$$dx^i = \frac{\varphi_0 X^i}{\varphi_\alpha X^\alpha} \tag{5.4}$$

The hyperplane at infinity for these affine coördinates dx^i is the intersection of $\varphi_\sigma Z^\sigma = 0$ with $Z^\infty = 0$. The result of combining (5.3) with (5.4) is

$$dx^i = \frac{\varphi^\infty Z^i - Z^\infty \varphi^i}{\varphi_\sigma Z^\sigma} \tag{5.5}$$

and these equations are inverse to (5.2).

[1] O. Veblen, *Projektive Relativitätstheorie*, Berlin, 1933.

[2] This matrix is, of course, a modification of the one introduced by T. Y. Thomas in these PROCEEDINGS, 12, 356 (1926).

[3] This pseudo group may be discussed in much the same way as what I called the enlarged conformal group in these PROCEEDINGS, 14, 738 (1928).

[4] J. L. Vanderslice, these PROCEEDINGS, 20, 674 (1934).

[5] These differ slightly from the formulas of T. Y. Thomas (see reference in Note 2) because of the difference in our matrices of transformation and because the second covariant index of our connection runs from 1 to n instead of from 0 to n as with Thomas. I am indebted to Dr. Vanderslice for making the computation.

[6] The references are given in Vanderslice's paper (Note 4 above).

SOME EXPANSIONS ASSOCIATED WITH BESSEL FUNCTIONS

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1. *An Expansion for the Product of Two Bessel Functions.*—1.1. An expansion for the product of two Bessel functions obtained by one of us[1] led to the discovery of a different expansion for the said product multiplied by the leading terms in the power series for the Bessel functions. Two proofs of this second expansion are given here. The first depends upon the fact that the function $V = J_a(u)u^a J_c(v)v^c$ satisfies the equation

$$u\frac{\partial^2 V}{\partial u^2} + (1-2a)\frac{\partial V}{\partial u} + uV + v\frac{\partial^2 V}{\partial v^2} + (1-2c)\frac{\partial V}{\partial v} + vV = 0$$

which is transformed by the substitutions $u = zx$, $v = z(1-x)$ into

$$z^2\frac{\partial^2 V}{\partial z^2} + (3-2r)z\frac{\partial V}{\partial z} + [z^2 - (2n+1)(2r+2n-1)]V$$

$$+ x(1-x)\frac{\partial^2 V}{\partial x^2} + [2s+2+(2r-3)x]\frac{\partial V}{\partial x} + (2n+1)(2r+2n-1)V = 0$$

where $r = a + c + {}^1/_2$, $s = -a - {}^1/_2$ and n is an arbitrary constant. A particular solution of the last equation is

$$V = V_n = z^{r-1} J_{r+2n} (z) D_x^{2n} [x^{2a+2n} (1 - x)^{2c+2n}].$$

If we assume that the former solution can be expressed as the sum of terms of type $A_n V_n$, where n runs from 0 to ∞, the constant A_n may be found by letting x approach zero and comparing our assumed series with Sonine's expansion[2]

$$z^{-s}J_{r+s}(z) = 2^{-s} \sum_{n=0}^{\infty} \binom{s}{n} \frac{(r + 2n)\, \Gamma\, (r + n)}{\Pi\, (r + s + n)} J_{r+2n} (z).$$

It is thus found that

$$J_a(u)u^a J_c(v)v^c = \sum_{n=0}^{\infty} A_n V_n$$

where

$$A_n = (-)^n \frac{(r + 2n)\, \Pi\, (r + n - 1)}{2^{2n-1/2}\, \Pi\, (a + n)\, \Pi(c + n) n!}.$$

1.2. An independent proof of the validity of the expansion is based upon the equation[1]

$$J_a(zx)J_c(z - zx) = x^a(1 - x)^c A \int_P J_{a+c}(zw)w^{-a-c}t^{-a-1} (1 - t)^{-c-1}dt$$

where

$$4\pi A = i + \cot \pi(a + c),\ w^2 = \frac{x^2}{t} + \frac{(1 - x)^2}{1 - t}$$

and the contour P is the double loop $(1+, 0+, 1-, 0-)$ of the Pochhammer type. The arguments of t and $(1 - t)$ are zero at the starting point which lies on the real axis between the points $t = 0$ and $t = 1$. By using the expansion[2]

$$w^{-a-c}J_{a+c}(zw) = (z/2)^{a+c-b} \sum_{n=0}^{\infty} \frac{\Pi\, (b + n - 1)(b + 2n)}{n!\, \Pi(a + c)}$$

$$F(b + n, -n;\ a + c + 1;\ w^2)J_{b+2n}(z) \qquad (1)$$

which is valid when b and $a + c$ are not negative integers, and integrating termwise we obtain terms involving integrals of type

$$I = A \int_P F(b + n, -n;\ a + c + 1;\ w^2)t^{-a-1}(1 - t)^{-c-1}dt.$$

The series may be shown to converge uniformly with respect to t when b is not an integer and so the integration of the series is permissible. Indeed, when n is a positive integer,

$$\Gamma(b+n)F(b+n, -n; a+c+1; w^2)/[\Pi(a+c)\Gamma(b+n-a-c)] = A' \int_P (\tau - w^2)^n d\tau/[\tau^{a+c+n+1}(1-\tau)^{b+n-a-c}] \quad (2)$$

where $A' = i + \cot \pi b$, as may be verified by expanding $(\tau - w^2)^n$. This relation defines the hypergeometric function for all values of w. Choosing the loops of the contour P to be circles of unit diameter with centers at 0, 1, respectively, we see that the total length of P is 4π and that on P we have the inequalities

$$|\tau - w^2| \leq \frac{3}{2} + |w^2|, \; |\tau^{-a-c-n-1}| \leq 2^{n+1}B_1, \; |(1-\tau)^{a+c-b-n}| \leq 2^n B_2$$

where $\log B_1 = |R(a+c)| \log 2 + 2\pi|I(a+c)|$, $\log B_2 = |R(b-a-c)| \log 2 + 2\pi|I(b-a-c)|$. Thus the modulus of the right-hand side of (2) is less than

$$|i + \cot(\pi b)| \left(\frac{3}{2} + |w^2|\right)^n 2^{2n+1}B_1B_2$$

and so the modulus of the nth term in (1) is less than

$$\left|(b+2n)\Gamma(b+n-a-c)\,(i+\cot(\pi b))\left(\frac{3}{2} + |w^2|\right)^n 2^{2n+1}B_1B_2J_{b+2n}(z)/n!\right|$$

As $n \longrightarrow \infty$, $\Gamma(b+2n+1)J_{b+2n}(z) \longrightarrow (z/2)^{b+2n}$ and so the series converges uniformly for all finite values of w^2 unless b is an integer. When b is an integer further investigation is needed to establish the uniform convergence but this may be avoided by excluding this case for the present and then returning later to an examination of the final result by the method of continuity.

1.3. The integral I may be expressed as a hypergeometric function of x when $b = a + c + {}^1/_2$. In order to carry through the proof we assume for the present that neither $a + c + {}^1/_2$ nor $a + c$ is an integer. We first use Kummer's relation

$$F(p, q; p + q + {}^1/_2; 4y - 4y^2) = F(2p, 2q; p + q + {}^1/_2; y)$$

wherein $p = a + c + n + {}^1/_2$, $q = -n$, $4y(1-y) = w^2$, $2y = 1 + i(t-x)t^{-1/2}(1-t)^{-1/2}$. The usual restrictions $|2y| < 1$, $|4y(1-y)| < 1$ are not necessary here since the series terminate and each side is merely a polynomial in y. The resulting series is then arranged in ascending powers of x and the hypergeometric function in the integrand becomes

$$\sum_{r=0}^{2n} \frac{(2a + 2c + 2n + 1)_r\, (-2n)_r}{r!(a + c + 1)_r} \left(\frac{-ix}{2\sqrt{t(1 - t)}}\right)^r A_r(t)$$

where

$$A_r(t) = F\left(2a + 2c + 2n + r + 1, r - 2n; a + c + r + 1; \frac{1}{2} + \frac{i}{2}\sqrt{t/(1-t)}\right)$$

$$= (-)^n i^r [t/(1 - t)]^{n - 1/2r} F\left(\frac{1}{2}r - n, \frac{1}{2}r + \frac{1}{2} - n;\ a + c + r + 1;\ 1/t\right).$$

The last step may be verified by using Kummer's formula combined with the relation between hypergeometric functions of arguments y and $y/(y - 1)$. It is convenient to perform the transformations at the point $t = N \exp i\pi$, $\arg (1 - t) = 0$, where N is a large positive constant, and then extend the result to the contour P by analytical continuation. When written in full the expression for I is

$$I=(-)^n A \int_P dt \sum_{r=0}^{2n} \sum_{s=0}^{m} \frac{(2a+2c+2n+1)_r\,(-2n)_r\,(^1/_2r-n)_s\,(^1/_2r+{}^1/_2-n)_s x^r}{r!(a+c+1)_r 2^r s!(a+c+r+1)_s t^{a+r-n+s+1}(1-t)^{c+n+1}}$$

where $m \leq (n - {}^1/_2r)$. Using now Pochhammer's extension of the Eulerian integral of the first kind and summing the resultant hypergeometric series in s of argument unity by means of the formula of Gauss, we have finally

$$I = (-)^n \frac{\Pi(a + c)\Pi(a + n - {}^1/_2)}{\Pi(c + n)} \sum_{r=0}^{2n} \frac{(2a + 2c + 2n + 1)_r\,(-2n)_r}{2^r\Pi(a + {}^1/_2r)\,\Pi(a + {}^1/_2r - {}^1/_2)} \frac{x^r}{r!} =$$

$$(-)^n \frac{\Pi(a + c)\Pi(2a + 2n)}{2^{2n}\Pi(a+n)\Pi(c + n)\Pi(2a)} F(2a + 2c + 2n + 1,\ -2n;\ 2a + 1;\ x).$$

When this hypergeometric function is expressed as a Jacobi polynomial and the corresponding value of I is placed in the infinite series for $J_a(zx)\, J_c(z - zx)$ we obtain the expansion given in section 1.1

$$x^a(1 - x)^c J_a(zx)\, J_c(z - zx) = \sum_{n=0}^{\infty} (-)^n \frac{(a+c+2n+{}^1/_2)\ \Pi(a+c+n-{}^1/_2)}{2^{2n-1/2}\,\Pi(a + n)\,\Pi(c + n)n!}$$

$$z^{-1/2}J_{a+c+2n+1/2}(z)D_x^{2n}[x^{2a+2n}(1 - x)^{2c+2n}].$$

The restriction that $a + c$ be not an integer and that $a + c + {}^1/_2$ be not a positive integer may be removed by the use of continuity but the restriction that $a + c + {}^1/_2$ be not a negative integer is necessary. Using Lommel's expansion to compare the expansion for $J_a(zx)\, J_c(z - zx)$, mentioned in the introduction, with the expansion developed here we obtain the equation

$$F(-2k, a + c + 1;\ a - k + 1;\ x) = \sum_{n=0}^{k} C_n F(-2n, 2a + 2c + 2n + 1;\ 2a + 1;\ x)$$

where

$$C_n = \frac{(-)^{n+k}(a+c+2n+{}^1/_2)\Pi(2a+2c+2n)\Pi(a+n-{}^1/_2)\Pi(a-k)\Pi(c+k)(2k)!}{2^{2k+2c}\Pi(a + c + k + n + {}^1/_2)\Pi(c + n)\Pi(2a)\Pi(a + c)(2n)!\,(k-n)!}$$

2. *Generating Functions for Bessel's Functions of Non-Integral Order.*—If $R(\nu) > 0$,

$$e^{-x\tau}\int_0^1 \exp\,((y - yt)/\tau)t^{\nu-1}\,dt = e^{-x\tau}\sum_{r=0}^{\infty} y^r\tau^{-r}/(\nu)_{r+1} = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}(-x)^s\tau^{s-r}y^r/[s!(\nu)_{r+1}].$$

Since the double series is absolutely convergent we may write it in the form

$$\sum_{n=0}^{\infty}(y/\tau)^n\sum_{s=0}^{\infty}(-xy)^s/[s!(\nu)_{n+s+1}] + \sum_{n=1}^{\infty}(-x\tau)^n\sum_{r=0}^{\infty}(-xy)^r/[(\nu)_{r+1}(n+r)!]$$

where in the first series $n = r - s$ and in the second series $n = s - r$, consequently when $R(\nu) > 0$

$$\frac{e^{-x\tau}}{\Gamma(\nu)}\int_0^1 \exp\,((y-yt)/\tau)\,t^{\nu-1}\,dt = \sum_{n=0}^{\infty}(y/\tau)^n(xy)^{-n/2-\nu/2}\,J_{n+\nu}(2\sqrt{xy}) + \sum_{n=1}^{\infty}(-x\tau)^n F(1;\ n + 1, \nu + 1;\ -xy)/[n!\Gamma(\nu + 1)].$$

If we multiply both sides of this equation by $\exp\,(-ux)\,x^{m-1}$, where $R(u)$, $R(m)$, $R(u + \tau) > 0$, and integrate between $x = 0$ and $x = \infty$ we obtain a convergent integral on the left. Moreover, when $|\tau| < |u|$ the series on the right may be integrated term by term and the factor $(u + \tau)^m$ arising from integration on the left may be expanded by the binomial theorem. Upon arranging the left-hand side as a Laurent series in τ and comparing it with the expression on the right we find

$$\frac{y^{\frac{n+\nu}{2}}\Gamma(m)}{u^m\Gamma(n + \nu + 1)}F(m;\ n+\nu+1;\ -y/u) = \int_0^{\infty} e^{-xu}x^{m-\frac{n+\nu}{2}-1}\,J_{n+\nu}\,(2\sqrt{xy})dx,$$

$$I^m_{n,\nu}\,(u) \equiv \int_0^{\infty} e^{-xu}x^{m+n-1}F(1;\ n + 1, \nu + 1;\ -xy)dx$$

$$= u^{-m-n}\Gamma(m+n)F(1, m+n;\; n+1, \nu+1;\; -y/u). \tag{3}$$

The first integral is well known and the second may be readily verified when $R(u) > 0$, by expanding the hypergeometric function, and integrating termwise.

When $u = 0$ the integral $I^m_{n,\nu}$ converges only if $0 < m + n < 1$, as may be seen by Barnes' asymptotic formula for ${}_1F_2$, assuming y real and positive. If, in addition, $m + n - \nu$ is a positive integer it may be shown that the integral is zero by letting $u \longrightarrow 0$ in the hypergeometric expression for $I^m_{n,\nu}$. The general value of Hardy's type[5] thus obtained is also the actual value since the integral converges. To do this it is convenient to use the reduction formula

$$(m + n - 1)yF(1, m + n;\; n + 1, \nu + 1;\; -y/u) = n\nu u[1 - F(1, m + n - 1;\; n,\nu;\; -y/u)]$$

to obtain

$$yI^m_{n,\nu} = n\nu u^{1-m-n}\Gamma(m + n - 1)F(1, 1 - n, 1 - \nu; 2 - m - n; u/y) + y^{1-n}u^{-m}n!\Gamma(m)(n - \nu - 1)_n F(m;\; \nu - n + 1; - y/u) \tag{4}$$

This is merely the result of a formal transformation of the function ${}_2F_2$ and holds without restriction. It is, in fact, a particular case of Barnes' asymptotic formula[4] for $F(a, c;\; u, v;\; z)$ which holds when $|\arg z| < 3\pi/2$. Making use of Kummer's relation

$$F(m;\, \nu - n + 1;\; -y/u) = \exp(-y/u)F(\nu - n - m + 1; \nu - n + 1; y/u)$$

we note that the confluent hypergeometric function on the right is a polynomial when $n + m - \nu$ is a positive integer. The last term in (4) then becomes zero as $u \longrightarrow 0$. If $m + n < 1$ the first term in (4) also tends zero and so in these circumstances the actual value of $I^m_{n,\nu}$ is zero.

This result may be compared with Orr's formula[5]

$$\int_0^\infty x^{m-1}F(a_1, a_2, \ldots, a_n;\, c_1, c_2, \ldots, c_n;\, -x)dx = \Gamma(m)\prod_{r=1}^{n}\frac{\Gamma(c_r)\,\Gamma(a_r - m)}{\Gamma(a_r)\,\Gamma(c_r - m)}$$

($m > a_l$ where a_l is the algebraical least a_r) and with the result of his remark that by putting $x = z/a_1$ and making $a_1 \longrightarrow \infty$, a corresponding formula may be derived for the case in which the number of the a's is less than the number of c's.

Several definite integrals which partake of the nature of generating functions are

$$\int_0^K J_\mu(z\, dn\, u) sn^{2\nu-1}u\, cn^{2\rho-1}u\, dn^{1+\mu}u\, du = \sum_{n=0}^\infty k^{2n}(-)^n\left(\frac{z}{2}\right)^n \frac{B(\nu+n,\rho)}{n!2} J_{\mu-n}(z)$$

$$\int^{K} J_\mu(z\,dn\,u)sn^{2\nu-1}u\,cn^{2\rho-1}u\,dn^{1-\mu}u\,du = \sum_{n=0}^{\infty} k^{2n}\left(\frac{z}{2}\right)^n \frac{B(\nu+n,\rho)}{n!2} J_{\mu+n}(z)$$

$$\int^{1} J_\mu(zx)x^{\mu+1}\exp[{}^1/_2tz(1-x^2)]dx = \sum_{n=0}^{\infty} t^n z^{-1} J_{\mu+n+1}(z)$$

$$\int^{1} J_\mu(zx)x^{\mu+1}\exp[{}^1/_2t(1-x^2)]\,dx = \sum_{n=0}^{\infty} t^n z^{-n-1} J_{\mu+n+1}(z).$$

In the first two expressions the Bessel functions of the first kind (J_ν's) may be replaced throughout by functions of the second kind (Y_ν's); and further, if the factor $(-)^n$ in the summation involving $J_{\mu-n}(z)$ be omitted the functions of the first kind may be replaced throughout by modified functions of the second kind (K_ν's). These equations, which hold for $R(\nu)$, $R(\rho) > 0$, may be verified by replacing $z\,dn\,u$ by $\sqrt{z^2 - z^2k^2sn^2u}$, expanding the Bessel functions in the integrand, using Lommel's expansion, and integrating termwise.

The last two equations, which hold when $R(\mu) > -1$ and which may be verified by using[6]

$$J_{\mu+n+1}(z) = \frac{z^{n+1}}{2^n n!}\int_0^1 J_\mu(zx)x^{\mu+1}(1-x^2)^n dx,$$

have probably been noticed before. The analogous series $\sum_{n=0}^{\infty} t^n z^n J_{n+\nu}(z)/n!$ has already been summed by W. Kapteyn[7] and some other Dutch mathematicians.

[1] S. O. Rice, "On Contour Integrals for the Product of Two Bessel Functions," *Quart. Jour. Math.*

[2] G. N. Watson, *Theory of Bessel Functions*, §5.21.

[3] *Ibid.*, §5.23.

[4] E. W. Barnes, *Trans. Roy. Soc. Lond.*, **206**, Part XI, 295 (1906).

[5] G. H. Hardy, *Trans. Camb. Phil. Soc.*, **21**, 13 (1908); W. McF. Orr, *Ibid.*, **17**, 171 (1898); **19**, 151 (1900).

[6] G. N. Watson, l. c., §12.11.

[7] W. Kapteyn, *Vraagstuk*, **132**, Solutions by G. R. Boogaerdt't, S. C. van Veen and others. *Wiskundige Opgaven*, **14**, 269–271 (1928).

SIMIMYS, A NEW NAME TO REPLACE EUMYSOPS WILSON, PREOCCUPIED.—A CORRECTION

BY ROBERT W. WILSON

A description of a new genus of cricetine-like rodent was published in the January, 1935, issue of the PROCEEDINGS of the National Academy of