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EXPERIMENTS IN EXPONENTIAL STABILIZATION OF A MOBILE ROBOT TOWING A TRAILER

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ABSTRACT

In this paper we apply some recently developed control laws for stabilization of mechanical systems with nonholonomic constraints to an experimental system consisting of a mobile robot towing a trailer. We verify the applicability of various control laws which have appeared in the recent literature, and compare the performance of these controllers in an experimental setting. In particular, we show that time-periodic, non-smooth controllers can be used to achieve exponential stability of a desired equilibrium configuration, and that these controllers outperform smooth, time-varying control laws. We also point out several practical considerations which must be taken into account when implementing these controllers.

1. INTRODUCTION

In this paper we present experimental results on the use of time-varying feedback controllers for stabilizing mechanical systems with nonholonomic constraints. In particular, we focus on the control of a two-wheeled mobile robot towing a trailer. We restrict ourselves to the point stabilization problem, although many of the techniques and experimental results described here are also applicable to more practical problems as parallel parking and backing into a loading dock.

The fundamental assumption in modeling the kinematics and dynamics of a mobile robot is that the wheels of the robot roll without slipping. This means that each wheel (or pair of wheels connected by an axle) is free to roll in the direction that it is pointing and spin around the vertical axis, without any losses due to friction. This is clearly an idealization and one of the questions which we hope to answer is to what extent this model is accurate enough for use in control design.

We represent a nonholonomic system as a control system of the form

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m. \qquad (1)$$

These systems arise in the context of mechanical systems when nonholonomic constraints restrict the allowable velocities of the system to the subspace of velocities spanned by the vector fields $g_1, \ldots, g_m$. An introduction to these systems can be found in the paper by Bloch *et al.* [1] and also [9]. We assume that the system represented by equation (1) is controllable. The system is referred to as completely nonholonomic in this case.

There is a large and increasing literature on control laws for stabilization of nonholonomic control systems. Part of the reason for the interest in these systems is that they fail to satisfy Brockett's necessary condition for smooth stabilizability [2]. The use of time-varying feedbacks was first studied in the context of mobile robots by Samson [11, 12, 13]. More recently, Coron showed that *all* controllable nonholonomic systems can be asymptotically stabilized to a point using periodic, smooth feedback laws [3], and exponentially stabilized to a point using periodic feedback laws which are smooth everywhere except at the origin [4]. We concentrate on the latter case, building on our previous work in this area [7, 8]. We also implement several smooth stabilizers, using controllers proposed by Teel *et al.* [14]. For the controllers which we implement, the kinematic equations are converted into a special normal form, called "chained form" [9].

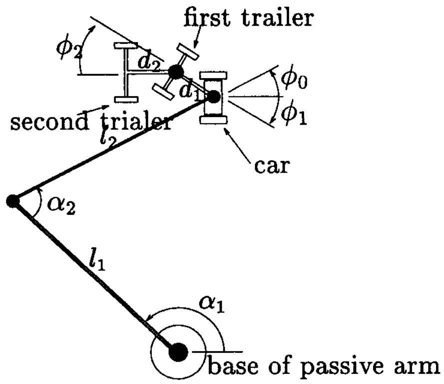

Figure 1: Experimental apparatus

2. DESCRIPTION OF EXPERIMENT

The cursory review of the technical framework in the following section depends very much on the particulars of the model being discussed and so it is useful for the reader to have in mind the physical experiment. The object of the experiments is to stabilize the system about a given position and orientation using feedback. The car is a two-wheeled device with each wheel driven separately by a stepper motor. The position and orientation of the system are sensed using a passive two link manipulator with the base fixed to the floor and the distal end attached to the car. Optical encoders at the manipulator joints and on the car return angle information. The black dots in Figure 1 are the encoder locations.

Once coordinate frames for the car and manipulator are chosen, the forward kinematics of the manipulator is computed to locate the position and orientation of the car. The orientation of the trailers is provided by encoders mounted on the car and first trailer. The important kinematic parameters of the aggregate system are listed in Table 1. The link lengths of the manipulator are denoted l_1 and l_2. The trailer lengths are denoted d_1 and d_2.

The optical encoders provide about 1 mm of resolution when the manipulator is fully extended. The car is powered by two 4-phase permanent magnet stepper motors. The motors are configured so that a single step is

Parameter	Length(cm)
l_1	88.9
l_2	84.6
d_1	19.0
d_2	19.0
car wheelbase	10.0
wheel radii	4.0

Table 1: Kinematic Parameters

0.9 degrees. The motors can handle a maximum step rate of approximately 500 steps per second and still provide a modicum of torque. Saturation of the motors occurs at about 600 steps per second. The step rates of the motors can be varied from more than 400 steps/sec to less than 1 step/sec in increments of less than 1 step/sec. When the stepper motors are used in this configuration they are controlled in an open-loop manner. For example, the control laws compute desired velocities based on the position and orientation of the system. The velocities are then converted into the equivalent "steps per second". The implicit assumption with this method is that the motors can apply the torque required to overcome inertial effects to maintain the proper speed. There is no direct way to verify that the desired velocity is actually achieved. However, since the control laws are continuous the input to the motors is naturally ramped. The experimental results demonstrate that the stepper motors perform quite well. The real-time control software implements a 200 Hz servo loop with a 5th order digital Butterworth filter to smooth all sensor inputs. The sample rate for the feedback control law is 20 Hz.

The kinematic models are presented below. The car with no trailer is represented by the following set of equations:

$$\begin{aligned} \dot{x} &= \cos\theta_0 v \\ \dot{y} &= \sin\theta_0 v \\ \dot{\theta}_0 &= \omega. \end{aligned} \tag{2}$$

The scalar v is the forward velocity of the car and ω is its angular velocity. These are inputs determined by the control law. The Cartesian position of the car is denoted (x, y). The car with a single trailer represents a 4-dimensional nonholonomic system with the model,

$$\begin{aligned} \dot{x} &= \cos\theta_1 v \\ \dot{y} &= \sin\theta_1 v \\ \dot{\theta}_0 &= w \\ \dot{\theta}_1 &= \tfrac{1}{d_1}\tan(\theta_0 - \theta_1)v. \end{aligned} \tag{3}$$

With this particular model x and y are the position of the *trailer*. The forward velocity of the trailer is denoted v and ω is the angular velocity of the car. The forward velocity of the car is computed as $v_{car} = \cos(\theta_0 - \theta_1)v$. The control law computes v and ω and then the car velocity, v_{car}, is determined using the previous expression. The next section briefly reviews the material required to understand the control laws.

3. Control Laws and Analysis

We now present the control laws used in the experiments. The theory used to develop these control laws is described in [7, 8]. We give a brief review of the technical machinery necessary to understand the control laws.

3.1. Dilations and Homogeneous Vector Fields.

This section reviews dilations and homogeneous vector fields. The introduction noted that continuous but not everywhere differentiable feedbacks are required for exponential stabilization. The exponential stabilization of nonholonomic systems is studied in the context of homogeneous vector fields. The idea is to construct a feedback which renders the closed loop system homogeneous of degree zero with respect to a dilation.

Let V be a real finite dimensional vector space of dimension n. Suppose a basis has been chosen for V such that its elements are represented by the following n-tuple:

$$x = (x_1, x_2, \ldots, x_n).$$

A dilation is defined by assigning n positive rationals $r = (1 = r_1 \leq r_2 \leq \cdots \leq r_n)$ and the following map $\delta_\lambda{}^r : V \to V$,

$$\delta_\lambda{}^r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n), \quad \lambda > 0.$$

We usually write δ_λ in place of $\delta_\lambda{}^r$

Definition 1. A continuous function $f : \Re^n \to \Re$ is *homogeneous of degree* $l \geq 0$ *with respect to* $\delta_\lambda{}^r$, denoted $f \in H_l$, if $f(\delta_\lambda{}^r x) = \lambda^l f(x)$. A continuous vector field X on $\Re^n$ is *homogeneous of degree* $m \leq 1$ *with respect to* δ_λ if $Xf \in H_{j-m}$ whenever f is smooth and $f \in H_j$ (Xf is the Lie derivative of the function f with respect to the vector field X).

The standard dilation ($r_i = 1, i = 1, \ldots, n$) on $\Re^n$ warps the space isotropically, that is it stretches each coordinate direction the same amount.

Definition 2. A continuous map from $\Re^n$ to $\Re$, $x \to \rho(x)$, is called a *homogeneous norm* with respect to the dilation δ_λ when

(1) $\rho(x) \geq 0, \quad \rho(x) = 0 \Leftrightarrow x = 0,$

(2) $\rho(\delta_\lambda x) = \lambda \rho(x) \quad \forall \lambda > 0.$

We are primarily interested in the convergence of time dependent functions using the homogeneous norm as a measure of their size. When a vector field is homogeneous it is most natural to use a corresponding homogeneous norm as the metric.

The concept of exponential stability of a vector field is now introduced in the context of a homogeneous norm. This definition was introduced by Kawski [6]. Let $f(t, x)$ be a continuous vector function of its arguments,

$$f(t, x) : \Re \times \Re^n \to \Re^n. \tag{4}$$

Without loss of generality we assume that $x = 0$ is an isolated equilibrium point of the system, $f(t, 0) = 0$, $\forall t$. A solution of the equations passing through x_0 at time t_0 is represented by $x(t, x_0, t_0)$.

Definition 3. The equilibrium point $x = 0$ is *locally exponentially stable with respect to the homogeneous norm* $\rho(\cdot)$ if there exist two strictly positive numbers α and β such that

$$\rho(x(t, x_0, t_0)) \leq \alpha \rho(x_0) e^{-\beta(t - t_0)} \quad \forall t \geq t_0,$$

provided $\rho(x_0)$ is sufficiently small.

We will see that this notion of stability is important when considering vector fields which are homogeneous with respect to a dilation. The convergence of trajectories is naturally studied using the corresponding homogeneous norm. This definition is not equivalent to the usual definition of exponential stability except when the dilation is the standard dilation ($r_i = 1$). However, even in the nonstandard dilation case, each state may be bounded by a decaying exponential envelope.

3.2. Control Laws. We now discuss application of these ideas to the stabilization of the car-trailer system. Consider the situation in which the input vector fields of the nonholonomic system are homogeneous of degree one with respect to some dilation. A feedback that is a homogeneous function of degree one makes the closed loop vector field homogeneous of order zero (using the convention described above). If this feedback is uniformly stabilizing in time then each state may be bounded by a decaying exponential envelope. For a car and trailer system the so-called chained form form coordinates of the input vector fields are homogeneous of degree one with respect to a dilation with powers assigned to a particular state corresponding to the number of Lie brackets of the input vector fields required to span that state direction.

The form of the stabilizing feedbacks for the systems in chained form are motivated from the discussions in [7] and [8]. The actual feedbacks are derived from optimizing the rate of convergence as observed in numerical simulations. There does not yet exist an easy *computational* method for generating Lyapunov functions that may be used for analysis of asymptotically stable homogeneous vector fields. Converse theorems do exist, however they are not useful for specific examples since knowledge of the flow is assumed in constructing the Lyapunov function. See Hahn [5] or the paper by Rosier [10] for more details. However lack of these results should not prevent the applied scientist or engineer from closing the loop on a physical system if extensive simulations indicate stability.

Recall the kinematic model of the car and no trailers. A transformation that converts equation (2) into a set of "almost" homogeneous vector fields is given by

$$\begin{aligned} z_1 &= x\cos\theta_0 + y\sin\theta_0 \\ z_2 &= \theta_0 \\ z_3 &= x\sin\theta_0 - y\cos\theta_0. \end{aligned} \tag{5}$$

This particular change of coordinates has the advantage of being a global diffeomorphism. One can confirm that the vector fields in these coordinates have the form

$$\begin{aligned} \dot{z}_1 &= u_1 - z_3u_2 \\ \dot{z}_2 &= u_2 \\ \dot{z}_3 &= z_1u_2, \end{aligned} \tag{6}$$

where $\omega = u_2$ and $v = u_1$. This system is nilpotent but not homogeneous because of the z_3u_2 present in the first equation. This term is actually improves the convergence properties of the system with the feedbacks given below. One may verify this by using center manifold analysis on the system with the smooth feedback. Hence, we essentially ignore this term when designing the feedbacks. The dilation that corresponds to these vector fields is

$$\delta_\lambda(z_1, z_2, z_3) = (\lambda z_1, \lambda z_2, \lambda^2 z_3) \quad \lambda > 0, \tag{7}$$

and the homogeneous norm

$$\rho(z) = (z_1^4 + z_2^4 + z_3^2)^{\frac{1}{4}}. \tag{8}$$

A control law motivated by [8] is

$$\begin{aligned} u_1 &= -c_{11}z_1 + c_{12}\frac{z_3}{\rho(z)}\cos\Omega t \\ u_2 &= -c_{21}z_2 + c_{22}\frac{z_3^2}{\rho^3(z)}\sin\Omega t, \end{aligned} \tag{9}$$

where the c_{ij} are positive real parameters which may be adjusted to modify the system response. Ω is the frequency of the time periodic component of the control. Note that these expressions are homogeneous functions of order 1 with respect to (8), are smooth on $\Re^n\backslash\{0\}$ and continuous at the origin. If the closed loop system is asymptotically stable then it is actually exponentially stable with respect to the homogeneous norm (8).

If one is interested in globally smooth feedback there are a number of results available. We compare our homogeneous feedback to a smooth controller derived from [14]. The smooth controller is

$$\begin{aligned} u_1 &= -c_{11}z_1 + c_{12}z_3\cos\Omega t \\ u_2 &= -c_{21}z_2 + c_{22}z_3^2\sin\Omega t, \end{aligned} \tag{10}$$

where the c_{ij} are parameters. More details on the properties of this feedback may be found in [7, 14]. The control law is written for the system in chained form so the preliminary coordinate transformation (5) is required.

The system with one trailer is now discussed. Recall the 4 dimensional set of kinematic equations describing the system (3). The diffeomorphism and input transformation that places the model into chained form is

$$\begin{aligned} z_1 &= x \\ z_2 &= \tfrac{1}{d_1}\sec^3\theta_1\tan(\theta_0 - \theta_1) \\ z_3 &= \tan\theta_1 \\ z_4 &= y, \end{aligned} \tag{11}$$

and the inputs are computed from

$$\begin{aligned} &u_1 = \cos\theta_1 v \\ &u_2 = \sec^3\theta_1\tan(\theta_0-\theta_1)\left(\frac{3}{d_1^2}\tan\theta_1\tan(\theta_0-\theta_1)\right. \\ &\qquad \left. -\frac{1}{d_1^2}\sec(\theta_0-\theta_1)\right)v + \frac{1}{d_1}\sec^3\theta_1\sec^2(\theta_0-\theta_1)\omega. \end{aligned} \tag{12}$$

The expression of the vector fields in these coordinates is

$$\begin{aligned} \dot{z}_1 &= u_1 \\ \dot{z}_2 &= u_2 \\ \dot{z}_3 &= z_2u_1 \\ \dot{z}_4 &= z_3u_1. \end{aligned} \tag{13}$$

This system is homogeneous of degree 1 with respect to the dilation

$$\delta_\lambda(z) = (\lambda z_1, \lambda z_2, \lambda^2 z_3, \lambda^3 z_4). \tag{14}$$

A particular choice of homogeneous norm is

$$\rho(z) = (z_1^{12} + z_2^{12} + z_3^6 + z_4^4)^{\frac{1}{12}}. \tag{15}$$

The feedback that is implemented has the form

$$\begin{aligned} u_1 &= -c_{11}z_1 + c_{12}\left(\frac{z_3^2}{\rho^3} + \frac{z_4^2}{\rho^5}\right)(\cos\Omega t - \sin\Omega t), \\ u_2 &= -c_{21}z_2 + c_{22}\frac{z_3}{\rho}\cos 2\Omega t + c_{23}\frac{z_4}{\rho^2}\cos 3\Omega t, \end{aligned} \tag{16}$$

where the c_{ij} are positive parameters. This feedback is homogeneous of degree 1 and so the closed loop vector field is homogeneous of degree 0 with respect to (14). A stabilizing feedback will necessarily stabilize at an exponential rate.

The smooth controller for the 4 dimensional system that is implemented is from [7, 14]. The system is written in chained form and the feedback takes the form,

$$\begin{aligned} u_1 &= -c_{11}z_1 + c_{12}\left(z_3^2 + z_3\right)(\cos\Omega t - \sin\Omega t), \\ u_2 &= -c_{21}^1z_2 + c_{22}z_3\cos 2\Omega t + c_{23}z_4\cos 3\Omega t, \end{aligned} \tag{17}$$

4. Experimental Results

The experimental results are presented in this section. The physical parameters in Table 1 were measured with a metal tape measure and so the accuracy of these measurements is limited to several millimeters. *This will* lead to errors in the computation of the position of the system. The most compelling reason to employ feedback is to make the system insensitive to such errors and so approximate measurement of the system position should be adequate if the feedback is "good". It is difficult to perform a detailed robustness analysis on these systems but the fact that the closed-loop systems perform quite well is testimony to some degree of robustness possessed by the feedback.

Some thought must be given to the interpretation of the results if a comparison between several types of controllers is made on the same system. The rate at which the system approaches its equilibrium position from different initial positions is a reasonable criterion to assess the controller performance. In any application the control effort is a real limitation on the achievable performance. This limitation is embodied in the fact that the stepper motors saturate at about 500 steps/sec. Therefore it is reasonable to choose, as a means of comparison between different controllers, a fixed neighborhood of the equilibrium point where it is desired that each control law stabilize the system with initial conditions in this neighborhood, but at the same time not saturate the motors. The individual control laws may be "tuned" to take full advantage of the actuator in this neighborhood. We compare the controllers in this manner. Outside the neighborhood, where the motors saturate, saturation functions may be used to increase the domain of attraction [15]. However, since we are interested in the long term behavior of the system, we need only consider initial conditions inside the neighborhood where the saturation function have no affect.

4.1. Stabilization of the car. Experimental results with feedback are now presented for the car. Figure 2 compares the exponentially stabilizing homogeneous controller and the smooth asymptotic controller, both of which use the coordinate change (5). Figure 3 compares the step rate of both controllers and also shows a log plot of the y variable. The car uses two motors and the step rate input to each motor is plotted for both experiments. Note that the peak step rate amplitude of the smooth asymptotic controller is higher than the peak amplitude of the exponential homogeneous control law.

Simulations are used to adjust the parameters of the controllers, the final tuning being performed on the actual system after the simulations yield the desired response. Note that the smooth controllers are asymptotically stabilizing the system but the rate is very slow. The control parameters used in these experiments are $c_{11} = 0.3$, $c_{12} = 0.4$, $c_{21} = 1.0$, and $\omega = 2.0$ for both the smooth and homogeneous controllers and $c_{22} = 3.0$ for the homogeneous controller and $c_{22} = 5.0$ for the smooth one.

4.2. Stabilization of a car with one trailer. This subsection presents results for the car and one trailer. Particular attention should be paid to the behavior of the y-variable. Figure 4 compare closed-loop behavior of the exponentially stabilizing homogeneous control law (16) and the smooth asymptotic control law (17) with the same initial conditions. We should emphasize that no specific initial condition was chosen to make one controller perform "better" than another. The peak step rate for both controllers is approximately 300 steps/sec. The $\log(|y|)$ plot, Figure 5, is useful for assessing the convergence rate of the system. This will be discussed in more detail in the next section. The parameters used in the experiments

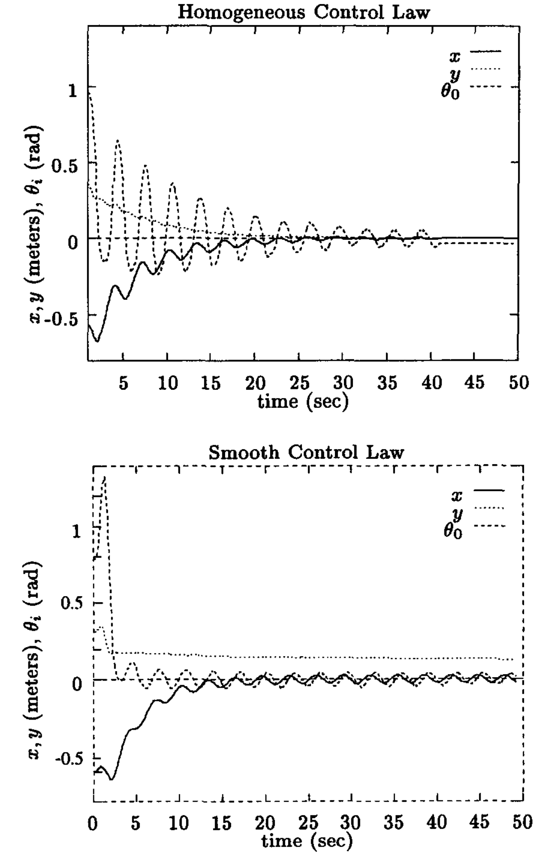

Figure 2: Response with (9) and (10)

with one trailer are the same for the homogeneous and smooth controllers. They are $c^1_{u_1} = c^1_{u_2} = c^2_{u_2} = c^3_{u_2} = 0.5$, $c^2_{u_1} = 0.6$, and $\omega = 0.5$.

5. Discussion

The first aspect of the experimental results to note is the rate at which y approaches zero. For the controllers which rely on chained form, the y variable is identified with the "hardest" state to control. Thus the rate at which this state decays is of practical interest. It is useful to plot $\log(|y|)$ to study this behavior. The fact that y in the homogeneous controllers' response may be bounded above by a straight line (see the log plots in Figures 3 and 5) indicates that y is approaching zero at an exponential rate. The average rate of convergence is equal to the average slope on the plots. The smooth controller in chained form decays at an algebraic rate. This is also evident from the log plots. The discrete nature of the motors places a lower bound on how close the system can come to the origin. This may cause ***hunting***. However this is a shortcoming of the hardware, not a limitation of the controller, and may be dealt with by ad hoc means (such as switching the controller off in some small neighborhood of the equilibrium point).

We now discuss control design related aspects for the individual problems. The controllers used in these experiments don't differentiate between length scales. For example, the (x, y) position of the car may be expressed

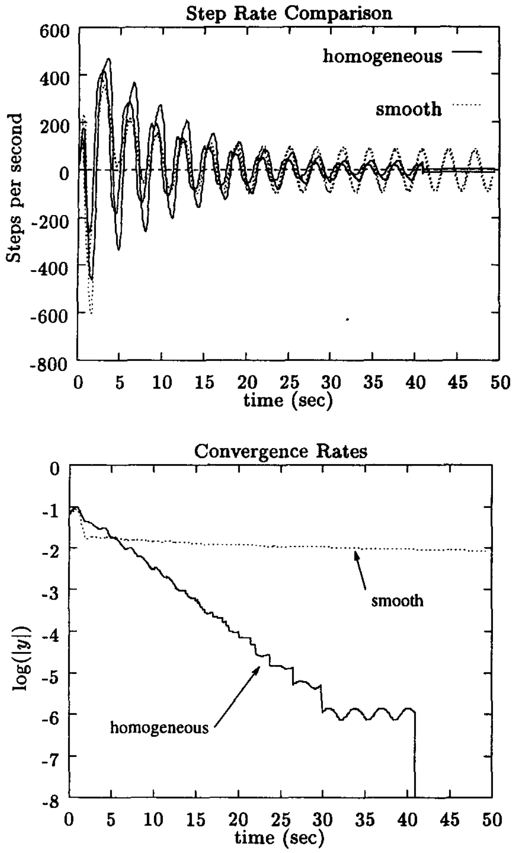

Figure 3: Controller effort and stabilization rate

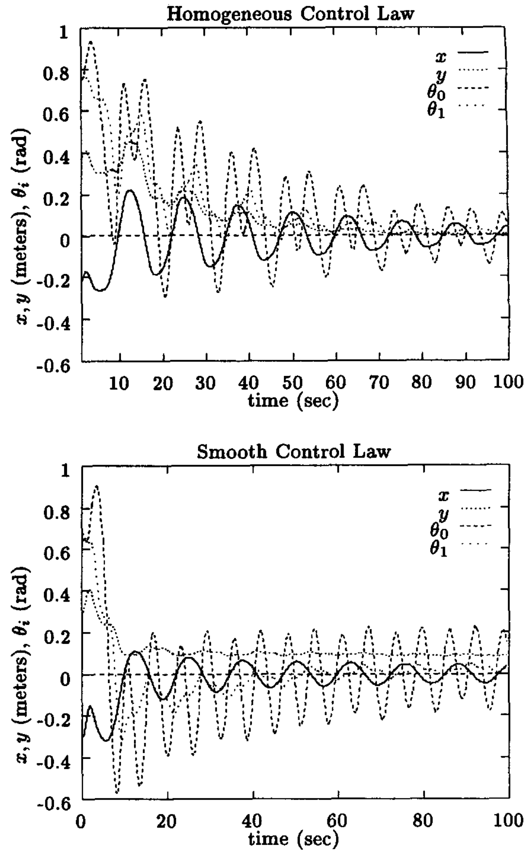

Figure 4: Response with (16) and (17).

in cm, m or even km. Hence as long as the actuators don't saturate, the region of convergence in terms of the linear variables is rather arbitrary. The response of the system depends critically on the length scale chosen though. For the homogeneous systems this is embodied by the *shape* of the corresponding homogeneous ball: homogeneous balls when the lengths are measured in kilometers and the angles in radians look much different than the balls with the lengths measured in meters. The length scale must be chosen so that the system response is satisfactory. The definition of "satisfactory" depends on the particular application .

The 3D system (car and no trailers) uses a length scale of 1 meter and angle scale of 1 radian. However the length scale for the system with the car and one trailer is the length of the trailer itself, i.e. one "unit" of length is 19 cm. A length scale of one meter leads to undesirable behavior because, for example, the homogeneous ball with $y = 1$ mm on its boundary also has $x = 10$ cm on its boundary! The finite precision of the actuators and sensors will invariably cause hunting in a neighborhood of the origin. This neighborhood is actually a homogeneous ball, for homogeneous closed-loop vector fields, and if the length scale is not chosen carefully can lead to large excursions of x with respect to small changes in y. This type behavior is characteristic of any homogeneous vector field. Our selection of the trailer length as the length scale mitigates this undesirable behavior for the homogeneous feedback.

Lastly, we discuss a very important concept that is germane to *any* control systems design requiring a diffeomorphism to place the model into a desired coordinate representation. This is nicely demonstrated by considering the car with one trailer model. The singular values of the linearization of the diffeomorphism (11), at various points in the phase space, indicates the amount of

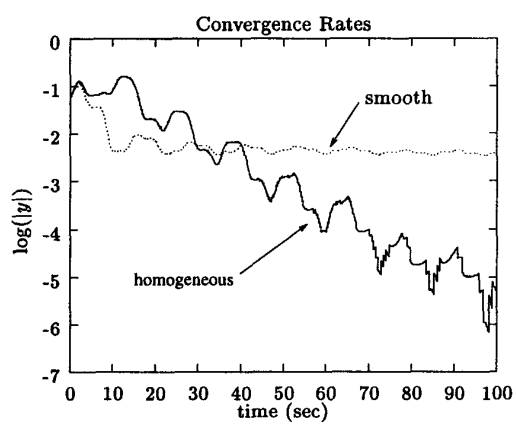

Figure 5: Stabilization rate.

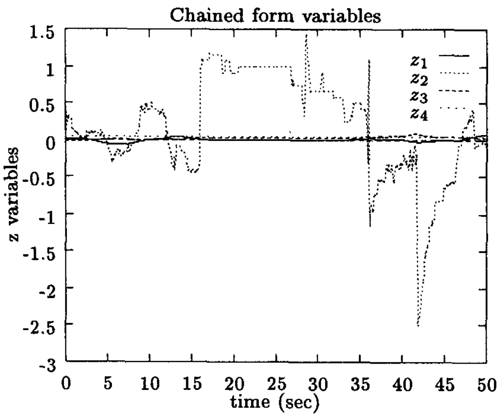

Figure 6: Affect of ill-conditioned diffeomorphism on system performance.

"stretching" performed on the variables by the transformation. A controller that depends on an ill-conditioned transformation may exhibit extreme sensitivity to small changes in certain state variables. This is exemplified in Figure 6. The analytical model is closed-loop stable however plotting the experimental data of the chained form variables shows that the z_2 variable is dominant and quite noisy. This results in very poor performance of the system since the control laws are written in terms of the chained form variables. The length scale chosen for this experiment is 5 meters and the controller is the homogeneous controller which uses transformation (11). However, this behavior is caused by the transformation and is observed with any controller implementation. The trailer length is actually $d_1 = 0.19/5 \approx 0.038$ as far as the diffeomorphism is concerned. The condition number of the diffeomorphism evaluated at the origin is 52.7. This is due primarily to a singular value with magnitude 37.2. The amplification of the physical data occurs in the $\theta_0 - \theta_1$ "input" direction to the z_2 "output" direction. This may demonstrated by performing a singular value decomposition on the linearization of the transformation at the origin.

We overcome the ill-conditioning by scaling the linear measurements with respect to the trailer length. Even for wheeled systems judicious choice of length scale may not solve the ill-conditioning problem. For example, consider the situation in which the ratio of two kinematic parameters is large: a length scale cannot be chosen to normalize both parameters to one. Finally, an important point to note is that the 3-dimensional system has no characteristic length associated with the kinematic model and the transformation specified by equations (5) has condition number 1 at all points in the phase space for any desired length scale.

The issue of transformation conditioning has not been addressed in the nonlinear systems literature but, as illustrated here, has a large impact on the performance. Control practitioners are well aware of the potential dangers of model inversion for linear systems. Our transformation may be interpretated as a kinematic inversion as opposed to the dynamic inversion often used in linear synthesis. One should expect the same problems to arise in the nonlinear setting as well.

6. Conclusions

We have presented in this paper an experimental comparison between smooth stabilizing control laws and continuous exponentially stabilizing controllers, demonstrating the superior performance of exponential stabilizers for mobile robots. Several design issues have been highlighted. First, our exponentially stabilizing controllers result in homogeneous vector fields. Signals are naturally measured with the homogeneous norm. Depending on the characteristic length of system (essentially a weighting between the linear and angular variables) the homogeneous balls can become quite skewed. In this case, controller hunting results in large motions about the equilibrium point. This phenomenon, which is special to our homogeneous feedbacks, was eliminated by rescaling the linear variables. Second, the condition number of the transformation, which takes the physical variables into chained form, may have a large impact on the closed loop system behavior. In particular, large singular values of the transform have an adverse affect on the system performance. This problem does not depend on the regularity of the feedback but is inherent in the transform itself. Renormalization of the linear variables, such that the transformation is well conditioned, improved the system response immensely for the car with a single trailer.

An expanded version of this paper is available from the authors in the form of a technical report.

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