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Proceedings of the 34th
Conference on Decision & Control
New Orleans, LA - December 1995

Configuration Controllability for a Class of Mechanical Systems*

Andrew D. Lewis
andrew@indra.caltech.edu

Richard M. Murray
murray@indra.caltech.edu

Control and Dynamical Systems
104-44 California Institute of Technology
Pasadena, CA 91125

17 August 1995

Abstract

We define a notion of controllability for mechanical systems which determines the configurations which are accessible from a given configuration. We derive sufficient conditions for this notion of controllability in terms of the given inputs, their Lie brackets, and their covariant derivatives.

1. Introduction

IN this paper we investigate the problem of determining the structure of the set of configurations which are reachable from a given configuration. This question falls into the realm of standard questions which can be answered using methods from nonlinear control theory. However, by using the special properties of mechanical systems, one can simplify the necessary computations and, more importantly, get an idea of the mechanical factors which contribute to accessibility or which hinder it. It will be worth noticing that the conditions we compute for our "configuration controllability" are expressible in terms of quantities defined on the configuration manifold. Also, our methods circumvent problems which arise when a mechanical control system is not accessible in the phase space, but is nevertheless accessible in the configuration space. This situation frequently arises when there are conservation laws present which are preserved by the inputs. In this case the system cannot be accessible in the phase space (it must satisfy the conservation law), but it may still be accessible in the configuration space. An example of this type is given in Section 2.

To simplify the problem, we restrict ourselves to the class of systems which evolve on Riemannian manifolds and whose Lagrangian is the kinetic energy with respect to the Riemannian metric. These systems have the feature that every configuration is an equilibrium point at zero velocity. This class of systems also includes a large number of applications in mechanics.

In [1] Bloch and Crouch study the same class of systems as we do. In their paper, extra structure in the form of system symmetries is introduced. It is shown that, under some conditions on these symmetries and on the inputs, the system is controllable. Their results draw on the work of San Martin and Crouch [2] on control systems on principal fibre bundles whose structure group is compact.

Another class of mechanical systems which has received some attention is systems with nonholonomic constraints. In [3], Bloch, et. al., study these systems under the conditions that the constraints be completely nonholonomic (meaning that all configurations are reachable from a given configuration with curves which satisfy the constraints) and that the inputs forces span a complement to the constraint forces. With these two assumptions, it is possible to demonstrate that the system is small-time locally controllable using methods of Sussmann [4]. An interesting example of a mechanical control system with nonholonomic constraints is the *Snakeboard*, first presented by Lewis, et. al., [5]. This system does not satisfy the assumptions of [3], but nevertheless may be shown to be small-time locally controllable [6].

We shall use the following symbols:

Q	: the configuration manifold which is n-dimensional
$\tau_Q : TQ \to Q$:	the tangent bundle of Q
$Z(TQ)$	: the zero section of TQ
$\mathfrak{X}(M)$	: the set of vector fields on Q
0_q	: the zero vector in the tangent space to Q at q

Throughout, the Einstein summation convention will be used where summation is implied over repeated upper and lower indices.

2. A Motivating Example

In this section we briefly describe a mechanical system which illustrates the need to refine the treatment of mechanical systems in nonlinear control theory. In particular, this example demonstrates that the nonlinear control calculations that one often performs do not provide a satisfactory resolution to the controllability problem for all mechanical systems. We propose that a weaker notion of controllability may be useful.

The example we consider is a rigid body with inertia J which is pinned to ground at its centre of mass. This example was first presented by Li, et. al., [7]. The body has an extensible massless leg attached to it, and the leg has a point mass, m, at its tip. The coordinate θ will describe the angle of the body and ψ will describe the angle of the leg from an inertial reference frame. The coordinate r will describe the extension of the leg. Thus the configuration space for this problem is $Q = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$. See Figure 1. The Lagrangian is

*Research supported by NSF Grant CMS-9502224

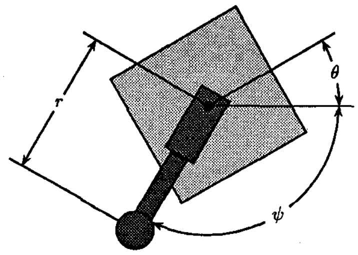

Fig. 1: The extensible leg

$$L = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\psi}^2).$$

If we consider forces applied in the r and $(\theta - \psi)$–directions, Lagrange's equations are

$$m\ddot{r} - mr\dot{\psi}^2 = u_2 \tag{1a}$$

$$J\ddot{\theta} = u_1 \tag{1b}$$

$$mr^2\ddot{\psi} + 2mr\dot{r}\dot{\psi} = -u_1. \tag{1c}$$

We may write these equations as a first order, affine, nonlinear control system and then perform the distribution calculations to obtain the accessibility distribution as

$$C(r,\theta,\psi,\dot{r},\dot{\theta},\dot{\psi}) = \left\langle 2mr\dot{\psi}\frac{\partial}{\partial\dot{\theta}} - J\frac{\partial}{\partial \dot{r}}, mr^2\frac{\partial}{\partial\dot{\theta}} - J\frac{\partial}{\partial\dot{\psi}}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\psi}, \frac{\partial}{\partial r}\right\rangle_{\mathbb{R}}.$$

Since this distribution does not span TQ, we conclude that the system is not accessible. Nevertheless, it is possible to steer the system from one configuration to another. Indeed we have the following result.

Claim: Select two configurations, $q_1 = (r_1, \theta_1, \psi_1)$, and $q_2 = (r_2, \theta_2, \psi_2)$. Suppose that the system starts at rest in configuration q_1. Then there exists inputs, u_1, u_2, which steer the system to rest at q_2.

Idea of Proof. We first note that the inputs leave the total angular momentum, $\mu = J\dot{\theta} + mr^2\dot{\psi}$, of the system conserved. Thus, when we start at rest at q_1, all consequent motions of the system will have zero angular momentum. This may be thought of as imposing a constraint given by

$$J\dot{\theta} + mr^2\dot{\psi} = 0. \tag{2}$$

Let us first answer the question: How many configurations are accessible from q_1 along paths which preserve zero angular momentum? This question may be formulated as a nonholonomic control problem and, as is shown by Murray and Sastry [8], all configurations are accessible from a given configuration q_1. Now, to prove the claim, we need to show that all motions of the system which preserve zero angular momentum are realisable using suitable inputs, u_1, u_2. Let c be a path in Q which satisfies the constraint (2) and which connects q_1 with q_2. We may suppose that c is reparameterised so that we start at rest at q_1 and end at rest at q_2. From (1a) and (1b) we immediately have $u_2 = m\ddot{r} - mr\dot{\psi}^2$ and $u_1 = J\ddot{\theta}$. We need only show that, so defined, u_1 satisfies (1c). From (2) we have

$$J\ddot{\theta} = -mr^2\ddot{\psi} - 2mr\dot{r}\dot{\psi}.$$

Therefore

$$mr^2\ddot{\psi} + 2mr\dot{r}\dot{\psi} = -u_1$$

which is simply (1c). This completes the proof. ∎

The claim indicates that we would like to be able to consider this problem controllable in some sense. The goal of this paper is to formulate a definition of controllability that would make this problem, and problems like it, controllable, and then determine a computable check of this controllability condition.

3. Review of Nonlinear Control Theory

In this section we provide a review of some of the basic concepts from nonlinear control theory as we shall need them for our study of mechanical control systems. Most of what we shall say in this section may be found in Nijmeijer and van der Schaft [9].

We consider a general affine control system of the form

$$\dot{x} = f(x) + u^a g_a(x) \tag{3}$$

where x evolves on a manifold M and $f, g_1, \ldots, g_m$ are vector fields on M. We consider the class of control inputs defined by

$$\mathscr{U} = \{u \colon \mathbb{R} \to \mathbb{R}^m \mid u \text{ is piecewise constant}\}.$$

We denote by $\mathcal{R}(x, T)$ the set of points reachable from x in time exactly T and

$$\mathcal{R}(x, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}(x, t).$$

We say that the system (3) is *locally accessible* at x if there exists $T > 0$ so that $\mathcal{R}(x, \leq t)$ contains a neighborhood of M for each $0 < t \leq T$.

Denote by $\mathscr{C}$ the accessibility algebra for the control system (3). Thus $\mathscr{C}$ is the smallest subalgebra of $\mathfrak{X}(M)$ which contains $\{f, g_1, \ldots, g_m\}$. This defines a distribution on M which we denote by C. If $\dim(C(x)) = \dim(M)$, then (3) is locally accessible at x. If the rank of C is constant, then C defines a foliation of M. Restricted to each leaf of this foliation, the control system (3) is locally accessible.

4. Problem Setup

The first part of the problem data is a Riemannian manifold (Q, g). Here Q is an n-dimensional manifold and g is a Riemannian metric on Q. The Lagrangian consists of the "kinetic energy" for the Riemannian metric. Thus

$$L(v) = \frac{1}{2}g(v, v). \tag{4}$$

Corresponding to the Riemannian metric is the *covariant derivative* defined for vector fields X and Y by

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j}X^j + \Gamma^i_{jk}X^jY^k\right)\frac{\partial}{\partial q^i}.$$

The Γ^i_{jk} are the *Christoffel symbols* and are related to the Riemannian metric with the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial q^k} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right)$$

where g^{ij} are the components of the inverse of g_{ij}. We shall see the covariant derivative arise as an important tool for expressing conditions for "configuration controllability" in Section 6.

Since we are considering mechanical *control* systems, we need to be clear about what an external force is for a mechanical system. Recall that external forces appear in Lagrange's equations as the right hand side of the equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = F_i. \tag{5}$$

If the components $F_1, \ldots, F_n$ depend only on position and not on velocity and time, then we may regard them as components of a one-form F on Q. Therefore, we shall consider our inputs to be determined by m linearly independent one-forms, $F^1, \ldots, F^m$. Corresponding to these m one-forms are m vector fields $Y_a = (F^a)^\sharp$, $a = 1, \ldots, m$, where $\sharp$ is the musical isomorphism defined by the Riemannian metric. The control problem is then given by

$$\nabla_{\dot{c}(t)} \dot{c}(t) = u^a Y_a(c(t)) \tag{6}$$

where $c\colon \mathbb{R} \to Q$ is a curve on Q. Note that the Riemannian metric, and hence the Lagrangian, enters in the definition of the control vector fields $Y_1, \ldots, Y_m$.

5. Controllability Definitions for Mechanical Systems

In this section we introduce our new notion of controllability on the configuration manifold. As notation we write 0_q for the zero vector in T_qQ.

Definition 1. Let $q \in Q$. We shall say that $q' \in Q$ is *reachable in time T* from q if there exists a solution (c, u) of (6) such that $\dot{c}(0) = 0_q$ and $\dot{c}(T) \in T_{q'}Q$. We denote the set of points reachable from q in time T by $\mathcal{R}_Q(q, T)$. We also denote

$$\mathcal{R}_Q(q, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}_Q(q, t).$$

The system (6) is *locally configuration accessible* at q if there exists $T > 0$ so that $\mathcal{R}_Q(q, \leq t)$ contains a neighborhood of Q for every $0 < t \leq T$.

The system (6) is *equilibrium controllable* if it is possible to steer the system between any two equilibrium points. •

Obviously equilibrium controllability is a feature we would like our systems to have. However, as we shall see in the examples in Section 8, there are some interesting cases when a system is locally configuration accessible, but not equilibrium controllable.

6. The Structure of the Accessibility Distribution

Here we shall compute a subset of the accessibility distribution for the system (6) restricted to the zero section of TQ. Our strategy will be to write the second order system (6) in first order form on TQ and perform standard distribution computations as one would for a nonlinear system of the form (3).

We will denote coordinates on TQ by $(q^1, \ldots, q^n, v^1, \ldots, v^n)$, departing from the usual notation of using $\dot{q}^i$ for the velocities.

To write the system in first order on TQ we need to *vertically lift* the control vector fields so that they enter the equations in the right way. We define

$$Y_a^{lift}(v) = \frac{d}{dt}\left(v + tY_a(\tau_Q(v))\right)\big|_{t=0}.$$

In coordinates we simply have

$$Y_a^{lift} = Y_a^i \frac{\partial}{\partial v^i}.$$

Thus the vertical lift is in "the v-direction" rather than in "the q-direction". The drift vector field for the system in first order form is called the *geodesic spray* [10] in Riemannian geometry. We shall denote this vector field by Z_g and in coordinates we have

$$Z_g = v^i \frac{\partial}{\partial q^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}.$$

As a first order system on TQ we may write (6) intrinsically as

$$\dot{v} = Z_g(v) + u^a Y_a^{lift}(v).$$

In local coordinates the system has the form

$$\begin{aligned} \dot{q}^i &= v^i \\ \dot{v}^i &= -\Gamma^i_{jk} v^j v^k + u^a Y_a^i. \end{aligned}$$

Now let's get to computing some brackets. The following computations are useful for determining what the accessibility distribution looks like when restricted to $Z(TQ)$. *Note that all quantities on the right hand side of the equations in the following lemma are defined only on Q and do not depend on velocity.*

Lemma 2. Let $X, Y \in \mathfrak{X}(Q)$. Then

i) $[X^{lift}, Y^{lift}] = 0$,

ii) $[Z_g, X^{lift}](0_q) = -X(q)$,

iii) $[Y^{lift}, [Z_g, X^{lift}]] = (\nabla_Y X + \nabla_X Y)^{lift}$, and

iv) $[[Z_g, Y^{lift}], [Z_g, X^{lift}]](0_q) = [Y, X](q)$.

Proof. Since the proof is just a computation, we prove ii and iii to demonstrate the calculations.

ii) In this case we have

$$Z_g = v^i \frac{\partial}{\partial q^i} - \Gamma^i_{jk}(q) v^j v^k \frac{\partial}{\partial v^i}, \quad X^{lift} = X^i(q) \frac{\partial}{\partial v^i}.$$

We may compute

$$[Z_g, X^{lift}] = \begin{bmatrix} 0 & 0 \\ \frac{\partial X^i}{\partial q^j} & 0 \end{bmatrix} \begin{pmatrix} v^j \\ -\Gamma^j_{kl} v^k v^l \end{pmatrix} - \begin{bmatrix} 0 & \delta^i_j \\ -\frac{\partial \Gamma^i_{kl}}{\partial q^j} & -2\Gamma^i_{jk} v^k \end{bmatrix} \begin{pmatrix} 0 \\ X^j \end{pmatrix}.$$

If we evaluate this at $(q, 0)$ we get the result.

iii) In this case we have

$$[Z_g, X^{lift}] = -X^i(q)\frac{\partial}{\partial q^i} + \left(\frac{\partial X^i}{\partial q^j} v^j + 2\Gamma^i_{jk} X^j v^k\right) \frac{\partial}{\partial v^i},$$

$$Y^{lift} = Y^i(q)\frac{\partial}{\partial v^i}$$

using ii. We may then compute

$$[Y^{lift}, [Z_g, X^{lift}]] = \begin{bmatrix} * & 0 \\ * & \frac{\partial X^i}{\partial q^j} + 2\Gamma^i_{kj} X^k \end{bmatrix} \begin{pmatrix} 0 \\ Y^j \end{pmatrix} - \begin{bmatrix} 0 & 0 \\ \frac{\partial Y^i}{\partial q^j} & 0 \end{bmatrix} \begin{pmatrix} -X^j \\ \frac{\partial X^j}{\partial q^k} v^k + 2\Gamma^j_{kl} X^k v^l \end{pmatrix}.$$

Reading off the coefficients gives

$$[Y^{lift}, [Z_g, X^{lift}]] = \left(\frac{\partial Y^i}{\partial q^j} X^j + \frac{\partial X^i}{\partial q^j} Y^j + 2\Gamma^i_{jk} X^j Y^k\right) \frac{\partial}{\partial v^i}$$

which is the coordinate representation of $(\nabla_X Y + \nabla_Y X)^{lift}$. ■

With these preliminary results in hand, we may now say some useful things about the form of the accessibility distribution restricted to $Z(TQ)$. We first need to construct some distributions on Q using the control vector fields $Y_1, \ldots, Y_m$. Define $\mathrm{Co}^{(0)}(\boldsymbol{Y})$ to be the collection of vector fields $\boldsymbol{Y} = \{Y_1, \ldots, Y_m\}$. Now iteratively define a sequence of collections of vector fields by

$$\mathrm{Co}^{(i+1)}(\boldsymbol{Y}) = \Big\{\nabla_{Y_a} X + \nabla_X Y_a \mid a = 1, \ldots, m \text{ and } X \in \mathrm{Co}^{(i)}(\boldsymbol{Y})\Big\}.$$

These collections of vector fields may be gathered up and so form

$$\mathrm{Co}^{(\infty)}(\boldsymbol{Y}) = \bigcup_{i=0}^{\infty} \mathrm{Co}^{(i)}(\boldsymbol{Y}).$$

Note that we may naturally regard T_qQ as a subset of $T_{0_q}TQ$. We shall use this identification now to obtain a subset of the accessibility distribution at 0_q. First of all, given Lemma 2 iv, it is clear that all iterated brackets of $Y_1, \ldots, Y_m$ are contained in $C(0_q)$. For example, the bracket

$$[Y_{a_k}, [Y_{a_{k-1}}, \ldots, [Y_{a_2}, Y_{a_1}]]](q) \in T_{0_q}TQ$$

is obtained from

$$\pm [[Z_g, Y^{lift}_{a_k}], [[Z_g, Y^{lift}_{a_{k-1}}], \ldots, [[Z_g, Y^{lift}_{a_2}], [Z_g, Y^{lift}_{a_1}]]]](0_q).$$

Also, given Lemma 2 ii and iii it is clear that $\mathrm{Co}^{(\infty)}(\boldsymbol{Y})(q) \subset C(0_q)$. Indeed, all elements of $\mathrm{Co}^{(\infty)}(\boldsymbol{Y})$ are generated by brackets of the form

$$[Y^{lift}_{a_k}, [Z_g, [Y^{lift}_{a_{k-1}}, Z_g, [\cdots, [Z_g, Y^{lift}_{a_1}]]]]]$$

Therefore, given both of these facts, and Lemma 2 ii and iv, we see that iterated brackets of vectors field in $\mathrm{Co}^{(\infty)}(\boldsymbol{Y})$, when evaluated at q, are in $C(0_q)$.

Remark 3. Note that what we have computed a strict subset of $\mathscr{C} \mid Z(TQ)$. There are many covariant derivative terms which are not captured by iterated brackets of vector fields from $\mathrm{Co}^{(\infty)}(\boldsymbol{Y})$. For a full description $\mathscr{C} \mid Z(TQ)$ with the inclusion of potential energy, see the dissertation of Lewis [11]. In that work the term *symmetric product* for the expression $\nabla_X Y + \nabla_Y X$ is introduced. •

7. Result

Here we state our sufficient condition for local configuration accessibility with comparative ease after the calculations of Section 6.

Proposition 4. Suppose that the involutive closure of the vector fields $\mathrm{Co}^{(\infty)}(\boldsymbol{Y})$ is equal to TQ. Then (6) is locally configuration accessible at each $q \in Q$.

Proof. Recall the notation that C is the accessibility distribution for the system (6). As stated at the end of Section 3, the control system is locally accessible when restricted to each leaf of the foliation defined by C. From Section 6 we know that, when evaluated at q, the brackets of vector fields from the collection $\mathrm{Co}^{(\infty)}(\boldsymbol{Y})$ lie in $C(0_q)$. By the hypotheses this means that $T_qQ \subset C(0_q)$ for each $q \in Q$. This implies that the zero section of TQ is an integral manifold of C. Let Λ be the unique *maximal* integral manifold of C which contains $Z(TQ)$. Note that the control system (6) leaves Λ invariant and is locally accessible when restricted to Λ. Thus $\mathcal{R}(0_q, \leq T)$ is open in Λ for each T sufficiently small. Therefore $\tau_Q(\mathcal{R}(0_q, \leq T)$ is open in Q. This proves the proposition. ■

It is interesting to note that our conditions depend only on objects defined on Q and not on TQ. Furthermore, the computations depend not only vector fields which have fewer components, but there are fewer operations to perform. For example the bracket

$$[Z_g, [Y^{lift}_a, [Z_g, Y^{lift}_b]]]$$

is represented in Proposition 4 by the covariant derivative

$$\nabla_{Y_a} Y_b + \nabla_{Y_b} Y_a.$$

8. Examples

In this section we present some simple examples to illustrate the use of the condition obtained in Section 7.

8.1. Robotic Leg

The first example we give is that of a robotic leg as discussed in Section 2 and shown in Figure 1. The configuration space for the system is $Q = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^+$ and we shall use coordinates

(θ, ψ, r) as indicated in the figure. The system has inputs defined by the one-forms

$$F^1 = d\theta - d\psi \quad \text{and} \quad F^2 = dr.$$

Thus we are allowed to apply a torque to change the angle between the main body and the leg, and we are allowed to apply a force to extend the leg. The Lagrangian for the system was given in Section 2.

We may compute the input vector fields to be

$$Y_1 = (F^1)^\sharp = \frac{1}{J}\frac{\partial}{\partial \theta} - \frac{1}{mr^2}\frac{\partial}{\partial \psi} \quad \text{and}$$
$$Y_2 = (F^2)^\sharp = \frac{1}{m}\frac{\partial}{\partial r}.$$

We also compute

$$[Y_1, Y_2] = -\frac{2}{m^2 r^3}\frac{\partial}{\partial \psi}.$$

This turns out to be the only bracket between the control vector fields that we shall need. The necessary covariant derivative is

$$2\nabla_{Y_1} Y_1 = -\frac{2}{m^2 r^3}\frac{\partial}{\partial r}.$$

We may also compute

$$[Y_1, 2\nabla_{Y_1} Y_1] = \frac{4}{m^3 r^6}\frac{\partial}{\partial \psi}.$$

With this example there are three possible combinations of inputs to consider.

Case 1. Inputs Y_1 and Y_2: In this case it is clear that the system is locally configuration accessible as the input vector fields and their bracket generate the maximal distribution on Q.

Case 2. Input Y_1: In this case the system is again locally configuration accessible since the vector fields $\{Y_1, 2\nabla_{Y_1} Y_1, [Y_1, 2\nabla_{Y_1} Y_1]\}$ generate the maximal distribution on Q.

Case 3. Input Y_2: In this case we only generate the direction Y_2 and so the system is not locally configuration accessible. Indeed, starting from rest and only applying force in the r-direction, the only behaviour that can be observed is motion back and forth of the mass on the end of the leg.

8.2. Planar Rigid Body

The system here is a planar rigid body with a force applied at some point in the body, and a torque applied at the centre of mass. The configuration space for the system is the Lie group $SE(2)$. To establish the correspondence between the configuration of the body and $SE(2)$, fix a point $O \in \mathbb{R}^2$ and let $\{e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}\}$ be the standard orthonormal frame at that point. Let $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ be an orthonormal frame attached to the body at its centre of mass. The configuration of the body is determined by the element $g \in SE(2)$ which maps the point O with its frame $\{e_1, e_2\}$ to the position, P, of the centre of mass of the body with its frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. Without loss of generality (by redefining our body reference frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$) we may suppose that the point of application of the force is a distance h along the $\boldsymbol{f}_1$ body-axis from the centre of mass. The situation is illustrated in Figure 2.

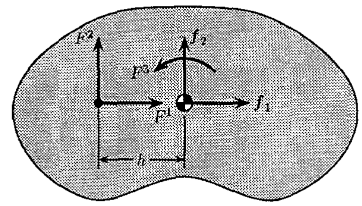

Fig. 2: Positions for application of forces after simplifying assumptions

The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2.$$

We will explore combinations of inputs defined by the one-forms

$$F^1 = \cos\theta dx + \sin\theta dy,$$
$$F^2 = -\sin\theta dx + \cos\theta dy - h d\theta, \quad F^3 = d\theta$$

which give the input vector fields

$$Y_1 = \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y},$$
$$Y_2 = -\frac{\sin\theta}{m}\frac{\partial}{\partial x} + \frac{\cos\theta}{m}\frac{\partial}{\partial y} - \frac{h}{J}\frac{\partial}{\partial \theta}, \quad Y_3 = \frac{1}{J}\frac{\partial}{\partial \theta}.$$

Since the system is obviously controllable (in any sense of the word) with all three inputs, we will only look at combinations of two or fewer inputs.

We present some computations. The Lie brackets which will be useful to us are

$$[Y_1, Y_2] = -\frac{h\sin\theta}{mJ}\frac{\partial}{\partial x} + \frac{h\cos\theta}{mJ}\frac{\partial}{\partial y},$$
$$[Y_1, Y_3] = \frac{\sin\theta}{mJ}\frac{\partial}{\partial x} - \frac{\cos\theta}{mJ}\frac{\partial}{\partial y},$$

and the interesting covariant derivative is

$$2\nabla_{Y_2} Y_2 = \frac{2h\cos\theta}{mJ}\frac{\partial}{\partial x} + \frac{2h\sin\theta}{mJ}\frac{\partial}{\partial y}.$$

We also compute

$$[Y_2, 2\nabla_{Y_2} Y_2] = \frac{2h^2\sin\theta}{mJ^2}\frac{\partial}{\partial x} - \frac{2h^2\cos\theta}{mJ^2}\frac{\partial}{\partial y}.$$

Case 1. Inputs Y_1 and Y_2: This corresponds to being able to apply a force at the point of application in any direction. In this situation we see that the vector fields $\{Y_1, Y_2, [Y_1, Y_2]\}$ generate all directions on TQ so the system is locally configuration accessible with these inputs.

Case 2. Input Y_2: This corresponds to being able to apply a force at the point of application in a direction perpendicular to the direction of the centre of mass. In this case the vector fields $\{Y_2, 2\nabla_{Y_2} Y_2, [Y_2, 2\nabla_{Y_2} Y_2]\}$ generate all directions in TQ and so the system is locally configuration accessible with this input.

Case 3. Input Y_1: Here we are able to apply a force at the point of application in the direction of the centre of mass. The only direction in TQ which may be obtained by bracketing is generated by $\{Y_1\}$. Clearly then the system is not locally configuration accessible.

Case 4. Inputs Y_1 and Y_3: With these inputs we apply a force in the direction of the centre of mass and a torque at the centre of mass. The set of vector fields $\{Y_1, Y_3, [Y_1, Y_3]\}$ generates all directions in TQ and so the system is locally configuration accessible.

8.3. Equilibrium Controllability of the Examples

Our sufficient conditions of Proposition 4 only determine whether a mechanical control system is locally configuration *accessible*. It is often more interesting to know when the system is *controllable*. In Section 5 we introduced the useful definition of *equilibrium controllability*. In [11] useful sufficient conditions for equilibrium controllability are given based on the small-time local controllability results of Sussmann [4]. Here we shall simply state the results of applying the test of [11] to the examples of this paper.

The Robotic Leg. This example turns out to be equilibrium controllable when both inputs are allowed. This is as one would expect given the discussion in Section 2. However, when only the input Y_1 is allowed, even though the system is locally configuration accessible, it does not satisfy the sufficient conditions for equilibrium controllability. This indicates that it may not be possible to steer the system from a given equilibrium to another. This is indeed the case and may be seen to be a consequence of the (uncontrolled) Coriolis force which causes the mass on the end of the leg to move outwards no matter what happens to the other variables.

The Planar Rigid Body. This example is equilibrium controllable when the input combinations Y_1 and Y_2 (Case 1) and Y_1 and Y_3 (Case 4) are allowed. Recall that the system is locally configuration accessible when the input Y_2 is used. However, it turns out that, with this input, the system does not satisfy the sufficient conditions of [11] for equilibrium controllability. In this case the metric is flat and so Coriolis forces are not a suitable explanation for the possible lack of controllability. It is not known at present whether this example is equilibrium controllable or not.

9. Conclusions and Future Work

In this paper we have embarked on an effort to provide controllability definitions for mechanical systems which are more suitable to their needs than are the general definitions. Easily computable sufficient conditions are also given which, in the examples given, provide a geometric interpretation of the factors that may go into providing controllability and what may cause one to lose it. These calculations are also a great deal simpler than the full distribution calculations required by general nonlinear control theory. Thus, by asking a weaker question more natural to mechanical systems, we are able to obtain an answer which is easier to get at and which can be interpreted in terms of the problem data. It is also significant that sufficient conditions for accessibility on the configuration manifold are computable in terms of quantities defined only on this manifold and which do not depend on velocities.

As was mentioned in Remark 3, the techniques introduced in this paper may be applied to obtain more complete conditions for a more general class of systems where potential energy is allowed. The reader is again referred to [11] for a discussion of these issues.

Ongoing work includes applying the methods in this paper to systems with symmetries and constraints. Here the geometry in the work of Bloch, et. al., [12] may provide some interesting connections with our work.

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