a power series in z into a series of type

$$f(x) = \sum_{n=0}^{\infty} A_n H_n(ix), \tag{13}$$

in which, under suitable conditions

$$A_n = \int_{-\infty}^{\infty} e^{-1/2x^2} H_n(-ix) f(x) dx. \tag{14}$$

Putting

$$f(x) = \sum_{m=0}^{\infty} B_m z^m \tag{15}$$

the equations of transformation are

$$B_m = \sum_{r=0}^{\infty} (-)^r (m + r, r) q^{1/2r} C_{m+r} A_{m+r}$$
$$A_n = \sum_{m=0}^{\infty} B_m \int_{-\infty}^{\infty} e^{-1/2x^2} H_n(-ix) z^m dx \tag{16}$$
$$= C_n^{-1} \sum_{s=0}^{\infty} (n + s, s) q^{1/2s^2} B_{n+s}$$

and these suggest the existence of a second inversion formula

$$X_m = \sum_{r=0}^{\infty} (-)^r (m + r, r) q^{1/2r} \, Y_{m+r} \tag{17}$$

$$Y_n = \sum_{s=0}^{\infty} (n + s, s) q^{1/2s^2} X_{n+s}.$$

A particular set of functions satisfying the Hermitian relation

$$J_{m,n} = \int_{-\infty}^{\infty} \text{Sech}^2 \left(\frac{1}{2} \pi x\right) F_m(ix) F_n(-ix) dx = 0 \quad m \neq n \tag{18}$$

is obtained by defining $F_n(u)$ to be a polynomial of the nth degree in u such that

$$F_n\left(\frac{d}{dt}\right) \text{Sech } t = \text{Sech } t \, P_n (\tanh t) \tag{19}$$

where $P_n(t)$ is the Legrendre polynomial. It has already been shown[4] that

$$F_m(ix) \text{ Sech} \left(\frac{1}{2} \pi x\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-ixz} \text{ Sech } z. \; P_m (\tanh z) \, dz \tag{20}$$

$$\therefore J_{m,n} = \frac{1}{\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \text{Sech } t. \; P_m (\tanh t) \, dt \int_{-\infty}^{\infty} e^{ix(t-z)} P_m(\tanh z) \text{Sech } z.dz$$

Changing the order of integration with respect to x and t, which is permissible since

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} e^{ixt} F_n(ix) \text{ Sech} \left(\frac{1}{2} \pi x\right) \text{Sech } t \, P_n (\tanh t) \, dt$$

is absolutely convergent, we have

$$\begin{aligned} J_{m,n} &= \frac{2}{\pi} \int_{-\infty}^{\infty} \text{Sech}^2 t. \; P_m (\tanh t) \, P_n (\tanh t) \, dt. \\ &= 0 \qquad m \neq n. \qquad (21) \\ &= \frac{4}{\pi} \frac{1}{2n + 1} \qquad m = n. \end{aligned}$$

[1] J. L. W. V. Jensen, *Nyt Tidsskr Math.*, **29**, 29 (1918). Jensen remarks that this generalized binomial coefficient plays an important part in Gauss' memoir "Summatie quarundam serierum singularium" (1811), in which the following relation is obtained

$$\begin{aligned} \sum_{r=0}^{n} (-)^r (n, r) &= (1 - q^n)(1 - q^{n-2}) \ldots (1 - q) \qquad n \text{ odd} \\ &= 0 \qquad n \text{ even} \end{aligned}$$

Jensen discusses the properties of the sums

$$\sum_{r=0}^{n} (-)^r q^{rs} (n, r), \qquad \sum_{r=0}^{n} q^{r(s + 1/2)} (n, r).$$

[2] R. Tambs Lyche, *Bull. Société mathématique France,* **55,** 102 (1927); *Compt. rend. t.* **186,** 1810 (1928); *Avhandlingar* Oslo (1928) No. 6; Forhandlingar Norske Videnskaber, Selskab 1, No. 35, 3 p. In the first paper Tambs Lyche shows that if

$$\phi_1(z) \equiv \phi(z) = qz + a_2z^2 + \dots a_pz^p + \dots$$

the nth iterated function is expressed formally by the series

$$\phi_n(z) = \phi[\phi_{n-1}(z)] = q^n z + a_2^{(n)}z^2 + \dots a_p^{(n)}z^p + \dots$$

where

$$a_p^{(n)} = \sum_{r=1}^{p-1} (-)^{p-r-1}\, q^{n-r+1/2(p-r)p-r-1}\, (n, r)(n-r-1, p-r-1)a_p^{(r)}$$

The notation $\begin{bmatrix} n \\ r \end{bmatrix}$ is preferable to either (n, r) or $\begin{bmatrix} n \\ r \end{bmatrix}_q$ but we use (n, r) here for convenience in printing. In 1897 F. H. Jackson used the notation $\begin{bmatrix} n \\ r \end{bmatrix}$ for the generalized binomial coefficient derived from p, where $p > 1$ but he also used it in a more general sense. See *Proc. London Math. Soc.* **28,** 475 (1897). The notation (n) for $1 - q^n$ was suggested by A. Cayley and adopted by P. A. MacMahon. *Proc. London Math. Soc.,* ser. 2, **15,** 314 (1916). L. J. Rogers, *Ibid.,* **16,** 315 (1917), uses the notation q_n for the same quantity and also writes $q_n!$ for the product $q_1\, q_2 \dots q_n$.

[3] F. H. Jackson, *Proc. London Math. Soc.,* ser. 2, **1,** 63 (1904); **1,** 361 (1904); **2,** 192 (1904); **3,** 1 (1905); *Trans. Roy. Soc. Edinburgh,* **41,** 1, 105, 399 (1904–5); *Proc. Edinburgh Math. Soc.,* **22,** 80 (1904); *Proc. Roy. Soc. Lond.,* **76,** 127 (1905); *Messenger Math.,* ser. 2, **37,** 123 (1907); **38,** 57, 62 (1908); **39,** 26, 145 (1910); **40,** 92 (1910); **47,** 57 (1917); **50,** 101 (1920); **57,** 169 (1928); *Rendiconti Palermo,* **29,** 340 (1910); *Amer. Jour. Math.,* **32,** 305 (1910); *Proc. Roy. Soc. Edinburgh,* **30,** 378 (1910). Many of these papers deal with a generalization of the binomial theorem and generalizations of the functions of Legendre and Bessel all of which are more or less closely connected with the theory of the q-hypergeometric series. This series has been studied also by C. G. J. Jacobi, *Jour. Math.,* **32,** 197 (1846); E. Heine, *Ibid.,* **32,** 210 (1846); **34,** 285 (1847); **39,** 288 (1850); J. Thomae, *Ibid.,* **70,** 258 (1869); G. N. Watson, Cambridge, *Phil. Trans.,* **21,** 281 (1910). A new use of the generalized binomial coefficients in the theory of numbers has been found recently by I. Schur, *Sitzungsberichte preuss. Akad. Wiss.,* 145 (1933).

[4] H. Bateman, *Tôhoku Math. Jour.,* **37,** 23 (1933).