A STUDY OF OPTIMAL ABSTRACT JAMMING STRATEGIES VS. NONCOHERENT MFSK.

Robert J. McEliece — California Institue of Technology, Pasadena, CA 91125

Eugene R. Rodemich — Jet Propulsion Laboratory, Pasadena, CA 91109

ABSTRACT

We introduce an abstract model for studying MFSK jammers. We conclude that Houston's partial-band tone jammers are optimal among all energy-restricted jamming threats vs. orthodox MFSK, but that if the communicator uses random amplitude modulation as a countermeasure, a gain of 3dB vs. optimal jamming (which is no longer tone jamming) is achievable.

I. INTRODUCTION. PROBLEM STATEMENT.

In this paper we study the performance of uncoded MFSK modulation in the presence of arbitrary additive jamming, the goal being to devise robust anti-jamming strategies. To do this we adopt the following abstract model.

The signal strength is a nonnegative real number X, which is transmitted as one component of an M-dimensional vector $\underline{X} = (X_1, X_2, \ldots, X_M)$; the remaining M-1 components of $\underline{X}$ are zero. The information transmitted by $\underline{X}$ is just the location of the nonzero component; we assume that this component is selected randomly according to a uniform distribution on $\{1,2\ldots,M\}$, so that each M-ary signal $\underline{X}$ conveys $\log_2 M$ bits of information. In the usual implementation, the signal strength X is a constant related to the available transmitter power. However, in this paper we shall allow X to be a random variable, and denote its distribution function by G(x). The randomness of X has nothing to do with the information being transmitted. It is introduced to give the communicator game-theoretic protection against certain jamming strategies. We call G the transmitter's strategy.

The jamming noise is an M-dimensional random vector $\underline{Z} = (Z_1, Z_2, \ldots, Z_M)$ independent of X, whose components are nonnegative random variables. We denote the M-dimensional distribution function of $\underline{Z}$ by $F(z_1, z_2, \ldots, z_M)$.

We assume that both the communicator and the jammer are subject to average power constraints, which we give in normalized form as follows:

$$E(X^2) = \lambda \tag{1}$$

$$\frac{1}{M}\sum_{j=1}^{M} E(Z_j^2) = 1. \tag{2}$$

The nonnegative number λ is the abstract symbol signal-to-noise ratio. (The abstract bit SNR is then $\lambda_{bit} = \lambda/(\log_2 M)$, but we will not use this quantity.)

In our model the receiver observes the M-dimensional random vector $\underline{R} = (R_1, R_2, \ldots, R_M)$, where $R_j = |X_j + e^{i\theta_j} Z_j|$, and $\theta_1, \theta_2, \ldots, \theta_M$ are independent random phase angles, uniformly distributed on $[0, 2\pi]$. R_j represents the output of the j-th energy detector of the standard noncoherent MFSK receiver. The receiver chooses the index j for which R_j is largest. (In case of ties, the receiver chooses randomly among the maximizing indices.) We denote $P_E = P_E(G,F)$ the error probability, i.e., the probability that the index selected is not the one containing the signal. For a given transmitter strategy G, we are interested in the worst case performance:

$$P_E^*(G) = \sup_F P_E(G,F), \tag{3}$$

where the jamming strategies F are restricted by (2). We are also interested in the 'minimax' value

$$P_E^* = \inf_G \sup_F P_E(G,F). \tag{4}$$

which represents the best performance the communicator, constrained by (1), can guarantee vs. an unknown jammer, constrained by (2).

In the next section, we summarize the previous work on this problem, and state our own results.

2. PREVIOUS WORK. STATEMENT OF RESULTS

For conventional MFSK vs. wideband gaussian noise, it is known that

$$P_E = \frac{1}{M} \sum_{k=2}^{M} (-1)^k \binom{M}{k} \exp\left(-\frac{k-1}{k}\lambda\right). \qquad (5)$$

(see [4], p. 489, Eq. (10-16)). This corresponds to our model with X equal to the constant value $\sqrt{\lambda}$, and $Z_1, Z_2, \ldots, Z_M$ independent normal random variables with mean zero and variance 1. Alternatively, (5) gives the symbol error probability when an MFSK frequency-hopped spread-spectrum system is used vs. a full-band noise jammer.

In a MFSK/FH spread-spectrum system v. an optimized partial-band noise jammer, Houston [2] showed that

$$P_E = \text{same as Eq. (5) for } \lambda \le \lambda_0$$

$$P_E = \frac{k}{\lambda} \qquad \lambda \ge \lambda_0, \qquad (6)$$

where k and λ_0 are given for M = 2, 4, 8, 16, 32 in the following table:

M	k	λ_0
2	.368	2.00
4	.466	2.34
8	.586	2.79
16	.721	3.49
32	.873	3.99

This worst-case jammer found by Houston for $\lambda \ge \lambda_0$ can be described in our model by again taking $X = \sqrt{\lambda}$, and the noise components Z_j independent, each with distribution given by

$$Z = \text{normal } (0;\lambda/\lambda_0 M)) \text{ with prob. } \lambda_0/\lambda$$

$$= 0 \qquad \text{with prob. } 1-(\lambda_0/\lambda)$$

In an MFSK/FH spread-spectrum system vs. optimized partial band tone jammer (restricted to jamming just one of the M-ary tones in each M-ary band), Houston [2] also showed that

$$P_E = \frac{M-1}{M}, \quad \text{for } \lambda \le M$$

$$= \frac{M-1}{\lambda}, \quad \text{for } \lambda \ge M. \qquad (7)$$

For $\lambda \ge M$, this corresponds to our model with $X = \sqrt{\lambda}$, and with Z selected to have just one nonzero component of magnitude $\sqrt{\lambda +}$ with probability $M/(\lambda +)$, and Z = (0,0,0,...,0) with probability $1-M/(\lambda +)$. For $\lambda \le M$, any Z-distribution which guarantees at least one component Z_j of magnitude $\sqrt{\lambda +}$ will achieve (7).

Here are the new results which we obtain in this paper:

• We will show that for a constant signal $X = \sqrt{\lambda}$ (i.e., orthodox MFSK), among all possible noise distributions satisfying (2), the one which gives the largest value of P_E is Houston's optimized tone jammer described by (7). This generalizes a recent result of Levitt [3], who showed that among a class of the jammers more general than those considered by Houston, Houston's remained the optimal one.

• We allow the transmitter the option of counteracting the jammer by using random amplitudes, i.e. by allowing the signal strength X to be a random variable (constrained by (1)). When this is allowed, the problem assumes a definite game-theoretic form. We find that subject to the restrictions (1) and (2), the minimax (saddle-point) strategies can be described as follows:

For $\lambda \le M$:

$$X_0: G(x) = \left(1 - \frac{\lambda}{M}\right) + \frac{\lambda}{M} \cdot \frac{x^2}{2M}, \qquad x \le \sqrt{2M} \qquad (8)$$

$$Z_0: F(z) = z^2/2M, \qquad z \le \sqrt{2M}. \qquad (9)$$

For $\lambda \ge M$:

$$X_0: G(x) = x^2/2\lambda, \qquad x \le \sqrt{2\lambda} \qquad (10)$$

$$Z_0: F(z) = \left(1 - \frac{M}{\lambda}\right) + \frac{M}{\lambda} \cdot \frac{z^2}{2\lambda}, \qquad z \le \sqrt{2\lambda} \qquad (11)$$

(The optimal random jamming vector Z will be shown to have at most one nonzero component. The random variable Z_0 described by (9) and (11) describes this nonzero component, it being understood that Z_0 is equally likely to appear in any of the M components of Z.)

Furthermore, we show that the saddlepoint value of P_E, attained when the strategies of the players (transmitter and jammer) are given by (8) - (11), is

$$P_E^* = \frac{M-1}{M}\left(1 - \frac{\lambda}{2M}\right), \qquad \lambda \le M \qquad (12)$$

$$P_E^* = \frac{M-1}{2\lambda}, \qquad \lambda \ge M. \qquad (13)$$

The optimal strategies (8) - (11) can be given a simple interpretation. For example (10) says that when $\lambda \ge M$, X_0^2 is uniformly distributed on $[0,2\lambda]$; and (11) says that Z_0^2 is uniformly distributed on $[0,2\lambda]$ with probability M/λ, and equal to zero with probability $1-M/\lambda$. Similarly (9) says that Z_0 is uniform on $[0,2M]$, and (8) says that X_0 is a mixture of a $U[0,2M]$ and the constant 0.

We note, comparing (7) and (13), that allowing random transmitter amplitudes gains a factor of 2 (3dB) in signal power relative to a worst case jammer, for $\lambda \geq M$. Furthermore the needed distribution of amplitudes (energy uniformly distributed on $[0,2\lambda]$) is not especially exotic and could be easily implemented.

3. PROOFS

Given fixed nonnegative real numbers $x, z_1, z_2, \ldots, z_M$, if θ is a random variable uniformly distributed on $[0,2\pi]$, define

$$L_j(x;z_1,z_2,\ldots,z_M) = \Pr\{|x+e^{i\theta}z_j| < \max_{k\neq j} z_k\} + \Pr\{|x+e^{i\theta}z_j| = \max_{k\neq j} z_k\} \cdot E\{[\tfrac{\mu_j}{1+\mu_j}] \,|\, |x+e^{i\theta}z_j| = \max_{k\neq j} z_k\}, \qquad (14)$$

where μ_j is the number of z_k's $(k\neq j)$ which take the maximum value. L_j is the value of P_E, conditioned on $X = x$, $Z_1 = z_1, \ldots, Z_M = z_M$, if the signal X is transmitted as the j-th component of $\underline{X}$, provided ties are broken by choosing randomly among the maximizing components of $\underline{R} = (R_1, R_2, \ldots, R_M)$. We can write L_j explicitly in terms of the following parameters:

$$z_j^* = \max_{k\neq j} z_k$$

$$m = \max_{1\leq k\leq M} z_k$$

ν = number of z_k's = m,

ν_j = number of z_k's $(k\neq j)$ = m.

Then for $x > 0$ it is easily seen that

$$L_j(x;z_1,z_2,\ldots,z_M) = 0 \quad \text{if } z_j + z_j^* < x \qquad (15a)$$

$$= 0 \quad \text{if } z_j - z_j^* > x \qquad (15b)$$

$$= 1 \quad \text{if } z_j^* - z_j > x \qquad (15c)$$

$$= \frac{1}{\pi}\cos^{-1}\frac{z_j^2+x^2-z_j^{*2}}{2z_jx} \qquad (15d)$$

$$\text{if } |z_j - z_j^*| < x < z_j + z_j^*$$

$$= \frac{\nu_j}{\nu}, \text{ if } (z_j, z_j^*) = (0,x) \qquad (15e)$$

Figure 1 may help the reader visualize Eqn. (15a)-(15e).

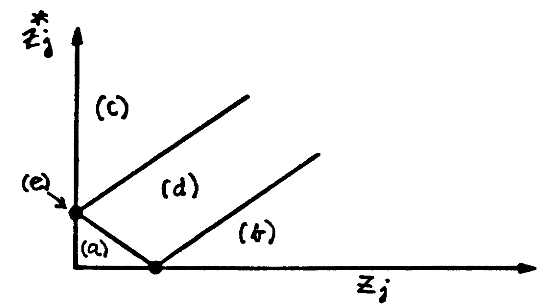


Figure 1. The regions specified by Eqn. (15).

For a fixed x, L_j is a continuous function of z_j and z_j^*, except at the point $E = (0,x)$.

For $x = 0$ (signal absent), we obviously have

$$L_j(0;z_1,z_2,\ldots,z_M) = 0, \text{ if } z_j > z_j^*$$

$$= 1, \text{ if } z_j < z_j^* \qquad (16)$$

$$= \frac{\nu_j}{\nu}, \text{ if } z_j = z_j^*.$$

The functions L_j allow us to give a convenient expression for P_E, in terms of the distribution functions (strategies) G and F introduced in Section 1. If

$$K(x;z_1,z_2,\ldots,z_M) = \frac{1}{M}\sum_{j=1}^{M} L_j(x;z_1,\ldots,z_M) \qquad (17)$$

then the error probability P_E is given by

$$P_E(G,F) = \iint K(x;z_1,z_2,\ldots,z_M)\,dG(x)\,dF(z_1,\ldots,z_M). \qquad (18)$$

The limiting values $P_E^*(G)$ and P_E^* (see (3) and (4)) need not be attained, since the kernel K defined in (17) is not continuous. However if we redefine the kernel (pessimistically, from the transmitter's viewpoint) so that ties are always broken in favor of the jammer, i.e.

$$\overline{K}_j(x;\underline{z}) = \begin{cases} 1 \text{ if } (z_j, z_j^*) = (0,x)\ ,\ x \neq 0 \\ L_j \text{ otherwise,} \end{cases} \qquad (19)$$

it follows that

$$\overline{L}_j(x;\underline{z}) = \limsup_{\underline{z}' \to \underline{z}} L_j(x;\underline{z}'). \qquad (20)$$

From this it easily follows that in fact

$$P_E^*(G) = \sup_F \overline{P}_E(G,F), \qquad (21)$$

where $\overline{P}_E$ is defined in terms of $\overline{L}_j$ instead of L_j, and that an extremal distribution in (21) does exist. From now on we consider only $\overline{L}_j$, $\overline{K}$, and $\overline{P}_E$.

The variational condition for a distribution F maximizing $\overline{P}_E(G,F)$ is

$$\int \overline{K}(x;z_1,\ldots,z_M)dG(x) - \lambda(z_1^2+\ldots+z_M^2) - \mu \leq 0, \qquad (22)$$

for all values of $(z_1, z_2, \ldots, z_M)$, with equality at all points of support of dF. In (22), λ and μ are Lagrange multipliers for the two side conditions on F, viz.

$$\frac{1}{M}\int (z_1^2+z_2^2+\ldots+z_M^2)dF = 1 \qquad (23)$$

$$\int dF = 1. \qquad (24)$$

The λ term in (22) is the only unbounded term as $z_1^2+\ldots+z_M^2 \to \infty$. Hence $\lambda \geq 0$. Furthermore the expression

$$\overline{P}_E(G,F) - \lambda \int (z_1^2+\ldots+z_M^2)dF - \mu \int dF \qquad (25)$$

is a convex functional of the distribution F, and so the variational condition (22) is a sufficient as well as a necessary condition for the attainment of the maximum in (21). We now state our basic result, which says that the worst-case jammer vs. <u>any</u> distribution of signalling amplitudes is a jammer which concentrates its energy in just <u>one</u> of the M-ary tones.

<u>Theorem 1.</u> For any fixed G(x), the error probability $P_E(G,F)$ is maximized by a distribution with $Z_2 = Z_3 = \ldots = Z_M = 0$. If $M \geq 3$, no maximizing distribution can have more than one $Z_j > 0$ simultaneously, unless $P_E = (M-1)/M$.

The proof of Theorem 1 depends on the following lemma.

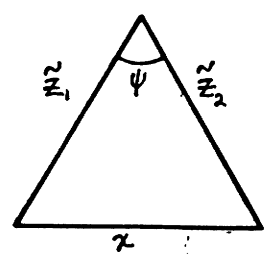
<u>Lemma</u>: $\overline{K}(x;\underline{z})$ is a non-increasing function of $x \geq 0$.

<u>Proof of Lemma</u>: Suppose $z_1, z_2, \ldots, z_M$ are a rearrangement of $\tilde{z}_1 \geq \tilde{z}_2 \geq \ldots \geq \tilde{z}_M$. Then for $x > 0$,

$$M\cdot K(x;\tilde{\underline{z}}) = \overline{L}_1(x;\tilde{\underline{z}}) + \overline{L}_2(x;\tilde{\underline{z}}) + \sum_{k=3}^{M} \overline{L}_k(x;\tilde{\underline{z}}). \qquad (26)$$

If $\tilde{z}_2 > 0$, the sum of the first two terms is

$$\overline{L}_{1,2}(x;\tilde{\underline{z}}) = \begin{cases} 0\ , \text{ if } x \geq \tilde{z}_1 + \tilde{z}_2 \\ 1\ , \text{ if } x \leq \tilde{z}_1 - \tilde{z}_2 \\ 1 - \frac{\psi}{\pi}, \text{ if } \tilde{z}_1 - \tilde{z}_2 < x < \tilde{z}_1 + \tilde{z}_2. \end{cases} \qquad (27)$$

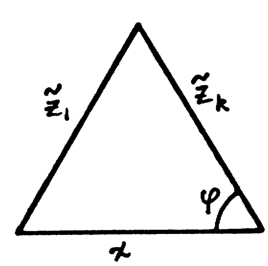


<u>Figure 2</u>. Illustrating (27).

The angle ψ is clearly an increasing function of x, so the term $(\overline{L}_1 + \overline{L}_2)$ in (26) is a decreasing function of $x > 0$.

For $k \geq 3$, if $\tilde{z}_k > 0$, we have

$$\overline{L}_k(x;\tilde{\underline{z}}) = \begin{cases} 0\ , \text{ if } x \geq \tilde{z}_1 + \tilde{z}_2 \\ 1\ , \text{ if } x \leq \tilde{z}_1 - \tilde{z}_2 \\ \frac{\phi}{\pi}, \text{ if } \tilde{z}_1 - \tilde{z}_2 < x < \tilde{z}_1 + \tilde{z}_2 \end{cases} \qquad (28)$$



<u>Figure 3</u>. Illustrating (28).

In Fig. 3, ϕ is a decreasing function of x, so $\tilde{L}_k$ is likewise decreasing for $x > 0$. If $\tilde{z}_k = 0 < \tilde{z}_1$,

$$L_k(x;\tilde{z}) = \begin{cases} 0, & \text{if } x > \tilde{z}_1 \\ 1, & \text{if } x \le \tilde{z}_1, \end{cases}$$

which is also decreasing. Similarly $\bar{L}_2$ is decreasing if $\tilde{z}_2 = 0 < \tilde{z}_1$. Finally, if $\tilde{z}_1 = 0$, $\bar{K} \equiv 0$ for $x > 0$, is also non-increasing. Since $\bar{K}(x;\underline{z}) \le (M-1)/M = \bar{K}(0;\underline{z})$, $\bar{K}$ is also non-decreasing at $x = 0$. This completes the proof of the lemma.

Proof of Theorem 1:

If $z_2 = \ldots = z_M = 0$, we have

$$\bar{K}(x;\underline{z}) = \begin{cases} (M-1)/M & x \le z_1 \\ 0 & x > z_1, \end{cases} \tag{30}$$

and so if we restrict the condition (22) to $z_2 = \ldots = z_M = 0$, we get

$$\frac{M-1}{M} G(z_1) - \lambda z_1^2 - \mu \le 0. \tag{31}$$

To prove Theorem 1, we need to show that (22) is implied by (31), and to investigate when equality cannot occur in (22).

For a given $z_1, z_2, \ldots, z_m$,

$$\int_0^\infty \bar{K}(x;\underline{z})dG(x) = \bar{K}(x;\underline{z})G(x)\Big|_{0-}^{\infty} + \int_{0-}^{\infty} G(x)\,[-d\bar{K}(x;\underline{z})]. \tag{32}$$

The integrated term in (32) is zero at both limits. If the integral in (32) is denoted by I, using (31) and the lemma we get

$$\begin{aligned} I &\le \frac{M}{M-1}\int_{0-}^{\infty}(\lambda x^2+\mu)[-d\bar{K}(x;\underline{z})] \\ &= \frac{M}{M-1}\left\{ -(\lambda x^2+\mu)\bar{K}(x;\underline{z})\Big|_{0-}^{\infty} + \lambda\int_0^\infty \bar{K}(x;\underline{z}).2xdx \right\} \\ &= \mu + \frac{M\lambda}{M-1}\int_0^\infty \bar{K}(x;\underline{z}).2xdx. \end{aligned} \tag{33}$$

If $\lambda = 0$, (32) and (33) give

$$\bar{K}(x;\underline{z})dG(x) \le \mu,$$

for all $(z_1,\ldots,z_M)$. Taking $z_2 = \ldots = z_M = 0$, and $z_1 \to \infty$, (30) implies that $\mu \ge (M-1)/M$. Then

$$\bar{P}_E(G,F) = \iint \bar{K}(x;\underline{z})dG(x)dF(\underline{z}) = \mu \ge \frac{M-1}{M},$$

which is an extreme case in which the communicator is completely overwhelmed by the jammer. We henceforth assume $\lambda > 0$. We have from (15)

$$\int_0^\infty L_j(x;\underline{z})2xdx = \int_0^{z_j^*-z_j+} 2xdx + \int_{|z_j^*-z_j|}^{z_j^*+z_j} \frac{1}{\pi}\cos^{-1}\frac{x^2+z_j^2-z_j^{*2}}{2xz_j}.2xdx. \tag{34}$$

The second integral in (34) is

$$\begin{aligned} &\frac{1}{\pi}\int_{|z_j^*-z_j|}^{z_j^*+z_j} \psi \cdot 2xdx \\ &= \frac{1}{\pi}\left(x^2\psi - z_j z_j^* \sin\psi + z_j^{*2}\psi\right)\Big|_{|z_j-z_j^*|}^{z_j+z_j^*} \\ &= z_j^{*2} - (z_j^* - z_j)_+^2 \end{aligned} \tag{35}$$

(see Figure 4).

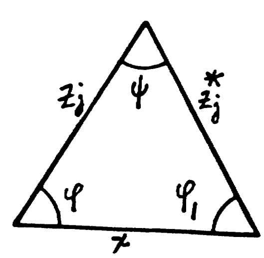


Figure 4. Illustrasting (35).

Hence

$$\int_0^{\infty} L_j\,(x;\underline{z})\cdot 2x\,dx = z_j^{*2}\text{, and} \qquad (36)$$

$$\int_0^{\infty} \bar{K}(x;\underline{z})\cdot 2x\,dx = \frac{1}{M}\sum_{j=1}^{M} z_j^{*2}$$

$$= \frac{1}{M}\,[(M-1)\tilde{z}_1^{\,2} + \tilde{z}_2^{\,2}]$$

$$\leq \frac{M-1}{M}\,(z_1^2 + \cdots + z_M^2)\,. \qquad (37)$$

Equation (37) combined with (33) is the desired condition (22), viz.

$$I \leq \mu + \lambda\,(z_1^2 + \ldots + z_M^2). \qquad (38)$$

This shows that a one dimensional distribution F_1 solves the extremal problem. Also , since quality cannot occur in (37) if $M \geq 3$ and $\tilde{z}_2 > 0$, any extremal distribution for $M \geq 3$ has at most one z_i at a time different from 0. This completes the proof of Theorem 1.

We conclude the paper with proof of the conditions (8) - (11) for the minimax problem (4). From the above work, the variational condition on F(z) is (22))

$$\frac{M-1}{M}\,G(z) - \lambda z^2 - \mu \leq 0, \qquad (39)$$

with equality at the points of support of F. The corresponding condition on G can be shown to be

$$\frac{M-1}{M}\,F(x) - \sigma^2 - \tau \leq 0, \qquad (40)$$

with equality at the points of support of G. These conditions are easily checked to be satisfied by the distributions in (8) - (11). A more direct proof of essentially the same theorem is given in [5], Theorem 4.2.

As a final remark, we note that, given the result of Theorem 1, the solution to the minimax problem (4) reduces to a simply-stated-game-theoretic problem. Two players, I and II, each declare a non-negative real number. If player I's number is larger, he or she is awarded one unit. If the players are restricted by placing an upper bound on the average value of their numbers, what is the value of the game? Our results (8) - (11) show that the value of the game is given by

$$1 - \frac{1}{2a} \quad \text{for } a \geq 1$$

$$\frac{a}{2} \quad \text{for } a \leq 1$$

where a is the ratio of the first player's mean to the second's. A similar result (for a = 1) was derived in an economic context by Bell and Cover [1].

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