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Stabilization of $\mathcal{LFT}$ Systems

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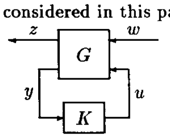
Abstract

The problem of parametrizing all stabilizing controllers for general LFT systems is studied. The LFT systems can be variously interpreted as multidimensional systems or uncertain systems, and the controller is allowed to have the same dependence on the frequency/uncertainty structure as plant. In the multidimensional systems case, this means the controller is allowed dynamic feedback, while the uncertain system case can be given a "gain scheduling" interpretation. Both μ and $\mathcal{Q}$ stability are considered, altough the latter is emphasized. In both cases, the output feedback problem is reduced by a separation argument to two simpler problems, involving the dual problems of full information (FI) and full control (FC). In the case of $\mathcal{Q}$ stability, the FI and FC stabilization problems can be characterized completely in terms of Linear Matrix Inequalities (LMIs). In the standard 1D system case with no uncertainty, the results in this paper reduce to the standard Youla parametrization, although the development here appears to be much simpler, and does not require coprime factorizations.

1 Introduction

The basic block diagram considered in this paper is



where G is the generalized plant with two sets of inputs: the exogenous inputs w and the control inputs u, and two sets of outputs: the measured outputs y and the regulated outputs z. The control problem in this setup is to design feedback controller K such that the closed loop structure is stabilized in some sense and the signal z is specified (cf. Doyle et al, 91; Glover and Doyle, 1989 and Zhou et al, 1990).

In this paper we are only concerned with the basic stabilization problem, so we will generally ignore z and w and focus on G as an LFT on Δ with input u and output y:

$$G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta\right).$$

The frequency/uncertainty structure Δ is in a set $\boldsymbol{\Delta} \in \mathbf{C}^{n \times n}$. For concreteness, we will assume that

$$\boldsymbol{\Delta} = \{\text{diag } [\delta_1 I_{q_1}, \ldots, \delta_r I_{q_r}] : \delta_i \in \mathbf{C}\}$$

$$\mathbf{B}\boldsymbol{\Delta} = \{\Delta \in \boldsymbol{\Delta} : \bar{\sigma}(\Delta) \leq 1\}$$

as in Doyle et al(1991). This notation is a direct generalization of the now standard notation for state-space realizations of transfer functions. One of the advantages of the use of LFTs with this notation is that it facilitates manipulation using state-space-like machinery. Thus we often refer to "state" and "state transformations" even when this terminology does not, strictly speaking, apply, because the meaning is always clear from context.

The aim of this paper is to apply the machinery from Doyle et al(1989 and 1991) to parametrize all stabilizing controllers for general *LFT* systems. In this setting, *both G and K* are LFTs on Δ, which may be interpreted in a number of ways. One is to view the Δ as transform variables in a multidimensional system. We then produce all stabilizing controllers for such systems. A more useful interpretation, as in Doyle et al(1991), is to view one part of Δ, (eg. $\delta_1 = 1/z$) as the transform variable in an uncertain system with the remaining part of Δ viewed as norm-bounded perturbations. The results in this paper could then be given a "gain-scheduled" interpretation, as the controllers depend on the same perturbations as does the plant. The terms frequency and uncertainty structures will be used interchangeably.

We will focus on $\mathcal{Q}$-stabilization under this general setting because the conditions for $\mathcal{Q}$-stability, stabilizability and detectability can be elegantly characterized using LMIs. Except for these LMI results, all the arguments and procedures can be applied to μ-stabilization case as well, although the set of controllers in the $\mathcal{Q}$ and μ case and their interpretations would be different.

The construction of stabilizing controllers for the *output feedback (OF)* problem is done via a sequence of special problems: *full information (FI)* problem, *disturbance feedforward (DF)* problem, *full control (FC)* problem and *output estimation (OE)* problem, together with a separation argument. These special problems are also of importance in their own right. The parameterization of all stabilizing controllers was first introduced for the one-dimensional case by Youla et al (1976) and extended to the multi-dimensional case by Guiver and Bose(1985). Both used the *coprime factorization* technique. In this paper we consider the general LFT setting without the need to introduce coprime factorizations.

In section 2 some notions related to $\mathcal{Q}$-(μ-)stability are defined and characterized. In section 3 we will examine the $\mathcal{Q}$-stabilizations of different special problems, the OF problem is solved via separation arguments. In section 4 the $\mathcal{Q}$-stabilizing controllers for OF problem will be characterized.

A longer version of this paper is available (Lu et al, 1991).

2 LFT Systems, Stabilizability and Detectability

2.1 Linear Systems and Stabilities

Consider $G(\Delta)$ and $\Delta \in \boldsymbol{\Delta}$ from above as well as the *commutative matrix set* $\mathcal{D}$ of $\boldsymbol{\Delta}$ defined as

$$\mathcal{D} = \{D \in \mathbf{C}^{n \times n} : D\Delta = \Delta D, \det[D] \neq 0\}.$$

The "state" variable transformation $x \mapsto x' := Tx$ is said to be *admissible* if the transformation matrix $T \in \mathcal{D}$. Then the corresponding "realization" is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \mapsto \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$$

Note that $G(\Delta)$ for $\Delta \in \boldsymbol{\Delta}$ is unchanged by the transformation, as are the stability properties. Because we will make frequent use of this fact,

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we will state it as a theorem. The proof follows immediately from the definitions in Doyle et al(1991)

Theorem 1 *The μ-stability and $\mathcal{Q}$-stability of LFT systems are invariant under admissible state variable transformations.*

Another important structural property of LFT systems is expressed in the following theorem.

Theorem 2 *Let A_1 and A_2 be two system matrices with corresponding uncertainty structures Δ_1 and Δ_2 respectively. Then the system $\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$ with any compatible dimensioned matrix A_{12} is $\mathcal{Q}$ (or μ)-stable with respect to structure $\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ if and only if A_1 and A_2 are $\mathcal{Q}$ (or μ)-stable.*

Note that in this theorem, the frequency structures Δ_1 and Δ_2 may or may not depend on each other. We can easily sketch the proof of this theorem. The state transformation matrix $T = \begin{bmatrix} dI & 0 \\ 0 & I \end{bmatrix}$ is admissible. By conducting this transformation, the transformed system matrix $TAT^{-1} = \begin{bmatrix} A_1 & dA_{12} \\ 0 & A_2 \end{bmatrix}$ tends to $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ as d tends to 0. By continuity, the stability of the resulting system is guaranteed by the stability of A_1 and A_2. So A is stable since the admissible state transformation does not change stability. This argument holds for both μ and $\mathcal{Q}$. This theorem also implies that a cascade system is $\mathcal{Q}$(or μ)-stable if and only if each subsystem is $\mathcal{Q}$(or μ)-stable.

2.2 Stabilizability and Detectability

The general stabilization problem is to design a (possibly dynamical) output feedback controller $K(\Delta) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ with frequency structure Δ_0 such that the feedback system with $G(\Delta)$ is μ-stable or $\mathcal{Q}$-stable with respect to the induced new frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$. Here the structure Δ_0 depends on Δ, that is, it is made up of possibly multiple copies of the elements ot Δ.

Now consider the following two special structures,

$$G_{SF}(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right] \qquad G_{OI}(\Delta) = \left[\begin{array}{c|c} A & I \\ \hline C & 0 \end{array}\right].$$

where the frequency structures in both cases are the same as the one for $G(\Delta)$.

We give following definitions:

Definition 1 *The system $G(\Delta)$ with frequency structure Δ is μ-stabilizable(or $\mathcal{Q}$-stabilizable) if there exists a dynamical controller for the corresponding system $G_{SF}(\Delta)$:*

$$K(\Delta) = \mathcal{F}_u(F, \Delta_0) = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right]$$

such that the closed loop system is μ-stable (or $\mathcal{Q}$-stable) with respect to the induced frequency structure.

We can characterize this property by following fact.

Lemma 1 *The given system is μ-stabilizable(or $\mathcal{Q}$-stabilizable), i.e. its corresponding system G_{SF} can be μ-stabilized(or $\mathcal{Q}$-stabilized) by some $K(\Delta) = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right]$ with frequency structure Δ_0 related to Δ if and only if the augmented system of $G_{SF}(\Delta)$*

$$G_a(\Delta) = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]$$

can be μ-stabilized(or $\mathcal{Q}$-stabilized) by static feedback $F = \begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix}$ with respect to frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$.

We can also define detectability in the obvious (dual) way.

2.3 Characterizations of $\mathcal{Q}$-Stabilizability and $\mathcal{Q}$-Detectability

Assume that $\text{rank}(B) = p \leq n$ and $\text{rank}(C) = q \leq n$ where p is the dimension of u and q is the dimension of y.

Proposition 1 *Consider the system $G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]$ with frequency structure Δ and $\text{rank}(B) = p < n$. Assume $B_\perp \in \mathbf{R}^{n\times(n-p)}$ is such that $B^* B_\perp = 0$ and $\begin{bmatrix} B & B_\perp \end{bmatrix}$ is invertible. Then there exists a static feedback F such that the closed loop system matrix $A + BF$ is $\mathcal{Q}$-stable with respect to frequency structure Δ if and only if there exists a matrix $P \in \mathcal{D}$ with $P = P^* > 0$ such that*

$$B_\perp^* APA^* B_\perp - B_\perp^* PB_\perp < 0.$$

Moreover, if there exists P satisfied above requirements, then state feedback matrix $F = -(B^* P^{-1} B)^{-1} B^* P^{-1} A$ $\mathcal{Q}$-stabilizes the given system.

The proof of this proposition needs the following lemma:

Lemma 2 *Consider the pair $(A, B) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p}$ and $\text{rank}(B) = p < n$. Let $B_\perp \in \mathbf{R}^{n\times(n-p)}$ and $B_0 \in \mathbf{R}^{p\times n}$ are such that $B_\perp^* B = 0$ and $\begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ is unitary. Then*

$$\inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(A + BF) = \bar{\sigma}(B_\perp^* A)$$

and the infimum is attained by $F = -(B_0^* B)^{-1} B_0^* A$.

Proof. Since $U := \begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ is unitary, then

$$\inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(A + BF) = \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(U^*(A + BF))$$

$$= \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma} \begin{bmatrix} B_0^* A + B_0^* BF \\ B_\perp^* A \end{bmatrix} = \bar{\sigma} \begin{bmatrix} 0 \\ B_\perp^* A \end{bmatrix} = \bar{\sigma}(B_\perp^* A).$$

Moreover the infimum is attained by $B_0^* A + B_0^* BF = 0$ or $F = -(B_0^* B)^{-1} B_0^* A$. □

Proof. of Proposition 1 There exists a static feedback F such that the closed loop system matrix $A + BF$ is $\mathcal{Q}$-stable with respect to frequency structure Δ if and only if

$$1 > \inf_{F,D} \bar{\sigma}(D(A + BF)D^{-1}) = \inf_{F,D} \bar{\sigma}(DAD^{-1} + DBFD^{-1})$$

where infimum is obtained over possible $F \in \mathbf{R}^{p\times n}$ and $D \in \mathcal{D}$.

Now take $V_\perp^* = (B_\perp^*(D^* D)^{-1} B_\perp)^{-\frac{1}{2}} B_\perp^* D^{-1}$ then it is easy to check that $V_\perp^* V_\perp = I$ and $V_\perp^*(DB) = 0$.

Now by lemma 2, we have

$$1 > \inf_{F,D} \bar{\sigma}(D(A + BF)D^{-1}) = \inf_{D} \bar{\sigma}(V_\perp^* DAD^{-1})$$

or there exists a $D \in \mathcal{D}$ such that

$$(V_\perp^* DAD^{-1})(V_\perp^* DAD^{-1})^* < I$$

Take $P = (D^* D)^{-1}$ then $P \in \mathcal{D}$ and $P = P^* > 0$, hence we have

$$(B_\perp^* PB_\perp)^{-\frac{1}{2}} B_\perp^* APA^* B_\perp (B_\perp^* PB_\perp)^{-\frac{1}{2}} - I < 0$$

or $B_\perp^* APA^* B_\perp - B_\perp^* PB_\perp < 0$ □

Using the above result we can easily get

Theorem 3 $G(\Delta) = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]$ *is $\mathcal{Q}$-stabilizable if and only if there exists a static feedback matrix F such that the closed loop system matrix $A + BF$ is $\mathcal{Q}$-stable with respect to the same frequency structure.*

Proof. If B is square and of full rank, then the result is trivial. Now we only consider the case where $rank(B) = p < n$.

Assume that the system can be $\mathcal{Q}$-stabilized by a dynamical controller $K(\Delta) = \mathcal{F}_u(F_0, \Delta_0)$ with $F_0 = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ and Δ_0 is related to the system frequency structure Δ. By lemma 1, this is equivalent to the augmented system

$$G_a(\Delta) = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]$$

being $\mathcal{Q}$-stabilized by static feedback $\begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix}$ with respect to the frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$. Now assume the commutative matrix set of Δ_N is $\mathcal{D}_N$, then by the above proposition, there exists a $P_N =: \begin{bmatrix} P & P_1 \\ P_1^T & P_0 \end{bmatrix} \in \mathcal{D}_N$, which is positive definite and $P \in \mathcal{D}$ (positive), such that

$$\begin{bmatrix} B\bot \\ 0 \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} P_N \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} B\bot \\ 0 \end{bmatrix} - \begin{bmatrix} B\bot \\ 0 \end{bmatrix}^* P_N \begin{bmatrix} B\bot \\ 0 \end{bmatrix} < 0$$

i.e. $B_\perp^* APA^* B_\perp - B_\perp^* PB_\perp < 0$ which means that the system can be $\mathcal{Q}$-stabilized by a static feedback matrix via the previous proposition. □

We can also get the dual results for $\mathcal{Q}$-detectability.

3 Stabilization and Special Problems

3.1 Problem Statement, Assumptions, and Special Problems

The stabilization problem is to *find a feedback mapping K such that the closed-loop system is well-posed and stable in some sense.* For the rest of this paper will focus on the $\mathcal{Q}$-stabilization problem, and controllers that $\mathcal{Q}$-stabilize systems will be said to be *admissible*. We call this general synthesis problem *output feedback(OF)*. All of the remaining results, including the proofs, hold for μ-stability as well, and the relationship will be discussed in the conclusions.

Assume $G(\Delta)$ has a realization of the form

$$G(\Delta) = \begin{bmatrix} G_{11}(\Delta) & G_{12}(\Delta) \\ G_{21}(\Delta) & G_{22}(\Delta) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

which has compatible dimensions with the related physical variables. Assume (A, B_2) *is $\mathcal{Q}$-stabilizable and* (C_2, A) *is $\mathcal{Q}$-detectable.*

In addition, let the state-space realization of $K(\Delta)$ be

$$K(\Delta) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

with the frequency structure Δ_0 a function of Δ, which will be assumed throughout to be *$\mathcal{Q}$-stabilizable and $\mathcal{Q}$-detectable.* The *well-posedness* of this interconnection implies $I - D_{22}\hat{D}$ is invertible.

We will see that (A, B_2) is $\mathcal{Q}$-stabilizable and (C_2, A) is $\mathcal{Q}$-detectable with respect to frequency structure Δ, is both sufficient and necessary for the solvability of OF problem.

Now, we consider some special problems which are related to the general OF problems, they all pertain to the standard block diagram, but with different structures from G. The problems are labeled as FI, DF, FC and OE which stand for *full information, disturbance feedforward, full control and output estimation* respectively. Their corresponding plants are

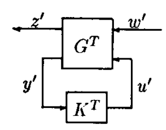
$$G_{FI}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \quad G_{DF}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & I & 0 \end{array}\right]$$

$$G_{FC}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_1 & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right] \quad G_{OE}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

Where all of these special systems have the same frequency structures as $G(\Delta)$. We assume all physical variables have the compatible dimensions.

The motivations for these different problems in one-dimensional case were given in Doyle et al (1989). To examine their relationships, we will use a simple notion of *algebraic duality.*

Consider the standard block diagram. Now consider another system shown below



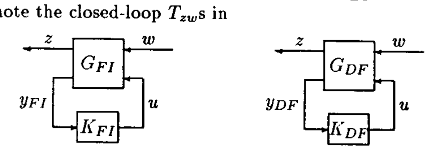
whose plant and controller are obtained by transposing $G(\Delta)$ and $K(\Delta)$. And we can check via LFT formula that $T_{zw}^T = [\mathcal{F}_\ell(G, K)]^T = \mathcal{F}_\ell(G^T, K^T) = T_{z'w'}$. It is not difficult to see that K $\mathcal{Q}$-stabilizes G with respect to the induced frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$ if and only if K^T $\mathcal{Q}$-stabilizes G^T with respect to the frequency structure Δ_N^T. And we say that these two control structures are *algebraically dual*, in particular, G^T and K^T are dual objects of G and K, respectively. So as far as stabilization or other synthesis problems are concerned, we can obtain the results for G^T from its dual object G if available.

So it is not difficult to see that the FI and FC structures (DF and OE structures) are algebraically dual. We will also see that FI and DF structures (FC and OE structures) are equivalent in a sense that will be made precise in the next subsection.

3.2 FI and DF Problems

For the FI problem, we only need to assume (A, B_2) is $\mathcal{Q}$-stabilizable to guarantee the solvability. And the conditions for the solvability of DF problem are both (A, B_2) is $\mathcal{Q}$-stabilizable and (C_2, A) is $\mathcal{Q}$-detectable.

Suppose that we have controllers K_{FI} and K_{DF} and let T_{FI} and T_{DF} denote the closed-loop T_{zw}s in



We have following theorem.

Theorem 4 *(i) Feedback matrix K_{DF} $\mathcal{Q}$-stablizes DF structure, then $K_{DF}[C_2\ I]$ can be used as a $\mathcal{Q}$-stabilizing feedback matrix for the corresponding FI structure, which makes $\mathcal{F}_\ell(G_{DF}, K_{DF}) = \mathcal{F}_\ell(G_{FI}, K_{DF}[C_2\ I])$.*

(ii) Suppose that $A - B_1C_2$ is $\mathcal{Q}$-stable. For any $\mathcal{Q}$-stabilizing feedback matrix K_{FI} for the FI structure, a $\mathcal{Q}$-stabilizing feedback controller for G_{DF} is $\mathcal{F}_\ell(P_{DF}, K_{FI})$ with

$$P_{DF}(\Delta) = \left[\begin{array}{c|cc} A - B_1C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ \begin{bmatrix} I \\ -C_2 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right].$$

Moreover, $\mathcal{F}_\ell(G_{FI}, K_{FI}) = \mathcal{F}_\ell(G_{DF}, \mathcal{F}_\ell(P_{DF}, K_{FI}))$.

Proof. (i) is easy. Now we prove (ii). Assume $\mathcal{F}_\ell(G_{DF}, \mathcal{F}_\ell(P_{DF}, K_{FI})) =: \mathcal{F}_\ell(G'_{FI}, K_{FI})$. We will first prove that $G'_{FI} = G_{FI}$.

We examine the system G'_{FI}, let x and $\hat{x}$ denote the state of G_{DF} and P_{DF}, respectively, take $e := x - \hat{x}$ and $\hat{x}$ as the states of the resulting interconnected system G'_{FI}, then its realization is

$$\left[\begin{array}{cc|cc} A-B_1C_2 & 0 & 0 & 0 \\ B_1C_2 & A & B_1 & B_2 \\ \hline C_1 & C_1 & D_{11} & D_{12} \\ \begin{bmatrix} 0 \\ C_2 \end{bmatrix} & \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

with respect to the frequency structure $\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$. The resulting transfer matrix is exactly G_{FI} as claimed. Since $A - B_1C_2$ is Q-stable, so the theorem follows.

□

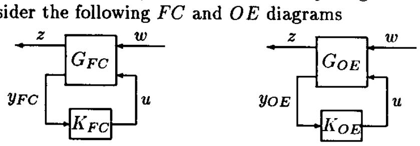
Remark 1 *Note that, given a feedback matrix K_{DF} for the DF structure, then the corresponding FI controller can be obtained from (i) as $K_{DF}\begin{bmatrix} C_2 & I \end{bmatrix}$; and now we return it to DF problem by (ii), then $\mathcal{F}_l(P_{DF}, K_{DF}\begin{bmatrix} C_2 & I \end{bmatrix}) = K_{DF}$. We also have the fact that $\mathcal{F}_l(P_{DF}, K_{FI})\begin{bmatrix} C_2 & I \end{bmatrix} = K_{FI}$.*

Remark 2 *This theorem shows that if $A - B_1C_2$ is Q-stable, then problems FI and DF are equivalent in the above sense.*

3.3 FC and OE Problems

FC(OE) problem is dual to the FI(DF) case and has the dual solvability condition to FI(FD) problem. The solutions to this kind of control problems can be obtained by first transposing G_{FC}(G_{OE} and solving the corresponding FI(DF) problem then transposing it back.

Consider the following FC and OE diagrams



we have the dual results to the ones in the last subsection:

Theorem 5 *(i) Given an OE feedback matrix K_{OE} which Q-stabilizes OEstructure, then the corresponding Q-stabilizing FC controller can be chosen as $\begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}$, which makes $\mathcal{F}_\ell(G_{OE}, K_{OE}) = \mathcal{F}_\ell(G_{FC}, \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE})$.*

(ii) Suppose that $A - B_2C_1$ is Q-stable. For any Q-stabilizing FC controller K_{FC}, a Q-stabilizing OE controller can be taken as $\mathcal{F}_\ell(P_{OE}, K_{FC})$ with

$$P_{OE}(\Delta) = \left[\begin{array}{c|c|c} A - B_2C_1 & 0 & \begin{bmatrix} I & -B_2 \end{bmatrix} \\ \hline C_1 & 0 & \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \\ C_2 & I & \end{array}\right].$$

Moreover, $\mathcal{F}_\ell(G_{FC}, K_{FC}) = \mathcal{F}_\ell(G_{OE}, \mathcal{F}_\ell(P_{OE}, K_{FC}))$.

Remark 3 *Given controllers K_{FC} and K_{OE} for FC and OE structures, respectively, we have the following facts:*

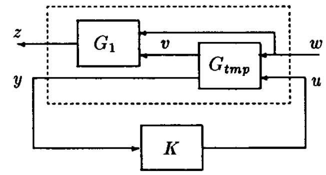
$$K_{FC} = \begin{bmatrix} B_2 \\ I \end{bmatrix} \mathcal{F}_\ell(P_{OE}, K_{FC}) \qquad K_{OE} = \mathcal{F}_\ell(P_{OE}, \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE})$$

Remark 4 *We can see that if $A - B_2C_1$ is Q-stable, then FC and OE problems are equivalent in the above sense.*

3.4 OF Problem and Separation Property

Consider the general OF problem. Without loss of generality, we further assume the realization of $G(\Delta)$ has $D_{22} = 0$.

Let x denote the "state" of system $G(\Delta)$. Since (A, B_2) is Q-stabilizable, there is a constant matrix F such that $A + B_2F$ is Q-stable. Note that $\begin{bmatrix} F & 0 \end{bmatrix}$ is actually a special FI stabilizing controller. Now let $v = u - Fx$, then the system can be broken into two subsystems G_1 and G_{tmp} as shown below



where

$$G_1(\Delta) = \left[\begin{array}{c|cc} A + B_2F & B_1 & B_2 \\ \hline C_1 + D_{12}F & D_{11} & D_{12} \end{array}\right]$$

is Q-stable, and

$$G_{tmp}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

From theorem 2, we know that K Q-stabilizes G if and only if K Q-stabilizes G_{tmp}. Now G_{tmp} has an OE structure, so let L be such that $A + LC_2$ is Q-stable then $\begin{bmatrix} L \\ 0 \end{bmatrix}$ is a Q-stabilizing controller for the corresponding FC problem. Since $A + B_2F$ is Q-stable by construction, by Theorem 5 we have a controller given by $K(\Delta) = \mathcal{F}_\ell(J, \begin{bmatrix} L \\ 0 \end{bmatrix})$, where

$$J(\Delta) = \left[\begin{array}{c|c|c} A + B_2F & 0 & \begin{bmatrix} I & -B_2 \end{bmatrix} \\ \hline -F & 0 & \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \\ C_2 & I & \end{array}\right]$$

which yields

$$K(\Delta) = \left[\begin{array}{c|c} A + B_2F + LC_2 & -L \\ \hline F & 0 \end{array}\right]$$

This proves the following theorem, which is stated without the assumption that $D_{22} = 0$.

Theorem 6 *Consider the general OF problem. Let F and L be such that $A + LC_2$ and $A + B_2F$ are Q-stable, then the controller*

$$K(\Delta) = \left[\begin{array}{c|c} A + B_2F + LC_2 + LD_{22}F & -L \\ \hline F & 0 \end{array}\right]$$

with the frequency structure Δ Q-stabilizes the given system.

We can also get the same result by the dual procedure to above construction.

Now we examine the separation property. We will denote the state variable of $G(\Delta)$ as x and state variable of resulting controller $K(\Delta)$ as $\tilde{x}$. We will take the state variable of the closed loop system as $\bar{x} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$, and the corresponding realization is

$$\left[\begin{array}{cc|c} A & B_2F & B_1 \\ -LC_2 & A + B_2F + LC_2 & -LD_{21} \\ \hline C_1 & D_{12}F & D_{11} \end{array}\right]$$

Now we conduct the admissible state transformation $\bar{x} \mapsto T\bar{x} = \begin{bmatrix} x \\ \tilde{x} - x \end{bmatrix}$, then $T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$. After the transformation, the realization is

$$\left[\begin{array}{cc|c} A + B_2F & B_2F & B_1 \\ 0 & A + LC_2 & B_1 - LD_{21} \\ \hline C_1 - D_{12}F & D_{12}F & D_{11} \end{array}\right]$$

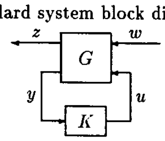
i.e. the system is decoupled into two separated subsystems which are Q-stable with respect to the same frequency structure Δ by assumption. Hence the closed-loop system after admissible state variable transformation is also Q-stable with respect to the new frequency structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$ by theorem 2, so is the original closed-loop system as desired.

The separation property is clear now, since the synthesis of OF problem can be reduced to FI and FC problems, i.e. the latter two problems can be designed independently.

4 The Characterization of All Q-Stabilizing Controllers For Output Feedback Problems

4.1 Problem Statement and Motivations

Consider again the standard system block diagram



with

$$G(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Suppose (A, B_2) is Q-stabilizable and (C_2, A) is Q-detectable, in this section we discuss the following problem:

Given a plant $G(\Delta)$, parametrize all controllers $K(\Delta)$ that Q-stabilize G.

More specially, the characterization problem is to find a suitable matrix $J(\Delta)$ such that any admissible controller $K(\Delta)$ for $G(\Delta)$ can be expressed as an LFT with respect to a Q-stable parameter matrix $Q(\Delta)$ with coefficient matrix $J(\Delta)$, i.e. $K = \mathcal{F}_l(J, Q)$.

Note that for a Q-stable parameter matrix $Q(\Delta)$, it will be assumed that *its realizations are Q-stabilizable and Q-detectable.*

In this section, we only consider the characterization of all admissible controllers for OF problems. For this purpose it is unnecessary to parametrize all admissible controllers for the special problems. Instead, we only parameterize a set of controller equivalence classes which generate the *same control action.* We say *two controllers, K and K', are of equivalent control actions if their corresponding closed loop transfer matrices are identical,* i.e. $\mathcal{F}_l(G, K) = \mathcal{F}_l(G, K')$, written as $K \cong K'$. The controller equivalence is an *equivalence relation.* We will see that, for different special problems, we have different refined versions of this relation. We will also see that the characterizations of equivalent classes of admissible controllers for special problems are good enough to construct the characterization of all admissible controllers for OF problem.

4.2 Admissible Controllers for FI and FC Problems

We first examine the FI structure as given in the last section. We say *two controllers K_{FI} and $\hat{K}_{FI}$ are equivalent if they produce the same closed loop transfer function from w to u* . Obviously, it is also guaranteed that $\mathcal{F}_l(G_{FI}, K_{FI}) = \mathcal{F}_l(G_{FI}, \hat{K}_{FI})$.

Since we have full information for feedback, our controller will have the following general form

$$K_{FI}(\Delta) = \left[\begin{array}{cc} K_1(\Delta) & K_2(\Delta) \end{array}\right]$$

with $K_1(\Delta)$ Q-stabilizing $\left[\begin{array}{c|c} A & B_2 \\ \hline I & 0 \end{array}\right]$ and arbitrary Q-stable $K_2(\Delta)$.

Proposition 2 *Let F be a constant matrix such that $A + B_2F$ is Q-stable. Then all admissible controllers, in the sense of generating all Q-stabilizing control, for FI can be parametrized as*

$$K_{FI}(\Delta) \cong \left[\begin{array}{cc} F & Q(\Delta) \end{array}\right]$$

with any Q-stable $Q(\Delta)$.

Proof. It is easy to see that the controller given in the above formula Q-stabilizes the system $G_{FI}(\Delta)$. Hence we only need to show that the given set of controllers parameterize all Q-stabilizing control action, u, i.e., there is a choice of Q-stable $Q(\Delta)$ such that the transfer functions from w to u for any stabilizing controller $K_{FI}(\Delta) = \left[\begin{array}{cc} K_1(\Delta) & K_2(\Delta) \end{array}\right]$ and for $K^0_{FI}(\Delta) = \left[\begin{array}{cc} F & Q(\Delta) \end{array}\right]$ are the same. To show that, make a change of control variable as $v = u - Fx$, where x denotes the state of the system $G_{FI}(\Delta)$. Consider the system with the controller $K_{FI}(\Delta)$, let $Q(\Delta)$ be the transfer matrix from w to v; it is Q-stable by the Q-stability of the closed loop system. Then $u = Fx + v = Fx + Qw$, so $K_{FI}(\Delta) \cong \left[\begin{array}{cc} F & Q(\Delta) \end{array}\right]$. □

Remark 5 *In the above proof, we have used a standard technique of changing variables. A similar change of variables appeared in Doyle et al(1989).*

Now we consider the dual FC problem whose diagram is given in the last section. We say *controllers K_{FC} and $\hat{K}_{FC}$ are equivalent in the sense that the same injection inputs y_{FC}'s produce the same outputs z's.* This also guarantees the identity of their resulting closed loop transfer matrices from w to z. We also have

Proposition 3 *Let L be a constant matrix such that $A + LC_2$ is Q-stable. Then the set of equivalent classes of all admissible controllers for FC in the above sense can be parameterized as*

$$K_{FC}(\Delta) \cong \left[\begin{array}{c} L \\ Q(\Delta) \end{array}\right]$$

with any Q-stable $Q(\Delta)$.

4.3 Admissible Controllers for Problems DF and OE

The DF block diagram is given as in the last section. We will further assume that $A - B_1C_2$ is Q-stable in this subsection. This will simplify the solution, and can be relaxed. We say that *the controllers $K_{DF}(\Delta)$ and $K'_{DF}(\Delta)$ are equivalent for the DF problem if the two transfer matrices from w to u in the above diagram are the same* . Of course the resulting two closed loop transfer matrices from w to z are identical.

Remark 6 *Since the equivalence between FI and DF problems as discussed in the last section, it is easy to show that if $K_{DF} \cong K'_{DF}$ in the DF structure, then $K_{DF}\left[\begin{array}{cc} C_2 & I \end{array}\right] \cong K'_{DF}\left[\begin{array}{cc} C_2 & I \end{array}\right]$ in the corresponding FI structure. We also have that if $K_{FI} \cong K'_{FI}$, then $\mathcal{F}_l(P_{DF}, K_{FI}) \cong \mathcal{F}_l(P_{DF}, K'_{FI})$.*

Now we construct the parametrization of equivalence classes of stabilizing controllers for DF structure. Let $K_{DF}(\Delta)$ be an admissible control for DF then $K_{FI}(\Delta) = K_{DF}(\Delta)\left[\begin{array}{cc} C_2 & I \end{array}\right]$ Q-stabilizes the corresponding $G_{FI}(\Delta)$. Assume $K_{FI}(\Delta) \cong K'_{FI}(\Delta) = \left[\begin{array}{cc} F & Q(\Delta) \end{array}\right]$ for some Q-stable $Q(\Delta)$, then $K'_{FI}(\Delta)$ Q-stabilizes $G_{FI}(\Delta)$ and $\mathcal{F}_\ell(J_{DF}(\Delta), Q(\Delta)) = \mathcal{F}_\ell(P_{DF}(\Delta), K'_{FI}(\Delta))$ where

$$J_{DF}(\Delta) = \left[\begin{array}{c|cc} A + B_2F - B_1C_2 & B_1 & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

with F such that $A + B_2F$ is Q-stable. Hence by Theorem 4, $K'_{DF}(\Delta) := \mathcal{F}_\ell(J_{DF}(\Delta), Q(\Delta))$ stabilizes $G_{DF}(\Delta)$ for any Q-stable $Q(\Delta)$. Since $K_{FI}(\Delta) \cong K'_{FI}(\Delta)$, so by Remarks 1 and 6 we know $K_{DF}(\Delta) \cong K'_{DF}(\Delta) = \mathcal{F}_\ell(J_{DF}(\Delta), Q(\Delta))$, this characterizes the equivalence class of all controllers for DF problem by the equivalence of FI and DF.

In fact, we have following proposition.

Proposition 4 *All admissible controllers for DF problem can be characterized by $K_{DF}(\Delta) = \mathcal{F}_\ell(J_{DF}(\Delta), Q_0(\Delta)$ with Q-stable $Q_0(\Delta)$, where $J_{DF}(\Delta)$ is given as above.*

Proof. It is easy to show the controllers expressed in the given LFT formula do Q-stabilize G_{DF} by transforming it to the corresponding FI problem. Now Let K_{DF} be any admissible controller for G_{DF}, then $\mathcal{F}_\ell(\hat{J}_{DF}, K_{DF})$ is Q-stable where

$$\hat{J}_{DF} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right]$$

Let $Q_0 := \mathcal{F}_\ell(\hat{J}_{DF}, K_{DF})$, then $\mathcal{F}_\ell(J_{DF}, Q_0) = \mathcal{F}_\ell(J_{DF}, \mathcal{F}_\ell(\hat{J}_{DF}, K_{DF})) =: \mathcal{F}_\ell(J_{tmp}, K_{DF})$, where

J_{tmp} can be obtained by using composition formula of LFT as

$$J_{tmp} = \left[\begin{array}{cc|cc} A - B_1C_2 + B_2F & -B_2F & B_1 & B_2 \\ -B_1C_2 & A & B_1 & B_2 \\ \hline F & -F & 0 & I \\ -C_2 & C_2 & I & 0 \end{array}\right] = \left[\begin{array}{cc|cc} A - B_1C_2 & -B_2F & B_1 & B_2 \\ 0 & A + B_2F & 0 & 0 \\ \hline 0 & -F & 0 & I \\ 0 & C_2 & I & 0 \end{array}\right] = \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array}\right]$$

Hence $\mathcal{F}_\ell(J_{DF}, Q_0) = \mathcal{F}_\ell(J_{tmp}, K_{DF}) = K_{DF}$. This shows that any admissible controller can be expressed in the form of $\mathcal{F}_\ell(J_{DF}, Q_0)$ for some Q-stable Q_0. □

Since for any admissible K_{DF}, we can get two parametrizations by the above two formulae. We relate them as

Proposition 5 *Let $K_{DF} = \mathcal{F}_\ell(J_{DF}, Q_0)$ for some Q-stable Q_0 be obtained from the constructive proof of Proposition 4 and let $K'_{DF} = \mathcal{F}_\ell(J_{DF}, Q) \cong K_{DF}$ for some Q-stable Q be obtained from its corresponding FI structure. Then $Q_0 = Q$. Moreover $K_{DF} = K'_{DF}$.*

Now we turn to the OE problem, the system structure is given in the last section. We will assume that $A - B_2C_1$ is Q-stable. We have the following result

Proposition 6 *All admissible controllers for OE problem can be characterized as $\mathcal{F}_\ell(J_{OE}, Q_0)$ with any Q-stable Q_0, where J_{OE} is defined as*

$$J_{OE} = \left[\begin{array}{c|cc} A - B_2C_1 + LC_2 & L & -B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array}\right]$$

with L such that $A + LC_2$ is Q-stable.

4.4 All Admissible Controllers for Problem OF

Consider the OF problem given in section 4.1, we have

Theorem 7 *Let F and L be such that $A + LC_2$ and $A + B_2F$ are Q-stable, then all controllers Q-stabilize $G(\Delta)$ can be characterized as $\mathcal{F}_u(J, Q)$ with*

$$J(\Delta) = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right]$$

and with any Q-stable $Q(\Delta)$ such that the structure is well-posed.

Proof. We will assume again $D_{22} = 0$ for simplicity.

Let x denote the state of system G. Since (A, B_2) is Q-stabilizable, there is a constant matrix F such that $A + B_2F$ is Q-stable. Note that $\left[\begin{array}{cc} F & 0 \end{array}\right]$ is actually a special FI Q-stabilizing controller. Now let $v = u - Fx$ as in the proof of Theorem 6, we have $K(\Delta)$ Q-stabilizes $G(\Delta)$ if and only if it Q-stabilizes

$$G_{tmp}(\Delta) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

Now $G_{tmp}(\Delta)$ is of the OE structure, let L be such that $A + LC_2$ is Q-stable then by proposition 6 all controllers Q-stabilizing $G_{tmp}(\Delta)$ are given by

$$K(\Delta) = \mathcal{F}_\ell(J(\Delta), Q(\Delta))$$

where

$$J(\Delta) = \left[\begin{array}{c|cc} A + B_2F + LC_2 & L & -B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right] = \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

This concludes our proof. □

This theorem shows that any admissible controller $K(\Delta)$ can be characterized as an LFT of a Q-stable parameter matrix $Q(\Delta)$, i.e., $K(\Delta) = \mathcal{F}_\ell(J(\Delta), Q(\Delta))$. There is an alternative direct proof that this parametrization produces *all* stabilizing controllers. To see this, recall from the inversion formulas for LFTs in Doyle et al(1991) that we can solve the equation $K(\Delta) = \mathcal{F}_\ell(J(\Delta), Q(\Delta))$ for $Q(\Delta)$ to give

$$Q = \mathcal{F}_u(J^{-1}, K) = \mathcal{F}_\ell(\hat{J}, K)$$

where a little algebra shows that

$$J^{-1} = \left[\begin{array}{c|cc} A & B_2 & L \\ \hline C_2 & D_{22} & I \\ -F & I & 0 \end{array}\right]$$

and

$$\hat{J} = \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array}\right] J^{-1} \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array}\right] = \left[\begin{array}{c|cc} A & L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & D_{22} \end{array}\right]$$

Note that Q is stable if and only if K stabilizes $\hat{J}_{22}$. But $\hat{J}_{22} = G_{22}$, so Q is stable if and only if K stabilizes G, as desired.

This last manipulation illustrates the power and simplicity of the LFT machinery. In contrast, the standard proof that the formula $K(\Delta) = \mathcal{F}_\ell(J(\Delta), Q(\Delta))$ yields all stabilizing controllers requires involved arguments using coprime factorizations. It is also possible to obtain similar results using the chain-scattering version of LFTs (Kimura, private communication).

5 Concluding Remarks

We have considered the problems of stabilization and the parametrization of all stabilizing controllers for LFT systems. Although the focus was on Q-stability, most of the results are identical in the μ-stability case via simple change of notation. The exception is that the FI and OE solutions in the μ case are not simply characterized by LMIs, and stabilization by dynamic feedback is not equivalent to stabilization by constant gain. The separation theory holds in the μ case exactly as in the Q case, although the notation is a bit more cumbersome.

The separation proof in this paper holds in greater generality than for just the Q and μ stability problems. All that is required for the separation proof is that the notion of stability satisfy two requirements: 1) stability invariance under a sufficiently rich set of similarity transformations, as in Theorem 1, and 2) a certain structural property as given in Theorem 2. It would clearly be possible to develop a more abstract axiomatic stabilization theory using these 2 properties.

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