FA-17 9:30

Proceedings of the 33rd
Conference on Decision and Control
Lake Buena Vista, FL - December 1994

Finite Time Horizon Robust Performance Analysis

Jorge E. Tierno* John C. Doyle
Electrical Engineering, 116-81,
California Institute of Technology,
Pasadena, CA 91125, U.S.A.

Abstract

Robust performance problems for linear time varying systems considered over a finite horizon, are reduced to the computation of the structured singular value of a finite matrix. Connections are established between the time domain and frequency domain tests.

Keywords: finite horizon robustness

1 Introduction

Most of the research done on robust performance analysis, with an $l_2 \to l_2$ performance measure, has been done in the frequency domain. If we are considering the system over an infinite time horizon and the system is linear and time invariant then the frequency domain approach is equivalent to the time domain one, but presents certain computational and technical advantages. However if we are working with finite time horizon, non-linear or time varying systems, the frequency domain approach can't be used and we are thus forced to set up our performance specifications, and uncertainty descriptions directly in the time domain.

In this paper we will explore different possible setups for uncertainty and performance requirements in the time domain based on quadratic constraints for discrete time systems, and we will show how they can be reduced to the computation of the structured singular value of a constant matrix.

An important issue in time domain based tests is computational complexity. It is to be expected that the complexity will grow with the length of the time horizon. It is important to exploit the structure of the matrices associated with the time domain tests to avoid this growth becoming prohibitive. We will discuss how the standard algorithm for computing the lower bounds for μ can be modified to take advantage of the structure of this particular problem.

We also investigate the behavior of the time domain tests in the limit when the time horizon tends to infinity. We establish connections between these limits and the frequency domain robustness tests. We expect these connections to shed light on the nature of the frequency domain tests for systems with uncertainty described by integral quadratic constraints.

*E-Mail: jorge@hot.caltech.edu

2 Preliminaries

The notation used here is fairly standard and is essentially taken from [1] and [2]. For any square complex matrix M we denote the complex conjugate transpose by M^*. The largest singular value and the structured singular value are denoted by $\overline{\sigma}(M)$ and $\mu(M)$ respectively. The spectral radius is denoted $\rho(M)$. For any complex vector x, then x^* denotes the complex conjugate transpose, and $|x|$ the Euclidean norm. $\{a(i)\}_{i=0}^{n}$ denotes the sequence $a(1)..a(n)$. $\mu_\Delta(M)$ denotes the structured singular value of M with respect to the uncertainty set Δ.

3 Performance Analysis

3.1 Uncertain Finite Time Horizon Systems

LTI systems are normally described as transfer functions. However, in order to derive computable tests we have to describe them in terms of constant matrices. This is achieved in state space by incorporating the delay operator into the uncertainty, and in the frequency domain case by doing a search over frequency, the analysis at each frequency point reducing to a constant matrix problem. None of these approaches can be applied to systems considered over a finite time horizon. However, for these systems a natural finite matrix representation can be achieved by mapping the temporal axis into the spatial one. To illustrate this concept consider the system that obeys the following equations,

$$\begin{aligned} x_{i+1} &= A(i)x_i + B(i)u_i \\ y_{i+1} &= C(i)x_i + D(i)u_i \end{aligned} \tag{1}$$

over a time horizon of length 4. These equations can be rewritten as:

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = M(i) \begin{pmatrix} x_i \\ u_i \end{pmatrix} \tag{2}$$

We can now define a mapping from the initial state and the time history of the inputs from $i = 0$ to 3, to the final state and the time history of the outputs from $i = 0$ to 3. Denote that mapping with the symbol $M_{[3]}$ (see figure 1). The system is now represented by a single constant matrix $M_{[3]}$, over which we can write our performance bounds. However the size of the matrix $M_{[k]}$ grows with the length of the time horizon. This means that a strong emphasis has to be put in the development of efficient algorithms for the computation of the stability tests.

We can add uncertainty to this system to model time varying or time invariant parameters or norm bounded unmodeled dynamics. As is the case with infinite horizon systems,

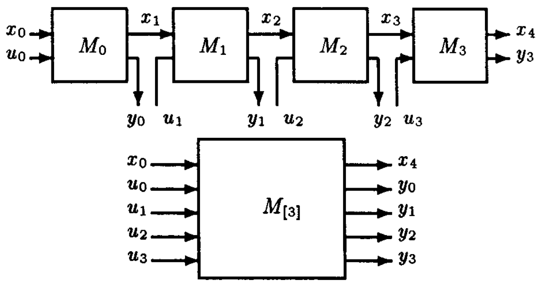

Figure 1: Conversion of the problem to a constant matrix

we will describe the uncertain model as a linear fractional transformation of a nominal plant,and a structured uncertainty operator. As an example, figure (2), shows how time varying parametric uncertainty can be added to the system in (2).

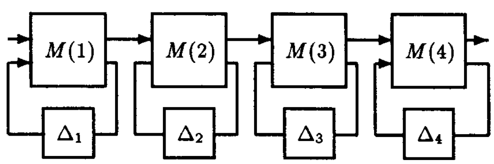

Figure 2: Adding uncertainty as an LFT

In the sections that follow we will show how to form the uncertain system as an LFT for the other classes of problems.

3.2 Robust Performance Problems

A wide class of system analysis problems can be characterized as noise rejection problems. In this case, given a bound on a particular norm of the inputs, it is desired to find the worst case norm of the output. The quotient between those two bounds is called the performance of the system. In infinite horizon systems, stability is a precondition for the norms of the outputs, and thus performance, to be defined. This is not the case in discrete time finite horizon systems.However, for systems described as LFT's we will still require as a precondition for performance that when the driving input (the one associated to the performance condition) and the initial state are zero, then all internal signals are zero.

In what follows we will describe several robust performance problems, and we will recast them as a μ computation for an adequate constant matrix and block structure. For simplicity we will write the equations for a first order, 2 input, 2 output, system considered over a 3 time steps horizon, however they can all be applied to an arbitrary system, and the generalization of the formulas is straightforward. For signals that are functions of time we will denote $a(k)$ its value at time k and $a = (a(1), \cdots a(T))$ the vector corresponding to its time history.

3.2.1 $l_2 \rightarrow l_2$ performance under parametric uncertainty

We will start with the discrete time version of the standard robust performance question that is answered with a μ test in the LTI, infinite time horizon case. Given a system as in (2) and a partition of the inputs and outputs:

$$\begin{aligned} u(k) &= (u_0(k), u_1(k))^t \\ y(k) &= (y_0(k), y_1(k))^t \end{aligned}$$

we would like to answer the following :

Question 1 *If*

$$\begin{aligned} u_1(k) &= \delta_1(k) y_1(k) \quad k = 0,1,2 \\ |\delta_i(k)| &\leq 1 \end{aligned}$$

is it true that the following two conditions hold:

$$\begin{aligned} x(0) = u_0(k) = 0 \quad k = 0,1,2 \Rightarrow \\ x(3) = u_1(k) = y_0(k) = y_1(k) = 0 \quad k = 0,1,2 \end{aligned}$$

and:

$$|y_0|^2 + |x(3)|^2 < |u_0|^2 + |x(0)|^2 \tag{3}$$

?

Remark: The first condition is similar to the requirement of stability in infinite time horizon. We require that if the system is not driven, then all internal signals should remain zero. The second condition is the counterpart of the performance condition in the standard setup.

When the answer to question (1) is yes we say that the system meets the robust performance requirement. The answer to question (1) is yes if and only if for every $\Delta_p \in \mathbb{R}^{4\times4}$ with $\|\Delta_p\|_2 \leq 1$, for all $\delta(i) \in [-1,1]$, the following set of equations has only the trivial solution:

$$\begin{aligned} (x_3, y_0, y_1)^t &= M_{[3]}(x_0, u_0, u_1)^t \\ (x_0, u_0)^t &= \Delta_p(x_3, y_0)^t \\ u_1(i) &= \delta(i) y_1(i) \quad i = 1,2,3 \end{aligned} \tag{4}$$

where $M_{[3]}$ is constructed as it was shown in the preceeding section. This is equivalent to $\Delta_1 M_{[3]}$ being invertible for every

$$\begin{aligned} &\Delta_1 \in \mathbf{\Delta}_1 = \\ &\{blockdiag(\Delta_p, \delta(1), \delta(2), \delta(3))\} \end{aligned}$$

and thus by the definition of the structured singular value the answer to question 1 is yes if and only if:

$$\mu_{\mathbf{\Delta}_1}(M_{[3]}) < 1 \tag{5}$$

Remark: It is not necessary for the parameters to vary with time. The temporal nature of the parameters is reflected by the uncertainty structure in $\mathbf{\Delta}_1$. This shows an important difference between the finite and infinite horizon cases. In the latter, the temporal nature of the uncertainty determines the test to perform (μ, frequency domain upper bound, state space upper bound.) In the former the temporal nature of the uncertain operators is reflected in the block structure.

3.2.2 $l_\infty \rightarrow l_2$ performance under parametric uncertainty

Another possible performance requirement, is to require a bound on the total energy in the output, given that the magnitude of the input, at each time instant is bounded. (The bound being either constant or a function of time). We will show how to set this performance requirement as a μ problem for a system with parametric uncertainty:

For the system in the preceeding section we would like to answer now the following:

Question 2 *If*

$$\begin{aligned} u_1(k) &= \delta_1(k) y_i(k) \\ |\delta_1(k)| &\leq 1 \\ |u_0(k)| &\leq 1 \quad k = 0, 1, 2 \\ |x(0)| &\leq 1 \end{aligned}$$

is it true that the following two conditions hold:

$$\begin{aligned} &x(0) = 0 \quad u_0 = 0 \Rightarrow \\ &\quad x(3) = u_1(k) = y_0(k) = y_1(k) = 0 \quad k = 0, 1, 2 \end{aligned}$$

and:

$$|y_0|^2 + |x(3)|^2 < 1 \tag{6}$$

?

This question can be answered by determining whether or not the the following set of equations has nontrivial solutions:

$$\begin{aligned} (x_3, y_0, y_1)^t &= M_{[3]}(x_0, u_0, u_1)^t \\ (x_0, u_0)^t &= blockdiag(\delta_x, \delta_0(0), \delta_0(1), \delta_0(2)) a \\ a &= \Delta(x_3, y_0(0), y_0(1), y_0(2))^t \\ u_1(i) &= \delta_1(i) y_1(i) \quad i = 1, 2, 3 \end{aligned} \tag{7}$$

for all $\Delta \in \mathbb{R}^{1\times 4}$ with $\|\Delta\|_2 \leq 1$, for all $\delta_j(i) \in [-1, 1]$, $\delta_x \in [-1, 1]$. Partition the matrix $R = M_{[3]}$ according to the partition in the input and output vectors:

$$\begin{matrix} [x_0, u_0(0), u_0(1), u_0(2)] & [u_1(0), u_1(1), u_1(2)] \\ [x_3, y_0(0), y_0(1), y_0(2)] & [y_1(0), y_1(1), y_1(2)] \end{matrix}$$

and build the matrix:

$$M_e = \begin{bmatrix} 0 & R_{11} & R_{12} \\ I_4 & 0 & 0 \\ 0 & R_{21} & R_{22} \end{bmatrix}$$

Again from the definition of μ, the system will only have trivial solutions, and therefore the answer to question 2 is yes if and only if:

$$\mu_{\Delta_2}(M_e) < 1$$

where:

$$\begin{aligned} \Delta_2 = \{&blockdiag(\Delta_y, \delta_x, \delta_0(i), \delta_1(i)), \\ &\Delta_y \in \mathbb{R}^{1\times 4}, \ \delta_j(i) \in [-1, 1]\} \end{aligned}$$

3.2.3 Affine Systems

In some applications we need to set an affine performance specification. For example suppose the desired response to the input u^o is y^o. Given that there is bounded noise added to the command signal u^o, we would like to know whether or not the maximum possible distance from the desired trajectory is smaller than a preset amount d. If y is the output corresponding to a given input u, our performance requirement is met if and only if:

$$\max_{|u - u^o| \leq n} |y - y^o| < d$$

In what follows we show how this question when asked of an uncertain system can can be recast as μ problem.

For the system in the preceeding sections, and a given set of signals u^o, x^i, y^o, x^f we would like to answer the following question:

Question 3 *If*

$$\begin{aligned} u_1(k) &= \delta_1(k) y_1(k) \quad k = 0..2 \\ |\delta_i(k)| &\leq 1 \end{aligned}$$

is it true that the following two conditions hold:

$$\begin{aligned} &x(0) = u_0(k) = 0 \quad k = 0, 1, 2 \Rightarrow \\ &\quad x(3) = u_1(k) = y_0(k) = y_1(k) = 0 \quad k = 0, 1, 2 \end{aligned}$$

and:

$$\begin{aligned} &|y_0 + y^o|^2 + |x(3) + x^f|^2 < \\ &\quad |u_0 + u^o|^2 + |x(0) + x^i|^2 \end{aligned}$$

?

We proceed in the same fashion as in the previous cases. Define the signals:

$$(x_3^1, y_0^1, y_1^1)^t = M_{[3]}(x^i, u^0, 0)^t \tag{8}$$

The answer to question (3) is yes if and only if the following system of linear equations has only the trivial solution:

$$\begin{aligned} (x_3^a, y_0^a, y_1)^t &= M_{[3]}(x_0^a, u_0^a, u_1)^t + \\ &\quad (x^f - x_3^1, y^o - y_0^1, -y_1^1) a \\ a &= \Delta_p (x_3^a, y_0^a)^t \\ u_0^a &= \Delta_i a \\ u_1(i) &= \delta_1(i) y_1(i) \quad i = 1, 2, 3 \end{aligned} \tag{9}$$

for all $\Delta_p \in \mathbb{R}^{1\times 4}$, $\Delta_1 \in \mathbb{R}^{4\times 1}$ with $\|\Delta_p\|_2 \leq 1$, $\|\Delta_i\|_2 \leq 1$ for all $\delta_1(i) \in [-1, 1]$. To prove this claim, note that if there exists a solution to these equations with $a \neq 0$ then there is one with $a = 1$. In this case the equations can be rewritten as:

$$\begin{aligned}
(x_3^a, y_0^a, y_1)^t &= M_{[3]}(x_0, u_0, u_1)^t + \\
&\quad (x^f, y^o, 0)^t \\
x_0 &= x_0^a - x^i \\
u_0 &= u_0^a - u^o \\
a &= \Delta_p(x_3^a, y_0^a)^t \\
u_0^a &= \Delta_i a \\
u_1(i) &= \delta_1(i) y_1(i) \quad i = 1,2,3
\end{aligned} \tag{10}$$

and thus the second condition of question (3) is violated. Similarly, the reader can verify that if a nontrivial solution with $a = 0$ exists, the first condition of the question is violated. Now build the matrix:

$$M_e = \begin{bmatrix} x^o - x_3^1 & R_{11} & R_{12} \\ y^o - y_0^1 & & \\ 1 & 0 & 0 \\ -y_1^1 & R_{21} & R_{22} \end{bmatrix}$$

and the set:

$$\begin{aligned}
\Delta_3 = \{&blockdiag(\Delta_p, \Delta_i, \{\delta_1(i)\}_{i=0}^{i=3}) \\
&\Delta_p \in \mathbb{R}^{1\times 4},\ \Delta_i \in \mathbb{R}^{1\times 4},\ \delta(i) \in \mathbb{R}\}
\end{aligned}$$

then the system (9) has only trivial solutions and therefore the answer to question (3) is yes if and only if

$$\mu_{\Delta_3}(M_e) < 1$$

3.2.4 Performance under uncertainty with memory

In the preceeding sections, we described how different performance requirements could be evaluated with respect to parametric uncertainty. We will now show how we can evaluate robustness with respect to dynamic uncertainty. The dynamic operators we will consider can be either time invariant (that is their matrix has a Toeplitz structure), or time varying (the matrix is lower triangular), and they are bounded in the $l_2 \rightarrow l_2$ induced norm. For simplicity we will describe the time invariant case with an l_2 into l_2 performance specification, but as in the preceeding sections the procedure is general.

Question 4 *If*

$$\begin{aligned}
u_1(k) &= \sum_{j=0}^{k} \delta_1(k-j) y_1(j) \quad k = 1,2,3 \\
\sum_{j=0}^{T} |\delta_1(j)|^2 &\leq 1
\end{aligned} \tag{11}$$

is it true that:

$$\begin{aligned}
&x(0) = u_0(k) = 0 \quad k = 0,1,2 \Rightarrow \\
&\quad x(3) = u_1(k) = y_0(k) = y_1(k) = 0 \quad k = 0,1,2
\end{aligned}$$

and:

$$|y_0|^2 + |x(3)|^2 < |u_0|^2 + |x(3)|^2 \tag{12}$$

?

Again, as in the preceeding sections, it can be shown that the answer to question (11) is yes if and only if the following system of equations has only trivial solutions:

$$\begin{aligned}
(x_3, y_0, y_1)^t &= M_{[3]}(x_0, u_0, u_1)^t \\
(x_0, u_0)^t &= \Delta_p(x_3, y_0)^t \\
v(i,j) &= \delta(i-j) y_1(i) \\
u_1(i) &= \sum_{j=0}^{i} v(i,j) \quad i = 0,1,2 \\
(u_1)^t &= \Delta_u(y_1)^t
\end{aligned} \tag{13}$$

for all $\Delta_p \in \mathbb{R}^{4\times 4}$ with $\|\Delta_p\|_2 \leq 1$, for all $\Delta_u \in \mathbb{R}^{3\times 3}$ with $\|\Delta_u\|_2 \leq 1$,for all $\delta(i) \in [-1,1]$ This system of equations can be represented by the diagram in figure (3)(for simplicity we only represent two time steps). This diagram is different as the one in the preceding sections since it includes implicit equations. Recent work ([5]) has extended the definition of μ to this kind of systems.

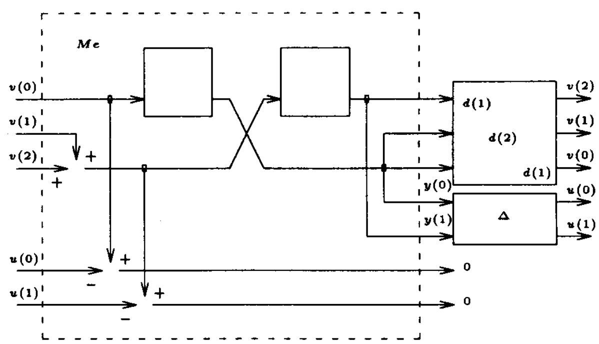

Figure 3: Dynamic uncertainty as an LFT with implicit equations

To build the matrix corresponding to figure (3), partition the matrix R as in the preceeding section, and introduce the variables:

$$\begin{aligned}
w &= [x_0, u_0, v, u_1]^t \\
z &= [x_3, y_0, y_1, y_1]^t
\end{aligned}$$

Define also the following two matrices:

$$M_e = \begin{bmatrix} M_{11} & 0 & M_{12} \\ M_{21} & 0 & M_{22} \\ M_{21} & 0 & M_{22} \end{bmatrix}$$

$$C = \begin{bmatrix} 0_{1\times 4} & 100000 & -1\ 0\ 0 \\ 0_{1\times 4} & 011000 & 0\ -1\ 0 \\ 0_{1\times 4} & 000111 & 0\ 0\ -1 \end{bmatrix}$$

It can be verified that the systems in (13) have only trivial solutions if and only if:

$$\ker\left(\begin{bmatrix} I - \Delta_4 M_e \\ C \end{bmatrix}\right) = \{\vec{0}\} \qquad (14)$$

for all $\Delta_4 \in \mathbf{\Delta_4}$ where:

$$\mathbf{\Delta_4} = \{blockdiag(\Delta_p, \{\delta(i,j)\}_{i=0,j=0}^{i=3,j=i}, \Delta_u), \\ \Delta_p \in \mathbb{R}^{4\times 4},\ \delta(i,j) \in \mathbb{R}, \Delta_u \in \mathbb{R}^{3\times 3}\}$$

Following the definition in [5], the answer to question (11) is yes if and only if:

$$\mu_{C,\mathbf{\Delta_4}}(M_e) = 0 \qquad (15)$$

Remark: For each performance requirement, we can add different kinds of uncertainty. Thus for all the performance questions described we can compute robustness with respect to any mix of dynamic and parametric uncertainty both time varying and time invariant over the horizon considered.

4 Computational Issues

4.1 Lower bound

It was shown in [2] that the computation of a lower bound for $\mu(M)$ can be tackled via a power iteration. Although in theory this algorithm can be used directly to establish a sufficient condition for the time domain performance specifications, it needs to be modified for practical considerations.

Extensive experimentation done with the power algorithm shows that the complexity of the algorithm is dominated by the multiplication of M and M^* with corresponding vectors. The cost of these operations is proportional to the square of the size of M; in time domain tests, the size of M grows linearly with the number of time steps considered. However, due to the special nature of the matrices involved, quadratic growth of the computation time with the number of time steps can be avoided.

In order to see how the structure of $M_{[k]}$ can be exploited to reduce the time complexity of the calculation, the multiplication:

$$M_{[k]} \begin{pmatrix} x_0 \\ u_0 \\ \vdots \\ u_{k-1} \end{pmatrix} \qquad (16)$$

is equivalent to computing the output of the system, when the initial condition is x_0 and the input is given by $u_0 \dots u_{k-1}$. Thus multiplication by $M_{[k]}$ can be done using the following sequence of operations:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M(1)\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \qquad (17)$$

$$\vdots$$

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = M(k)\begin{pmatrix} x_{k-1} \\ u_{k-1} \end{pmatrix}$$

The computation time for this set of operations grows linearly with the number of time steps considered. The multiplication by $M_{[k]}^*$ is similar, and involves simulating backwards in time the transpose of the original system.

4.2 Upper bound

The same computational complexity problems arising in the lower bound arise in the computation of the upper bound. However, it is not as straightforward to use the structure in the matrix $M_{[k]}$ to reduce the growth of the computation time, when using current state of the art optimization algorithms for LMI's.

Using gradient search, we can develop an algorithm that is slower on small problems when compared to the LMI methods, but whose computation time doesn't degrade as much with the number of time samples. This is due to the fact that the complexity in computing the gradient depends on the number of repeated singular values, and this apparently is more strongly related to the order of the system than to the number of time samples taken. However our tests of this algorithm are still preliminary and more extensive experimentation and further research is needed in this area.

5 Connections to the frequency domain tests

If the system under consideration is LTI, the uncertainty description is repeated at each time instant and the performance condition is given as a full block mapping the final state to the initial state, we can develop some connections between the time domain tests and the corresponding frequency domain ones. From these connections we expect to derive a better understanding of the nature of the frequency domain tests for LTI systems.

Consider an n-dimensional linear time invariant system, defined by the equations:

$$x(k+1) = Ax(k) + Bu(k) \qquad (18)$$

$$y(k) = Cx(k) + Du(k) \qquad (19)$$

Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (20)$$

$$G(z) = zI_n * M \qquad (21)$$

where $*$ denotes the Redheaffer star-product and let $\mathbf{\Delta}$ be an uncertainty structure. Let $M[k]$ be the time domain mapping associated to G, with the following uncertainty structure:

$$\mathbf{\Delta}_{[k]} = \{blockdiag(\underbrace{\Delta, \Delta, \cdots, \Delta}_{k}), \Delta \in \mathbf{\Delta}\}$$

With this definitions we will have:

$$M_{[k]} * \Delta_{[k]} = (M * \Delta)^k \qquad (22)$$

Denote

$$td\mu(M,k) = \mu_{\mathbf{\Delta_1}}(M_{[k]}) \qquad (23)$$

$$tdub(M,k) = ub_{\mathbf{\Delta_1}}(M_{[k]}) \qquad (24)$$

where

$$\mathbf{\Delta}_1 = \{blockdiag(\Delta_p, \Delta_u), \Delta_p \in \mathbb{C}^{nxn}, \Delta_u \in \mathbf{\Delta}_{[k]}\}$$

and $ub_{\mathbf{\Delta}}$ denotes the $\mu_{\mathbf{\Delta}}$ upper-bound as defined in [3]. Finally, for an uncertainty set $\mathbf{\Delta}$ we will define the set $\mathbf{O_N \Delta}$ of operators with same block diagonal structure than the elements of $\mathbf{\Delta}$ but with each entry in the set $\mathcal{O}(\mathbb{C}^N)$ of operators defined from $\mathbb{C}^N$ into $\mathbb{C}^N$.

The following is adapted from a theorem by A. Packard ([3]) It connects the robust stability question posed for an LTI system in finite horizon, to the performance of a finite time horizon system.

Theorem 1 *[3] Given a system M an uncertainty structure Δ_u, then there exists $K > 0$ such that for all $k > K$, $td\mu(M(\cdot), k) < 1$, if and only if $\mu_\Delta(M) < 1$*

By using results on the lossless-ness of the S-procedure when the system is a constant matrix ([4], [5], [6]) we can generalize the preceding result to the upper bounds in the frequency and time domain respectively. The following theorem is essentially from [6]:

Theorem 2 *[6] Given a constant matrix M and an uncertainty structure $\mathbf{\Delta}$ then $ub_\Delta(M) < 1$ if and only if $M * \Delta$ is well posed for all $\Delta \in \mathcal{O}_N \mathbf{\Delta}$, for all $N \in \mathbb{N}$*

We will then have the following:

Theorem 3 *There exists $K > 0$ such that for all $k > K$, $tdub(M(\cdot), k) < 1$ only if $fdub_{\Delta_u}(G(z)) < 1$*

PROOF.
$\Rightarrow$ Assume $tdub(M(\cdot), k) < 1$.
Then according to the small gain theorem for all N for all $\Delta_{[k]} \in \mathcal{O}_N \Delta_{[k]}$:

$$\|M_{[k]} * \Delta_u\| < 1$$

From equation (22) it follows that for all Δ in $\mathcal{O}_N \mathbf{\Delta}$, for all $k > K$:

$$\|(M * \Delta)^k\| < 1 \tag{25}$$

Since for any operator A ,

$$\rho(A)^k = \rho(A^k) \leq \|A^k\|$$

equation (25) implies that for all Δ in $\mathcal{O}_N \mathbf{\Delta}$

$$\rho(M * \Delta) < 1 \tag{26}$$

and thus for all $\delta \in \mathbb{C}$, $|\delta| = 1$, for all $\Delta \in \mathcal{O}_N \mathbf{\Delta}$

$$(\delta I_n * M) * \Delta_u$$

is well posed. Thus according to (2):

$$ub_\Delta(\delta I * M) < 1 \; \forall \; \delta \in \mathbb{C}, \; |\delta| = 1$$

and thus

$$fdub_{\mathbf{\Delta}}(M) < 1$$

6 Conclusions

We present a setup in which performance of a time varying, finite time horizon linear system, under uncertainty described by quadratic constraints, can be tested. The performance conditions take the form of standard μ tests on constant matrices. However it is not desirable to use directly the usual computation schemes for the μ lower and upper bounds, since those do not exploit the special structure of the matrix derived from the finite time horizon problem. We discussed a modification to the lower bound power algorithm, that achieves linear growth in the computation time with the number of time steps considered. We also discussed some possible modification to the upper bound algorithms. Our results in this area are however preliminary and we are carrying out further research in this problem. By using lossless-ness results for the S-procedure applied to constant matrices we were also able to establish connections between the frequency domain tests, and the limits of time domain tests.

We expect to extend the setup presented in this paper to the robustness analysis in trajectories of uncertain nonlinear systems and to treat model validation and system ID of uncertain systems with observed data.

Acknowledgements:

The authors would like to thank Richard Murray for many helpful suggestions, and AFOSR, NASA and NSF for partial funding of this work.

References

[1] M. K. H. Fan, A. L. Tits, and J. C. Doyle, "Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics," *IEEE Transactions on Automatic Control*, vol. AC-36, pp. 25–38, 1991.

[2] P. M. Young and J. C. Doyle, "Computation of μ with real and complex uncertainties," in *Proceedings of the 29th Conference on Decision and Control*, pp. 1230–1235, IEEE, 1990.

[3] A. Packard and J. C. Doyle, "The complex structured singular value" in *Automatica* vol 29(1), pp. 71 -109, 1993.

[4] A. Megretski and S. Treil, "S-Procedure and Power Distribution inequalities: A new method in optimization and robustness of uncertain systems," *Mittag-Leffler Institute*, Rep. No. 1, 1990/1991.

[5] F. Paganini and J. C. Doyle, "Necessary conditions for analysis of uncertain behaviors" submitted to the *33rd IEEE Conference on Decision and Control*, 1994.

[6] H. Bercovici , C. Foias and A. Tannenbaum "Structured Interpolation Theory", in *Operator Theory: Advances and Applications* vol. 47 pp. 195 -220, 1990.

[7] A. Rantzer, "Uncertainties with Bounded Rates of Variation," in *Proceedings of the American Control Conference*, pp. 29-30, 1993.

[8] Poola K, Tikku A., "Robust Performance against Time-Varying Structured Perturbations," submitted to *IEEE Transactions on Automatic Control.*