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PROGRESSIVE WAVES OF FINITE AMPLITUDE AND SOME STEADY MOTIONS OF AN ELASTIC FLUID

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1. *Progressive Plane Waves.*—When a gas is initially stationary and at uniform temperature and pressure the density ρ may be regarded as also uniform initially with a value ρ_0 which for convenience may be taken as unity. The velocity of sound at this time will also be independent of position and equal, say, to c_0.

If now the gas is set in motion by a piston so that it moves parallel to the x-axis, then at any subsequent time t the pressure p at the place x will depend only on the density so long as no shock waves have passed over this place and the velocity potential ϕ will satisfy the partial differential equation

$$\frac{\partial^2\phi}{\partial t^2} + 2\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial t} + \left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2} = c^2\frac{\partial^2\phi}{\partial x^2} \qquad (1)$$

which is a particular case of a more general equation given by Lagrange.[1]

It is known that this equation may be treated with the aid of the well-known contact transformation

$$X = \frac{\partial\phi}{\partial x} \equiv u,\ T = \frac{\partial\phi}{\partial t} = f'(\rho) - \frac{1}{2}u^2 \qquad (2)$$

$$W = xX + tT - \phi = ux + t[f'(\rho) - \frac{1}{2}u^2] - \phi \qquad (3)$$

usually associated with the name of Legendre[2] but apparently known also to Euler[3] and Monge.[4] It is thus converted into a linear partial differential equation

$$\frac{\partial^2 W}{\partial X^2} - 2X\frac{\partial^2 W}{\partial X\partial T} + X^2\frac{\partial^2 W}{\partial T^2} = c^2\frac{\partial^2 W}{\partial T^2} \qquad (4)$$

which, as Poisson remarks,[5] is of a type which, when c is constant, can be solved by means of definite integrals but the resulting expression does not readily give the type of solution corresponding to a progressive wave which he found by an ingenious direct method of solution involving the use of an inequality to show that one part of the complete solution is zero in a progressive wave.

This apparent failure of Legendre's transformation to give all the solutions of the partial differential equation arises from the fact that X and T are not independent quantities in the progressive wave and so cannot be treated as new independent variables for the formation of a new partial differential equation. The solution for the progressive wave is not a singular solution in the sense in which this term is usually used;[6] the nature of the solution is discussed by McCowan[7] who regards it as special, i.e., a limiting case of the general solution.

The failure is only apparent because the relation

$$dW = xdX + tdT \tag{5}$$

which is true in any case, shows that when X and T are functions of a single parameter τ, W is also a function of τ and we have the equations

$$\phi = xX(\tau) + tT(\tau) - W(\tau) \tag{6}$$

$$0 = xX'(\tau) + tT'(\tau) - W'(\tau) \tag{7}$$

which are precisely those which define the progressive waves. The equations may be written in other forms and we shall find it convenient to indicate other ways of finding them.

If x_0 is the initial coördinate of the particle which is at x at time t we may write

$$\rho = \frac{\partial x_0}{\partial x}, \quad \rho u = -\frac{\partial x_0}{\partial t} \tag{8}$$

and the partial differential equation for x_0 is

$$\left(\frac{\partial x_0}{\partial x}\right)^2 \frac{\partial^2 x_0}{\partial t^2} - 2\frac{\partial x_0}{\partial x}\frac{\partial x_0}{\partial t}\frac{\partial^2 x_0}{\partial x \partial t} + \left(\frac{\partial x_0}{\partial t}\right)^2 \frac{\partial^2 x_0}{\partial x^2} = c^2 \frac{\partial^2 x_0}{\partial x^2}\left(\frac{\partial x_0}{\partial x}\right)^2. \tag{9}$$

This may also be reduced to a linear equation by the transformation of Legendre, the appropriate equations being

$$\rho = \frac{\partial x_0}{\partial x}, \ m = \rho u = -\frac{\partial x_0}{\partial t}, \ \sigma = \rho(x - ut) - x_0 \tag{10}$$

$$\rho^2 \frac{\partial^2 \sigma}{\partial \rho^2} - 2m\rho \frac{\partial^2 \sigma}{\partial \rho \partial m} + m^2 \frac{\partial^2 \sigma}{\partial m^2} = \rho^2 c^2 \frac{\partial^2 \sigma}{\partial m^2}. \tag{11}$$

The solution corresponding to progressive waves is now

$$x_0 = \rho(\tau)[x - tu(\tau)] - \sigma/\tau) \tag{12}$$

where τ is defined by the equation

$$0 = \rho'(\tau)[x - tu(\tau)] - t\rho(\tau)u'(\tau) - \sigma'(\tau). \qquad (13)$$

This equation must be the same as before because a relation between ρ and ρu implies a relation between the quantities X and T given by equations (2). Comparing (13) with (7), which may be written in the form

$$0 = xu'(\tau) - t[u(\tau)u'(\tau) + c^2(\tau)\rho'(\tau)/\rho(\tau)] - W'(\tau) \qquad (14)$$

we find that $$c^2(\tau)\, \rho'^2(\tau) = \rho^2(\tau)\, u'^2(\tau).$$

This is precisely the condition that the quantity x_0 defined by (12) should be a solution of the partial differential equation (9). A comparison of (13) and (14) also gives the relation

$$\rho'(\tau)W'(\tau) = u'(\tau)\sigma'(\tau). \qquad (15)$$

It should be noticed that when x is taken as dependent variable (9) transforms into Lagrange's differential equation[8]

$$\left(\frac{\partial x}{\partial x_0}\right)^2 \frac{\partial^2 x}{\partial t^2} = c^2 \frac{\partial^2 x}{\partial x_0{}^2} \qquad (16)$$

which has been frequently treated with the aid of Legendre's transformation.[9] The solution for progressive waves is now

$$\begin{aligned} x &= tu(\tau) + xv(\tau) - w(\tau) \\ 0 &= tu'(\tau) + xv'(\tau) - w'(\tau) \end{aligned} \qquad (17)$$

where $v(\tau)$ is the reciprocal of $\rho(\tau)$. We have in fact

$$u = \frac{\partial x}{\partial \tau},\ v = \frac{\partial x}{\partial x_0} = \frac{1}{\rho}. \qquad (18)$$

A comparison of (17) and (12) shows at once that $\rho(\tau)w(\tau) = -\sigma(\tau)$. Indeed, in the Legendre transformation in which u and v are new independent variables the new dependent variable is $w = -v\sigma = -\sigma/\rho$.

If we introduce the variable ψ which Kirchhoff calls the stream-function[10] and write

$$x_0 = \frac{\partial \psi}{\partial x},\ \rho = \frac{\partial^2 \psi}{\partial x^2},\ \rho u = -\frac{\partial^2 \psi}{\partial x \partial t},\ p + \rho u^2 = \frac{\partial^2 \psi}{\partial t^2}, \qquad (19)$$

the equation for ψ involves only the second derivatives and may be treated by the method of Legendre[11] in which the equation is first solved for the derivative $\frac{\partial^2 \psi}{\partial t^2}$ and then differentiated with respect to x. The result is, of course, equation (9). If, on the other hand, the equation for ψ is first subjected to a Legendre transformation in which the quantities $x_0 = \frac{\partial\psi}{\partial x}$ and $y_0 = \frac{\partial\psi}{\partial t}$ are taken as new independent variables, the new equation is of the same type as before, involving only second derivatives of the new dependent variable z. When this equation is treated by Legendre's method of differentiation so as to provide an equation that can be transformed into a linear equation by Legendre's transformation the new dependent variable is $\frac{\partial z_0}{\partial y_0}$ which is t and since

$$\frac{\partial t}{\partial x_0} = \frac{u}{p}, \quad \frac{\partial t}{\partial y_0} = \frac{1}{p} \tag{20}$$

a relation between $\frac{\partial t}{\partial x_0}$ and $\frac{\partial t}{\partial y_0}$ implies a relation between p and u. The solution for progressive waves is thus obtained in the form

$$t = [y_0 + x_0 u(\tau)]/p(\tau) - s(\tau) \tag{21}$$

where τ is defined by the equation

$$u'(\tau)x_0 = tp'(\tau) + p'(\tau)s(\tau) + p(\tau)s'(\tau) \tag{22}$$

which is equivalent to one derived from (12) and (13) by eliminating x.

If, in the differential equation for t we take y_0 as new dependent variable, the new independent variables being x_0 and t, we obtain the equation

$$\frac{\partial^2 y_0}{\partial t^2} = c^2\rho^2 \frac{\partial^2 y_0}{\partial x_0^2}. \tag{23}$$

This equation may be treated by Legendre's transformation, the new independent variables being

$$\frac{\partial y_0}{\partial t} = p, \quad \frac{\partial y_0}{\partial x_0} = -u \tag{24}$$

while the new dependent variable is $tp - ux_0 - y_0 = -ps$. Calling this quantity R we have the equations

$$\frac{\partial^2 R}{\partial u^2} = c^2\rho^2 \frac{\partial^2 R}{\partial p^2} \tag{25}$$

$$dR = tdp - x_0 du$$

and in adiabatic flow the partial differential equation can be reduced to the equation of Euler and Poisson[12] and treated by the method of Riemann.

The wave-solution given by the failure of Legendre's transformation is obtained again in the form (21).

The wave-solutions have been discussed at some length by Hugoniot[13] and Hadamard,[14] particular attention being paid to the generation of the wave-motion from a state of rest by the accelerated motion of a piston and the formation of singularities or discontinuities in the motion as indicated by Stokes.[15] By means of a representation in a space of three dimensions the solution is interpreted as a developable surface which touches the plane $z = 0$ along a characteristic of the partial differential equation. The edge of regression of this developable indicates where singularities occur, for beyond this edge z becomes non-existent or two-valued. If $x = \zeta(\sigma)$, $y = \eta(\sigma)$, $z = \rho(\sigma)$ are the equations of the edge the developable can be regarded as the envelope of the plane

$$[x - \zeta(\sigma)][\eta'(\sigma)\rho''(\sigma) - \rho'(\sigma)\eta''(\sigma)] + [y - \eta(\sigma)][\rho'(\sigma)\zeta''(\sigma) - \zeta'(\sigma)\rho''(\sigma)] + [(z - \rho(\sigma)][\zeta'(\sigma)\eta''(\sigma) - \eta'(\sigma)\zeta''(\sigma)] \tag{26}$$

and the equation obtained by differentiating with respect to σ keeping x, y, z constant can be expressed in various forms which are all included in the equations

$$\frac{x - \zeta(\sigma)}{\zeta'(\sigma)} = \frac{y - \eta(\sigma)}{\eta'(\sigma)} = \frac{z - \rho(\sigma)}{\rho'(\sigma)} \tag{27}$$

obtained by combining any one form of the equation with (26).

This result will be used in §2; it is mentioned here because the general results of Hugoniot and Hadamard for the case of the equation of wave-propagation are applicable also to the equation for steady two-dimensional flow, the solution for progressive waves being replaced by a solution giving a generalization of the well-known flow discussed by Prandtl and Meyer. A few words must be said about a type of solution of the equation for ψ in which the second derivatives of ψ are all related and are thus functions of a single variable s. The general solution for such a case which will be derived in §2, gives the relations

$$\begin{aligned} x_0 &= \frac{\partial \psi}{\partial x} = e(s)x + st + n(s), \quad 0 = e'(s)x + t + n'(s) \\ y_0 &= \frac{\partial \psi}{\partial t} = sx + t\int ds/e'(s) + \int n'(s)ds/e'(s) \end{aligned} \tag{28}$$

and in the adiabatic flow in which $p = k\rho^{\gamma}$ we have

$$s = Ce(s) - \frac{2}{\gamma - 1}\sqrt{k\gamma}\left[e(s)\right]^{\frac{\gamma+1}{2}}$$

$$\rho = e(s), \quad \rho u = -s, \quad p + \rho u^2 = \int ds/e'(s), \tag{29}$$

where C is an arbitrary constant.

It may be remarked that Lagrange's equation (16) is derivable by differentiation with respect to x_0 from an equation of type

$$\frac{\partial^2 w}{\partial t^2} = F\left(\frac{\partial^2 w}{\partial x_0^2}\right) \tag{16'}$$

wherein $x = \frac{\partial w}{\partial x_0}$, $F'(z) = c^2(z)/z^2$. This equation had a particular solution of type

$$\frac{\partial w}{\partial t} = t \int ds_0/e_0'(s_0) + s_0 x_0 + \int n_0'(s_0) ds_0/e_0'(s_0),\ 0 = t + x_0 e_0'(s_0) + n_0'(s_0),\ \frac{\partial w}{\partial x_0} = s_0 t + x_0 e_0(s_0) + n_0(s_0) \text{ if } \int ds_0/e_0'(s_0) = F[e_0(s_0)].$$

In adiabatic flow

$$\sqrt{k\gamma}\left(\frac{1-\gamma}{2}\right)^{-1}\left[e_0(s_0)\right]^{\frac{1-\gamma}{2}} = C \pm s_0$$

$$\frac{1}{\rho} = e_0(s_0), \quad c = \sqrt{k\gamma}\left[e_0(s_0)\right]^{\frac{1-\gamma}{2}}.$$

2. *Steady Flow in Two Dimensions.*—The dynamical equations

$$\begin{aligned} \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) &= 0 \\ \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p) &= 0 \end{aligned} \tag{30}$$

can be satisfied in a general way by writing

$$\rho u^2 + p = -\frac{\partial^2 S}{\partial y^2},\ \rho uv = \frac{\partial^2 S}{\partial x \partial y},\ \rho v^2 + p = -\frac{\partial^2 S}{\partial x^2} \tag{31}$$

where S is a function which Neumann[16] calls the stream-function but which will be called here the stress-function. It must be chosen so that the equation of continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{32}$$

is satisfied and that p is a specified function of ρ.

In the Prandtl-Meyer type of flow, the existence of which follows from the general remarks made by Hadamard[14] on the solution of equations of type

$$A\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial x^2} + 2H\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial x \partial y} + B\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial y^2} \tag{33}$$

the quantities p, ρ, u and v are all functions of a single parameter τ and so there must be solutions of Neumann's equation for S in which the second derivatives of S are all functions of a single parameter s. Replacing S, for the moment by z and using r, s, t for the second derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$, respectively, we may ask under what condition the equations $r = e(s)$, $t = f(s)$ are compatible. Denoting third derivatives by k, l, m, n we find by differentiating

$$k = l'(s)l, \quad l = e(s)m, \quad m = f'(s)l, \quad n = f'(s)m \tag{34}$$

and the condition of compatibility is

$$e'(s)f'(s) = 1. \tag{35}$$

It will be noticed that we have identified the parameter s with $\frac{\partial^2 z}{\partial x \partial y}$. But this can be done without much loss of generality.

The equations $r = e(s)$, $t = f(s)$ may be regarded as partial differential equations of the first order for $\frac{\partial S}{\partial x}$ and $\frac{\partial S}{\partial y}$, respectively. Solving these we have

$$\begin{aligned} \frac{\partial S}{\partial x} &= xe(s) + ys + g(s) \\ \frac{\partial S}{\partial y} &= xs + yf(s) + h(s) \end{aligned} \tag{36}$$

where s is given respectively by the equations

$$\begin{aligned} 0 &= xe'(s) + y + g'(s) \\ 0 &= x + yf'(s) + h'(s). \end{aligned} \tag{37}$$

These must be the same and so $h'(s) = f'(s)g'(s)$, $f'(s)e'(s) = 1$.

[Though the analysis of Hadamard gives some of the general features of the flow specified by these equations and the flow has already been studied in a general way by S. Lees[17] and the present author,[18] the analysis can now be presented in an improved form which indicates more clearly its relation to the graphical method of solving problems in supersonic flow used by Prandtl,[19] Ackeret[20] and Busemann.[21] The improvements depend upon advantageous choices of the parameter τ on which the principal quantities depend. In the above form of the solution $\tau = s$ where s represents the physical quantity ρuv. We shall now choose τ so that the velocity of sound is the function $c(\tau)$ which will be regarded as the derivative of a function $a(\tau)$ so that $c(\tau) = a'(\tau)$.]

The parameter τ is, however, not yet defined because the functions are unspecified. We shall suppose further that the components of velocity u, v, are functions $u(\tau)$, $v(\tau)$ given by the equations

$$u(\tau) = a(\tau) \cos \tau - a'(\tau) \sin \tau, \quad v(\tau) = a(\tau) \sin \tau + a'(\tau) \cos \tau \qquad (38)$$

and supposed not to be constants
then

$$u'(\tau) = - \sin \tau[a(\tau) + a''(\tau)], \; v'(\tau) = \cos \tau[a(\tau) + a''(\tau)]. \qquad (39)$$

The velocity potential ϕ may now be expressed in the form

$$\phi = xu(\tau) + yv(\tau) - w(\tau) \qquad (40)$$

where τ is defined by the equation

$$0 = xu'(\tau) + yv'(\tau) - w'(\tau). \qquad (41)$$

Lagrange's partial differential equation for ϕ, when written in Rayleigh's form

$$c^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = u \left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x}\right) + v \left(u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y}\right) \qquad (42)$$

is seen to be satisfied on account of the equation $c = a'(\tau)$ provided $u''(\tau)$, $v''(\tau)$ and $w''(\tau)$ are not all zero, for then we may write

$$M \frac{\partial \tau}{\partial x} \equiv u'(\tau), \; x \frac{\partial \tau}{\partial y} = v'(\tau), \quad M = w''(\tau) - xu''(\tau) - yv''(\tau) \qquad (43)$$

where M is different from zero.

Since $q^2 = u^2 + v^2 = a'^2(\tau) + a^2(\tau)$, we have $a^2(\tau) = q^2 - c^2$ and so $q^2 \geq c^2$, a result obtained before. Writing $\rho = \rho(\tau)$, the equation of continuity (31) gives

$$\rho'(\tau)a'(\tau)[a(\tau) + a''(\tau)] + \rho(\tau)[a(\tau) + a''(\tau)]^2 = 0 \qquad (44)$$

and since, by hypothesis u and v are not both constant, we have $a(\tau) + a''(\tau) \neq 0$ and so

$$\rho'(\tau)a'(\tau) + \rho(\tau)a''(\tau) + \rho(\tau)a(\tau) = 0. \qquad (45)$$

In adiabatic flow the relation between c and q is

$$\frac{2c^2}{\gamma - 1} + q^2 = b^2 \qquad (46)$$

where b is a constant. This gives the equation

$$a^2(\tau) + \frac{\gamma + 1}{\gamma - 1} a'^2(\tau) = b^2 \qquad (47)$$

whose solution is

$$a(\tau) = b \sin(\lambda\tau + \alpha) \qquad (48)$$

where α is a constant and $\lambda = \left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2}$.

To obtain the stream-lines we put

$$x = R \cos\tau - T(\tau)\sin\tau, \quad y = R\sin\tau + T(\tau)\cos\tau. \qquad (49)$$

Substituting in (41) we find that

$$\begin{aligned} w'(\tau) &= T(\tau)[a(\tau) + a''(\tau)], \quad \phi = Ra(\tau) + T(\tau)a'(\tau) - w(\tau) \\ dx &= [dR - T(\tau)d\tau]\cos\tau - [R + T'(\tau)]\sin\tau d\tau \\ dy &= [dR - T(\tau)d\tau]\sin\tau + [R + T'(\tau)]\cos\tau d\tau. \end{aligned} \qquad (50)$$

The equation for the stream-lines

$$\frac{dy}{dx} = \frac{v}{u} = \frac{a(\tau)\sin\tau + a'(\tau)\cos\tau}{a(\tau)\cos\tau - a'(\tau)\sin\tau} \qquad (51)$$

thus takes the form

$$a'(\tau)[dR - T(\tau)d\tau] = a(\tau)[R + T'(\tau)]d\tau. \qquad (52)$$

Making use of (45) it becomes

$$d[R\rho a'(\tau)] = \rho d[T(\tau)a(\tau)]. \qquad (53)$$

Putting $T(\tau)a(\tau) = h'(\rho)$ we have

$$R\rho c = \rho h'(\rho) - h(\rho) + n \qquad (54)$$

where n is a constant associated with the particular stream-line under consideration. The equations of the stream-line may thus be expressed in the parametric form

$$\begin{aligned} x &= \frac{\cos\tau}{a'(\tau)}\left[h'(\rho) - \frac{1}{\rho}h(\rho) + \frac{n}{\rho}\right] - \frac{\sin\tau}{a(\tau)}h'(\rho) \\ y &= \frac{\sin\tau}{a'(\tau)}\left[h'(\rho) - \frac{1}{\rho}h(\rho) + \frac{n}{\rho}\right] + \frac{\cos\tau}{a(\tau)}h'(\rho) \end{aligned} \qquad (55)$$

where ρ is a function of τ given by equation (45). We may also write

$$\psi = \rho c[x\cos\tau - y\sin\tau] + h(\rho) - \rho h'(\rho). \qquad (56)$$

It should be noticed that the points on different stream-lines that have the same value of τ all lie on a straight line

$$0 = xu'(\tau) + yv'(\tau) - w'(\tau) \tag{57}$$

which cuts each stream-line at the Mach angle β corresponding to this value of τ. Equations (55) indicate that

$$\tan \beta = \frac{a'(\tau)}{a(\tau)}, \qquad \sin \beta = \frac{c}{q} \tag{58}$$

and that the distance d along this line between two stream-lines with parameters n_1 and n_2 is

$$d = \frac{n_1 - n_2}{\rho a'(\tau)} = \frac{n_1 - n_2}{\rho c}. \tag{59}$$

Thus $d\rho c$ is constant for the stream-lines. This relation, when written in the form $d\rho q \sin \beta$ = constant, is merely another way of expressing the equation of continuity.

The lines τ = constant generally have an envelope E as indicated in Ackeret's diagram, reproduced with full explanations by Taylor and McColl.[22] This envelope may be regarded as the projection of the edge of the developable obtained by regarding ϕ as a third coördinate z and marks out a boundary of the region in which the flow can be represented by this solution of the equations. Taking σ to be the length of the arc of E measured from some fixed point, we have $\zeta'(\sigma) = \cos \tau$, $\eta'(\sigma) = -\sin \tau$, $r\zeta''(\sigma) = -\sin \tau$, $r\eta''(\sigma) = \cos \tau$, where r is the radius of curvature at the point σ on E. Comparing (40) and (41) with (26) and (27) we obtain the relations $a(\tau) = -\rho'(\sigma)$,

$$\begin{aligned} w'(\tau) &= [a(\tau) + a''(\tau)][\eta(\sigma) \cos \tau - \zeta(\sigma) \sin \tau] \\ &\qquad - X[a(\tau) + a''(\tau)] \\ w(\tau) &= Y\rho'(\sigma) + rX\rho''(\sigma) - \rho(\sigma) \end{aligned} \tag{60}$$

where X and Y are the distances of the origin of coördinates from the tangent and normal at the point σ on E. If, on the other hand, we compare (40) and (41) with (36) and (37) we obtain the relations

$$e'(s) = -\tan \tau, \quad f'(s) = -\cot \tau, \quad ds/d\tau = \rho'(\tau)q^2(\tau) \sin \tau \cos \tau. \tag{61}$$

3. *Spiral Flow.*—Keeping the notation of §2 and using polar coördinates (q, ω) in the uv-plane as independent variables, Tschapliguine[23] obtained linear partial differential equations for ϕ and ψ the solutions of which when expanded in series of particular solutions involve linearly a set of constants. We now wish to know how these constants of integration enter into the expression for the stress-function S. It has been found

that in the case of the fluid motion which Taylor[24] calls spiral flow S is a homogeneous quadratic function of the constants. We have in fact

$$\phi = m\omega - n\int q\,dq\,Q'/Q^2, \quad \psi = n\omega - m\int Q\,dq/q^2$$

$$S = -mn\omega + \int dq\left[\frac{m^2p}{q^3} + \frac{n^2pQ'}{Q^3} + \frac{n^2qQ'}{Q^2}\right] \qquad (62)$$

$$\frac{\partial S}{\partial x} = mn\frac{y}{r^2} - pk - \frac{n^2x}{pr^2}, \qquad \frac{\partial S}{\partial y} = -mn\frac{x}{r^2} - py - \frac{n^2y}{pr^2}$$

$$\varphi \equiv \arctan(y/x) = \omega - \arctan(mQ/nq),$$

$$r^2 \equiv x^2 + y^2 = \frac{m^2}{q^2} + \frac{n^2}{Q^2}$$

where m and n are the constants and $Q = \rho q$.

A diagram in the Qq-plane may be used with advantage in the study of this and other types of flow. The curve C giving the relation between

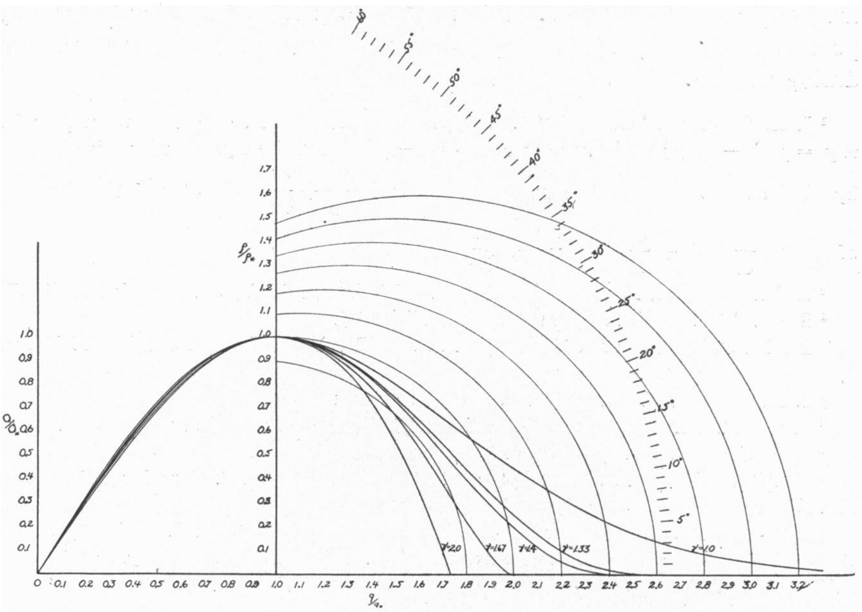

FIGURE 1

Q and q in adiabatic flow starts from O, runs up to a maximum value of Q and then slopes downward until a point is reached at which Q and ρ are both zero. If now a tangent at a point P on this curve meets the axis of q in T, the circle on OT as diameter meets the ordinate PN in a point R such that the angle between OR and Oq is Mach's angle β. We have in fact

$$q^2/c^2 = 1 - qQ'/Q, \ p = -\int Qdq, \quad \rho = Q/q, \qquad (63)$$

where Q' is the derivative of the function $Q(q)$. The velocity of sound c is represented as the distance from OR of N, the foot of the ordinate of P. The pressure p is represented by the area between this ordinate and the portions of the curve and q-axis which meet at the point where the density is zero.

Figure 1, which shows the form of the Q—q curves for adiabatic flow for different values of γ has been drawn for me by Francis and Milton Clauser. Some of the circles have been drawn and a scale included so that the Mach angle can be read off by placing a straight edge through the origin and the point R on the circle. (The points are not marked on the diagram.) In spiral flow the curves along which ω is constant may be obtained by a rotation of the curve C for we have the equations

$$x \cos \omega + y \sin \omega = n/Q, \quad x \sin \omega - y \cos \omega = m/q. \qquad (64)$$

The transverse velocity and the density of radial momentum are both inversely proportional to r. The moment of momentum about O of any portion of the fluid is proportional to its mass and so remains constant during motion.

[1] J. L. Lagrange, "Oeuvres," t. 12, pp. 323-340. If the relation between p and ρ is $p = f(\rho) - \rho f'(\rho)$ we have $c^2 = dp/d\rho = -\rho f''(\rho)$ and the equation for ϕ is obtained by eliminating ρ between the equation of continuity $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$ and the equation of motion $\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = -c^2 \frac{\partial \rho}{\partial x}$.

[2] A. M. Legendre, *Mém. Paris* (1787) 347, L'Histoire de l'Acad. des sciences (1787).

[3] See P. Stäckel, *Bibl. Math.* (3) **1,** 275, 517 (1900).

[4] See M. Chasles, Aperçu Historique . . . , p. 376, A. de Morgan, *Camb. Phil Trans.,* **8,** 606–613 (1848).

[5] S. D. Poisson, *J. Éc. Polyt.,* **7,** 312–392 (1808).

[6] See A. R. Forsyth, Theory of Differential Equations, Part IV, vol. 6, p. 3.

[7] J. McCowan, *Proc. Edinburgh Math. Soc.,* **11,** 2–6 (1893).

[8] See. S. D. Poisson, *J. Éc. Polyt.,* cah. 21, 187–204 (1832), S. Earnshaw, *Proc. Roy. Soc. London,* **9,** 500–501 (1858).

[9] J. Hadamard, "Lecons sur la propagation des ondes" (1903), J. R. Wilton, *Phil. Mag.* (6) **30,** 761–779 (1915), A. E. H. Love and F. B. Pidduck, *Trans. Roy. Soc. London,* A, **222,** 167–226 (1922)

[10] W. Kirchhoff, *J. für Math.,* **164,** 163–195 (1930).

[11] A. M. Legendre, loc. cit.; J. R. Wilton, *Mess. of Math.,* **43,** 58–63 (1914). Wilton solves an equation from which (16) may be derived by differentiation, the dependent variable θ being such that $x = \frac{\partial \theta}{\partial x_0}$.

[12] See Darboux, "Théorie des surfaces," t. 2, chapters 3, 4 and 9. Another method of treating the linear differential equations derived from (16) and (23) by Legendre's transformation is given by R. Liouville, *Comptes Rendus,* **98,** 723–726 (1884).

[13] H. Hugoniot, *J. Éc. Polyt.*, cah. 57, 1–97 (1887), cah. 58, 1–125 (1889).

[14] J. Hadamard, loc. cit., "Cours d'Analyse," t. 2.

[15] G. G. Stokes, *Phil. Mag.*, (3) **23,** 349 (1848); "Math. Phys. Papers," vol. 2, 51–55. A good illustration of the formation of a shock wave is given by W. Hope-Jones, *Math. Gazette*, **14,** 173–186(1928–9).

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