Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance Objectives I: Robust Performance Analysis

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***Abstract*—This paper introduces an induced-norm formulation of a mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance criterion. It is shown that different mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms arise from different assumptions on the input signals. While most mixed norms can be expressed explicitly using either transfer functions or state-space realizations of the system, there are cases where the explicit formulas are very hard to obtain. In the later cases, examples are given to show the intrinsic nature and difficulty of the problem. Mixed norm robust performance analysis under structured uncertainty is also considered in the paper.**

I. Introduction

THIS paper considers the system in Fig. 1 where G is a linear system, w_0 is a signal of bounded spectrum and w_1 is a signal of bounded power. These signal sets are defined in Section II. We are interested in the induced norm on G when the inputs are from these sets and z is taken to be of bounded power. This is called a mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem because if only w_0 were present, this induced norm would be the standard $\mathcal{H}_2$ norm on G and if only w_1 were present, it would be the standard $\mathcal{H}_\infty$ norm.

Motivation for this problem comes from several sources. The most general motivation is that we would like to develop a theory of robust $\mathcal{H}_2$ performance with $\mathcal{H}_\infty$ norm-bounded structured uncertainty similar to the μ-analysis theory for robust $\mathcal{H}_\infty$ performance. While the $\mathcal{H}_\infty$ norm is natural for norm-bounded perturbations, in many applications the natural norm for the input–output performance is the $\mathcal{H}_2$ norm. The mixed problems considered in this paper provide a starting point for a theory of robust $\mathcal{H}_2$ performance.

A second motivation arises from the paper by Doyle *et al.* [5], where standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems are treated as separate problems, but in a unified state-space framework. A natural continuation of this work is to find a single problem formulation that has the standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theories as special cases. Additional motivation came from Bernstein and Haddad [2], who consider a mixed framework with an apparent "duality" to the framework proposed here.

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Fig. 1.

Although certain formulas are transposes of each other, the nature of this duality is not operator theoretic. The present paper, and the companion paper on synthesis, could be viewed as an attempt to extend and formalize the work in the two papers above. The connections between these papers will be considered in more detail in the synthesis paper.

A final and somewhat peripheral motivation is that we wish to suggest a theory of noise signals which does not require stochastics, although it is clear that the theory developed here could be done entirely within a conventional stochastic framework. We pursue a slightly different, more operator theoretic course in the spirit of the $\mathcal{L}_2$ motivation for $\mathcal{H}_\infty$ optimal control, but using signals bounded in power or spectrum rather than energy. We believe that this course will eventually lead to a framework for modeling signals that will be simpler and easier to motivate than conventional stochastics, although much more work will be needed before this goal will be realized. In order to avoid a long and technical preliminary section, signal sets of bounded power and spectrum are defined and developed in an informal and heuristic manner. While this approach greatly shortens and simplifies the paper, we recognize that a rigorous treatment will require that the preliminaries in Section II be revisited.

Section III presents the main results of the paper where the system's performance under various inputs is quantified. In particular, the mixed analysis problems seems to divide naturally into cases where w_0 is white or not, and where w_1 is causally dependent on w_0 or not. We say w_1 is causally dependent on w_0 if $w_1 = Ww_0$ for some $W \in \mathcal{H}_2$. The analysis results were given without proof in Zhou *et al.* [13] for the white and causal case, which is the case that is treated in the companion synthesis paper. Section IV gives our results on the mixed robust performance analysis problem, which we consider a step in the direction of developing a robust $\mathcal{H}_2$ theory.

II. Preliminaries

This section reviews some elementary mathematical and system theoretic results, and presents the notation, which is fairly standard.

A. Notation

The Hardy space $\mathcal{H}_2$ ($\mathcal{H}_2^{\perp}$) consists of square-integrable functions on the imaginary axis with analytic continuation into the right- (left-) half plane. The Hardy space $\mathcal{H}_\infty$ consists of bounded functions with analytic continuation into the right-half plane. The Lebesgue spaces $\mathcal{L}_2$ and $\mathcal{L}_\infty$ consist of, respectively square-integrable and bounded functions on $(-\infty, \infty)$.

All integrals are Lebesgue integrals. In general, $u(t) : \mathbb{R} \to \mathbb{R}^m$ and $w_i(t) : \mathbb{R} \to \mathbb{R}^{m_i}$ will be used to denote signals which are inputs to systems, $z(t) : \mathbb{R} \to \mathbb{R}^q$ and $y(t) : \mathbb{R} \to \mathbb{R}^p$ denote signals which are the outputs of a system, and $x(t) : \mathbb{R} \to \mathbb{R}^n$ denote signals which are the states of a system. Let $*$ denote the convolution operator, superscript $*$ denote the adjoint operator, and $\langle x, y \rangle$ the usual inner product on $\mathbb{C}^n$ or $\mathbb{R}^n$. In most cases, we will omit all vector and matrix dimensions and assume that all quantities have compatible dimensions.

A transfer matrix in terms of state-space data is denoted

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D.$$

For a matrix $M \in \mathbb{C}^{p\times r}$ or $\mathbb{R}^{p\times r}$, M' denotes its conjugate transpose and $\bar{\sigma}(M)$ denotes its maximum singular value. The prefix $\mathcal{B}$ denotes the closed unit ball and the prefix $\mathcal{R}$ denotes real-rational. The unsubscripted norm $\|\cdot\|$ will denote the standard Euclidean norm on vectors. Finally, if $X = X'$ is the stabilizing solution to the algebraic Riccati equation

$$A'X + XA + XRX + Q = 0$$

with $A + RX$ stable, then we will denote the solution by $X = Ric(H)$ where

$$H = \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix}$$

is the associated Hamiltonian. The matrix H for which $Ric(H)$ is defined is the domain of the Riccati operator and will be denoted by $dom(Ric)$. For more details on this notion for Riccati equations and Hamiltonian matrices, see [5].

B. Signals and Norms

All signals and systems considered in this paper are assumed to be deterministic. The development of the signal sets here is somewhat peripheral to the main theme of this paper, and will be quite informal and heuristic. Some relevant background material may be found in [9]. The objective is to motivate certain induced norms, which are mixtures of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. These mixed norms could also be motivated in a stochastic framework.

$\mathcal{L}_2$ and $\mathcal{L}_\infty$ Signals: These classes of functions (signals) are well understood and widely used in control community; we remind the reader that the 2 and ∞ norms of a signal

$$u = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m$$

are defined, respectively, as

$$\|u\|_2 := \sqrt{\int_{-\infty}^{\infty} \|u(t)\|^2 dt}$$

and

$$\|u_\infty\| := \operatorname*{esssup}_t \|u(t)\|.$$

Bounded Power Signals: Given a signal $u(t)$, we define its autocorrelation matrix as

$$R_{uu}(\tau) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t+\tau)u(t)'dt$$

if the limit exists for all τ. It can be shown that $R_{uu}(\tau) = R_{uu}(-\tau)' \geq 0$.

For the purpose of this paper, we further assume the Fourier transform of the signal's autocorrelation matrix function exists (but may contain impulses). This Fourier transform is called the spectral density of u, denoted $S_{uu}(j\omega)$

$$S_{uu}(j\omega) := \int_{-\infty}^{\infty} R_{uu}(\tau)e^{-j\omega\tau} d\tau.$$

Then $R_{uu}(\tau)$ can be obtained from $S_{uu}(j\omega)$ by inverse Fourier transform as

$$R_{uu}(\tau) := \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{uu}(j\omega)e^{j\omega\tau} d\omega.$$

Note that spectral density matrices are Hermitian ($S_{uu}(j\omega) = S_{uu}^*(j\omega)$) and positive semidefinite ($S_{uu}(j\omega) \geq 0$).

We will consider the set of signals satisfying the following conditions:

A1) $u(t) \in \mathcal{L}_\infty$;

A2) the autocorrelation matrix $R_{uu}(\tau)$ exists for all τ;

A3) the power spectral density function $S_{uu}(j\omega)$ exists (it need not be bounded and may include impulses).

A signal u satisfying the above conditions is said to have bounded power if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|^2\, dt < \infty.$$

The set of all signals having bounded power is denoted by

$$\mathcal{P} := \{u(t) : u(t) \text{ satisfies A1)-A3) and } \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|^2\, dt < \infty\}.$$

A seminorm can be defined on the space of signals of bounded power, i.e.,

$$\|u\|_{\mathcal{P}} = \sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|^2\, dt} = \sqrt{\text{Trace}\,[R_{uu}(0)]}.$$

The script "$\mathcal{P}$" is used to differentiate this power semi-norm from the usual Lebesgue $\mathcal{L}_p$ norm. The power norm of a signal can also be computed from its spectral density function

$$\|u\|_{\mathcal{P}} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}[S_{uu}(j\omega)] d\omega}.$$

We note that if $u \in \mathcal{P}$ and $\|u(t)\|_\infty < \infty$, then $\|u\|_\mathcal{P} \leq \|u\|_\infty$. Not every $\mathcal{L}_\infty$ signal, however, is in $\mathcal{P}$, because the limit in the definition of the autocorrelation matrix may not exist. Note also that signals of bounded power may be persistent signals in time such as sines or cosines. Clearly an $\mathcal{L}_2$ signal has zero power so $\|\cdot\|_\mathcal{P}$ is only a semi-norm, not a norm.

The cross-correlation between two signals u and v is defined as

$$R_{uv}(\tau) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t+\tau)v(t)'dt$$

if it exists for all τ. It is easy to show that the cross-correlation has the following property

$$R_{uv}(\tau) = R_{vu}(-\tau)'.$$

The Fourier transform of $R_{uv}(\tau)$ is called the cross-spectral density and is denoted as $S_{uv}(j\omega)$.

Bounded Spectrum Signals: Note that a bounded power signal need not have bounded spectral density; for example, a sine function has an impulse as a spectral density. On the other hand, some signals having bounded spectral density need not have bounded power; in particular, a signal u having bounded spectrum $S_{uu} = I$ has unbounded power.

The set of signals having bounded spectrum is denoted as

$$\mathcal{S} := \{\, u(t) : u(t) \text{ satisfies A1)-A3) and } \|S_{uu}(j\omega)\|_\infty < \infty\}.$$

The quantity $\|u\|_s := \sqrt{\|S_{uu}(j\omega)\|_\infty}$ is a seminorm on $\mathcal{S}$.

The engineering relevance of the set $\mathcal{S}$ is that it can be used to model signals with fixed or bounded spectral characteristics. Similarly, $\mathcal{P}$ could be used to model signals whose spectrum is not bounded but which are bounded in power. In both cases, these signals can be passed through weighting filters to produce signals with desired frequency content. We will primarily view the signals in $\mathcal{S}$ and $\mathcal{P}$ directly in the frequency domain in terms of their spectra.

Since $u \in \mathcal{L}_\infty$ we have that $R_{uu}(\tau) < \infty$, and hence $S_{uu}(j\omega)$ cannot be constant for all ω. When we refer to white signals we mean the limits of sequences of signals in $\mathcal{BS}$ that approach a constant spectrum. Some of the manipulations that we will make using white signals in subsequent sections require essentially an interchange of this limit process with others. A rigorous treatment of this material would justify the details of these interchanges of limits.

We have not demonstrated that $\mathcal{S}$ is nonempty. One solution to this would be to note that sample paths of stationary stochastic processes satisfy the assumptions for $\mathcal{S}$. A more satisfactory solution would be to exhibit deterministic signals that satisfy $\mathcal{S}$, but this is not trivial and is beyond the scope of this paper. This is an important issue and must be addressed before the nonstochastic theory suggested here can be considered to be established.

Spectral Analysis and Induced Norms We now list some useful spectral analysis facts for a linear system G with convolution kernel $g(t)$, input u, and output z as shown in Fig. 2.

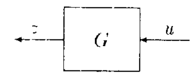

Fig. 2.

TABLE I
INDUCED SYSTEM GAINS

Input	Output	Signal Norms	Induced Norms
$\mathcal{L}_2$	$\mathcal{L}_2$	$\|u\|_2^2 = \int_{-\infty}^{\infty} \|u\|^2 dt$	$\|G\|_\infty$
$\mathcal{S}$	$\mathcal{S}$	$\|u\|_\mathcal{S}^2 = \|S_{uu}\|_\infty$	$\|G\|_\infty$
$\mathcal{S}$	$\mathcal{P}$	$\|u\|_\mathcal{P}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{S_{uu}(j\omega)\}d\omega$	$\|G\|_2$
$\mathcal{P}$	$\mathcal{P}$	$\|u\|_\mathcal{P}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{S_{uu}(j\omega)\}d\omega$	$\|G\|_\infty$

The following standard properties are assumed

$$\begin{aligned} R_{zu}(\tau) &= g(\tau) * R_{uu}(\tau) \\ R_{zz}(\tau) &= g(\tau) * R_{uu}(\tau) * g(-\tau)' \\ S_{zu}(j\omega) &= G(j\omega)S_{uu}(j\omega) \\ S_{zz}(j\omega) &= G(j\omega)S_{uu}(j\omega)G^*(j\omega). \end{aligned}$$

A more complete development of this material would prove these results using only the earlier assumptions. For $g(t)$ which are exponentially bounded, this should be entirely straightforward. These properties are useful in establishing several input and output relationships; in particular we have the relationships listed in Table I. Note that the induced norms from energy ($\mathcal{L}_2$) to energy, power to power, and spectrum to spectrum are all ∞-norms, while the induced norm from spectrum to power is the 2-norm. In particular, if the input signal is white with unit spectral density, then the power of the output equals the 2-norm of the transfer matrix.

C. Computing $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Norms

This section reviews some results on the computation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a transfer matrix G. Consider a realization

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1}$$

with A stable (i.e., all eigenvalues in the left-half plane). Let L_c denote the controllability gramian of (A, B) and L_o the observability gramian of (C, A), then

$$AL_c + L_cA' + BB' = 0 \qquad A'L_o + L_oA + C'C = 0$$

and for $D = 0$, the $\mathcal{H}_2$ norm of G can be computed by

$$\|G\|_2^2 = \text{Trace}(CL_cC') = \text{Trace}(B'L_oB). \tag{2}$$

Note that this computation involves the solution of a linear equation and can be done in a finite number of steps.

Computing the $\mathcal{H}_\infty$ norm of G is much harder. A recent effort involves using a Hamiltonian matrix. Given $\gamma > \bar{\sigma}(D)$,

Fig. 3.

define the Hamiltonian matrix

$$H := \begin{bmatrix} A + BR^{-1}D'C & \gamma^{-2}BR^{-1}B' \\ -C'(I + DR^{-1}D')C & -(A + BR^{-1}D'C)' \end{bmatrix} \quad (3)$$

where $R := \gamma^2 I - D'D > 0$.

The following lemma is essentially from [1], [12], [3].

Lemma 1: The following conditions are equivalent:

a) $\|G\|_\infty < \gamma$

b) $\bar{\sigma}(D) < \gamma$ and H has no eigenvalues on the imaginary axis

c) $\bar{\sigma}(D) < \gamma$ and $H \in dom(Ric)$

d) $\bar{\sigma}(D) < \gamma$, $H \in dom(Ric)$, and $Ric(H) \geq 0$ ($Ric(H) > 0$ if (C, A) is observable)

To determine $\|G\|_\infty$ numerically, select a positive number γ; determine if $\|G\|_\infty < \gamma$ by calculating the eigenvalues of H and using the above theorem. Increase or decrease γ accordingly, and refine the iteration until the desired precision is reached.

III. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Norm Performance Analysis

In general, in any analysis problem, our objective is to determine a system's performance under certain specified criteria with a fixed controller. The performance criteria may be bandwidth, overshoot, tracking error, robustness against uncertainties and disturbance, and so on. The criteria we are interested in this paper are related to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. It can be argued that the $\mathcal{H}_2$ norm, traditionally called a quadratic (functional) criteria, is a more natural and more suitable measure for system performance than the $\mathcal{H}_\infty$ norm. If there are uncertainties in the system model, however, then it is not a suitable measure for the system robustness. On the other hand, system robustness can be and has been very effectively described using $\mathcal{H}_\infty$ related criteria. It is thus natural that some quantity that combining the $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ norm is a desirable measure of a system's robust performance. The mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm introduced in this line of research is an attempt to achieve this goal.

A. Problem Formulations

To set up our mixed norm analysis problem, let us consider a system shown in Fig. 3.

The norms induced on G when G is subjected to two different classes of inputs, $w = \begin{bmatrix} w_0(t) \\ w_1(t) \end{bmatrix}$, are of particular interest to us. Specifically, we assume that the signal $w_0(t)$ is a signal with spectral density $S_{w_0 w_0}(j\omega)$ and the spectrum is bounded, i.e., $w_0(t) \in \mathcal{S}$ and the signal $w_1(t)$ is a bounded power signal, i.e., $w_1(t) \in \mathcal{P}$ with power spectrum $S_{w_1 w_1}(j\omega)$. We will be concerned with problems when w_0 and w_1 are independent and when w_1 has a causal or noncausal dependence on w_0, and these give differing assumptions on the

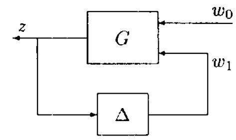

Fig. 4.

signal space for w and its spectrum S_{ww}. We shall measure the system performance by the power of the output $z(t)$.

Problem I: Let $\mathcal{W} \subset \mathcal{BS} \times \mathcal{P}$ and let $\mathcal{BW} = \mathcal{W} \cap (\mathcal{BS} \times \mathcal{BP})$Compute the induced norm

$$\sup_{w \in \mathcal{BW}} \|z\|_{\mathcal{P}}^2. \quad (4)$$

The exact form of the set $\mathcal{W}$ depends on the assumptions on w_0 andw_1, which will be specified later. This problem has been referred to as the "mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$" problem in our previous research because, from the table shown in the last section, if we ignore w_1 then the norm induced on G from w_0 to z is the $\mathcal{H}_2$ norm; similarly, if we ignore w_0 then the norm induced on G from w_1 to z is the $\mathcal{H}_\infty$ norm. Hence when both w_0 and w_1 act on the system, the induced norm will be a mixture of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.

The following alternative problem will be formulated to address the norm evaluation of Problem I.

Problem II: Given $\gamma > \|G_1\|_\infty$ and $\mathcal{W}$ as above, compute

$$J := \sup_{w \in \mathcal{W}} \left(\|z\|_{\mathcal{P}}^2 - \gamma^2 \|w_1\|_{\mathcal{P}}^2 \right).$$

The term γ^2 can be considered as a Lagrange multiplier for Problem I which has the constraint $\|w_1\|_{\mathcal{P}} \leq 1$. The following lemma illustrates the relation between Problems I and II.

Lemma 2: Suppose that γ_o is such that

$$J(\gamma_o) = \sup_{w \in \mathcal{W}} \left\{ \|z\|_{\mathcal{P}}^2 - \gamma_o^2 \|w_1\|_{\mathcal{P}}^2 \right\} = \|z^o\|_{\mathcal{P}}^2 - \gamma_o^2 \|w_1^o\|_{\mathcal{P}}^2 \quad (5)$$

with $\|w_1^o\|_{\mathcal{P}} = 1$. Then

$$\sup_{w \in \mathcal{BW}} \|z\|_{\mathcal{P}}^2 = \|z^o\|_{\mathcal{P}}^2.$$

Proof: Equation (5) implies that z^o is produced by $w \in \mathcal{BW}$ and

$$\|z^o\|_{\mathcal{P}}^2 - \gamma_o^2 \|w_1^o\|_{\mathcal{P}}^2 \geq \sup_{w \in \mathcal{BW}} \left\{ \|z\|_{\mathcal{P}}^2 - \gamma_o^2 \|w_1\|_{\mathcal{P}}^2 \right\}$$

and is hence optimal since

$$\|z^o\|_{\mathcal{P}}^2 \geq \sup_{w \in \mathcal{BW}} \left\{ \|z\|_{\mathcal{P}}^2 + \gamma_o^2 (1 - \|w_1\|_{\mathcal{P}}^2) \right\} \geq \|z\|_{\mathcal{P}}^2.$$

□

Hence a solution to Problem II will give a solution to Problem I if such a γ_o can be found. It is unfortunately not always the case that such a γ_o can be found (e.g., in the case of Theorem 2) and in Section III-C this will be discussed in more detail.

Another motivation for introducing Problem II is its relation to the following robust performance problem:

Problem III: Let $G = [G_0 \quad G_1]$ with $\|G_1\|_\infty < \gamma \leq 1$ be a nominal system and $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty \leq 1$ be the system uncertainty as shown in Fig. 4. Evaluate the system's worst performance

$$J_0 := \sup_{w_0 \in \mathcal{BS}, \|\Delta\|_\infty \leq 1} \|z\|_{\mathcal{P}}^2 = \sup_{w_0 \in \mathcal{BS}, \|\Delta\|_\infty \leq 1} \|(I - G_1\Delta)^{-1} G_0 w_0\|_{\mathcal{P}}^2.$$

The following theorem shows that the robust $\mathcal{H}_2$ performance J_0 can be bounded above.

Theorem 1: Suppose w_1 depends causally on w_0 and

$$J = \sup_{w \in \mathcal{W}} \left(\|z\|_{\mathcal{P}}^2 - \gamma^2 \|w_1\|_{\mathcal{P}}^2 \right).$$

Then

$$J_0 \leq \frac{J}{1 - \gamma^2}.$$

Proof: Note that for any $w_1 \in \mathcal{P}$ depending causally on w_0, we have

$$\|z\|_{\mathcal{P}}^2 - \gamma^2 \|w_1\|_{\mathcal{P}}^2 \leq J.$$

From the setup of Problem III, $w_1 = \Delta z$, so w_1 depends causally on w_0, hence

$$\begin{aligned} \|z\|_{\mathcal{P}}^2 &\leq \gamma^2 \|w_1\|_{\mathcal{P}}^2 + J \\ &= \gamma^2 \|\Delta z\|_{\mathcal{P}}^2 + J \\ &\leq \gamma^2 \|z\|_{\mathcal{P}}^2 + J. \end{aligned}$$

Therefore, for any $\Delta \in \mathcal{RH}_\infty$

$$\|z\|_{\mathcal{P}}^2 \leq \frac{J}{1 - \gamma^2}.$$

□

Hence the performance index J gives not only the system performance under two different kinds of disturbances but also an upper bound for the robust $\mathcal{H}_2$ performance. Since most of our analysis will be done in the frequency domain, we shall first give a frequency domain characterization of $\|z\|_{\mathcal{P}}$. Denote the cross spectral density of w_0 and w_1 by $S_{w_0 w_1}(j\omega)$. Now assume G is stable and partition G compatibly with w_0 and w_1 as $[G_0 \; G_1]$, where G_0 is assumed strictly proper (otherwise the output signal will have unbounded power if w_0 is white). In terms of the state-space matrices, this can be represented as

$$G(s) = \left[\begin{array}{c|cc} A & B_0 & B_1 \\ \hline C & 0 & D_1 \end{array} \right] =: [G_0 \quad G_1].$$

The spectral density matrix of w is positive semidefinite and hence it can be written as

$$\begin{aligned} S_{ww} &= \begin{bmatrix} S_{w_0 w_0} & S_{w_0 w_1} \\ S_{w_0 w_1}^* & S_{w_1 w_1} \end{bmatrix} \\ &= \begin{bmatrix} S_{w_0 w_0} & S_{w_0 w_0} W^* \\ W S_{w_0 w_0}^* & W S_{w_0 w_0} W^* + S_{11} \end{bmatrix}, \qquad (6) \\ &\qquad \text{for some } S_{11} \geq 0 \text{ and } W. \end{aligned}$$

Using this formula and the facts from spectral analysis shown before, we get

$$S_{zz} = [G_0(j\omega) \quad G_1(j\omega)] \begin{bmatrix} S_{w_0 w_0} & S_{w_0 w_1} \\ S_{w_0 w_1}^* & S_{w_1 w_1} \end{bmatrix} \begin{bmatrix} G_0^*(j\omega) \\ G_1^*(j\omega) \end{bmatrix} \qquad (7)$$

and

$$\|z\|_{\mathcal{P}}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Trace}[S_{zz}(j\omega)] d\omega. \qquad (8)$$

We will say that w_1 depends causally on w_0 if $w_1 = W w_0$ for some $W \in \mathcal{H}_2$, which implies $S_{w_0 w_1} = S_{w_0 w_0} W^*$ and $S_{11} = 0$. This is a very narrow notion of causality, but it is appropriate for the purposes of this paper. We will say that w_1 has a noncausal dependence on w_0 when no specific constraint on W is imposed, that is, w_1 may or may not depend causally on w_0.

We shall consider several different cases for our analysis problem:

a) Orthogonal: w_0 and w_1 are orthogonal, i.e., $S_{w_0 w_1} = 0$ and $W = 0$ (but S_{11} is not necessarily zero);
b) White and causal: w_0 is white and w_1 is causally dependent on w_0;
c) Nonwhite and causal: w_0 is nonwhite and w_1 is causally dependent on w_0;
d) Nonwhite and noncausal: w_0 is nonwhite and w_1 is not necessarily causally dependent on w_0.

Each case then corresponds to different assumptions on the signal set $\mathcal{W}$. Note that by nonwhite we mean not necessarily white.

Let us first consider the analysis problem when w_0 and w_1 are orthogonal. In this case we have the following theorem.

Theorem 2: If w_0 and w_1 are orthogonal, i.e., $S_{w_0 w_1} = 0$, then

$$\sup_{w \in \mathcal{BW}} \|z\|_{\mathcal{P}}^2 = \|G_0\|_2^2 + \|G_1\|_\infty^2 \qquad (9)$$

and for $\gamma > \|G_1\|_\infty$

$$\sup_{w \in \mathcal{W}} \left(\|z\|_{\mathcal{P}}^2 - \gamma^2 \|w_1\|_{\mathcal{P}}^2 \right) = \|G_0\|_2^2.$$

Proof: Since $S_{w_0 w_1} = 0$, we have

$$\begin{aligned} \|z\|_{\mathcal{P}}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ &\mathrm{Trace}[G_0(j\omega) S_{w_0 w_0} G_0^*(j\omega)] \\ &+ \mathrm{Trace}[G_1(j\omega) S_{w_1 w_1} G_1^*(j\omega)] \} d\omega \end{aligned}$$

and

$$\begin{aligned} \sup_{w \in \mathcal{BW}} \|z\|_{\mathcal{P}}^2 &= \sup_{w_0 \in \mathcal{BS}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Trace}[G_0(j\omega) S_{w_0 w_0} G_0^*(j\omega)] d\omega \\ &\quad + \sup_{w_1 \in \mathcal{BP}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Trace}[G_1(j\omega) S_{w_1 w_1} G_1^*(j\omega)] d\omega \\ &= \|G_0\|_2^2 + \|G_1\|_\infty^2 \end{aligned}$$

and the worst signal w_0 is white noise with unit spectral density matrix, $S_{w_0 w_0} = I$, while the worst signal for w_1 is as given in the Appendix.

On the other hand

$$\sup_{w\in\mathcal{W}}\left(\|z\|_{\mathcal{P}}^2-\gamma^2\|w_1\|_{\mathcal{P}}^2\right)=\sup_{w_0\in\mathcal{BS}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Trace}[G_0(j\omega)S_{w_0w_0}G_0^*(j\omega)]\times d\omega+\sup_{w_1\in\mathcal{P}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Trace}\,S_{w_1w_1}\times[G_1^*(j\omega)G_1(j\omega)-\gamma^2 I]d\omega = \|G_0\|_2^2$$

with a worst-case signal $w_1=0$. □

In each of the cases a)-d) we have

$$\|z\|_{\mathcal{P}}=\|G_0w_0+G_1w_1\|_{\mathcal{P}}\le\|G_0w_0\|_{\mathcal{P}}+\|G_1w_1\|_{\mathcal{P}}.$$

Hence

$$\sup_{w\in\mathcal{BW}}\|z\|_{\mathcal{P}}\le\|G_0\|_2+\|G_1\|_\infty\le\sqrt{2(\|G_0\|_2^2+\|G_1\|_\infty^2)}.$$

Thus the relationships among the costs of $\|z\|_{\mathcal{P}}$ in different cases can be summarized as the following theorem.

Theorem 3:

$$\begin{aligned}\sup\{\|z\|_{\mathcal{P}}: S_{w_0w_1}=0\}&\le\sup\{\|z\|_{\mathcal{P}}: w_0\text{ is white and } w_1 \text{ depends causally on } w_0\}\\&\le\sup\{\|z\|_{\mathcal{P}}: w_0\text{ is nonwhite and } w_1 \text{ depends causally on } w_0\}\\&\le\sup\{\|z\|_{\mathcal{P}}: w_0\text{ is nonwhite and } w_1 \text{ is noncausal}\}\\&\le\sqrt{2}\sup\{\|z\|_{\mathcal{P}}: S_{w_0w_1}=0\}.\end{aligned}$$

We will show later that

$$\sup\{\|z\|_{\mathcal{P}}: w_0\text{ is white and } w_1\text{ is noncausal}\}$$

$$=\sup\{\|z\|_{\mathcal{P}}: w_0\text{ is nonwhite and } w_1\text{ is noncausal}\}.$$

Hence the cost $\sup_{w\in\mathcal{BW}}\|z\|_{\mathcal{P}}$ for different cases makes very little difference in the actual induced norm. For engineering purposes, it is probably adequate to choose whichever case is easiest to work with. On the other hand, the different cases have features which are interesting from a theoretical point of view, so the relationship between the different cases will be considered further.

B. Preliminary Manipulations for Problem II

To compute J for the other cases b)-d), we first need to establish some key formulas for Problem II. Let $\gamma>0$ be such that $\|G_1\|_\infty<\gamma$, then

$$\sup_{w\in\mathcal{W}}\left\{\|z\|_{\mathcal{P}}^2-\gamma^2\|w_1\|_{\mathcal{P}}^2\right\}=\sup_{w\in\mathcal{W}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Trace}(S_{zz}-\gamma^2S_{w_1w_1})d\omega$$

and substituting from (6)–(7) we obtain,

$$\mathrm{Trace}(S_{zz}-\gamma^2S_{w_1w_1})=\mathrm{Trace}\,S_{w_0w_0}\{\gamma^2G_0^*(\gamma^2I-G_1G_1^*)^{-1}G_0-\Gamma^*(\gamma^2I-G_1^*G_1)\Gamma\}-\mathrm{Trace}\,S_{11}(\gamma^2I-G_1^*G_1)$$

where

$$\Gamma:=W-(\gamma^2I-G_1^*G_1)^{-1}G_1^*G_0.$$

Since we wish to maximize this expression and $(\gamma^2I-G_1^*G_1)>0$, the maximizing S_{ww} will always have $S_{11}=0$. Thus for all the cases considered here, $S_{w_1w_1}=WS_{w_0w_0}W^*$, and w_1 is completely correlated with w_0.

Let $N_1\in\mathcal{RH}_\infty$ be a spectral factor of $\gamma^2I-G_1^*G_1$ such that $N_1^{-1}\in\mathcal{RH}_\infty$ and $N_1^*N_1=\gamma^2I-G_1^*G_1$, then

$$J=\sup_{w\in\mathcal{W}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Trace}\,S_{w_0w_0}\{\gamma^2G_0^*(\gamma^2I-G_1G_1^*)^{-1}G_0-\Gamma^*N_1^*N_1\Gamma\}d\omega. \tag{10}$$

Let $R=\gamma^2I-D_1'D_1$ and

$$X=Ric\begin{bmatrix}A+B_1R^{-1}D_1'C & B_1R^{-1}B_1'\\ -C'(I+D_1R^{-1}D_1')C & -(A+B_1R^{-1}D_1'C)'\end{bmatrix}. \tag{11}$$

Then it can be shown that

$$N_1(s)=\left[\begin{array}{c|c}A & B_1\\\hline -R^{-1/2}(D_1'C+B_1'X) & R^{1/2}\end{array}\right] \tag{12}$$

and

$$N_1(\gamma^2I-G_1^*G_1)^{-1}G_1^*G_0=(N_1^*)^{-1}G_1^*G_0=N_2+N_3$$

where

$$N_2(s)=\left[\begin{array}{c|c}A & B_0\\\hline R^{-1/2}(D_1'C+B_1'X) & 0\end{array}\right]\in\mathcal{RH}_2 \tag{13}$$

$$N_3(s)=\left[\begin{array}{c|c}-(A+B_1R^{-1}D_1'C+B_1R^{-1}B_1'X)' & -XB_0\\\hline R^{-1/2}B_1' & 0\end{array}\right]\in\mathcal{RH}_2^\perp. \tag{14}$$

Hence

$$N_1\Gamma=N_1W-N_2-N_3 \tag{15}$$

and without loss of generality, we can assume

$$W=N_1^{-1}N_2+Q$$

for some $Q\in\mathcal{Q}\subset\mathcal{L}_2$, since the mapping from Q to W is bijective and where $\mathcal{Q}$ depends on the set of assumptions. Hence

$$N_1\Gamma=Q-N_3.$$

and the following lemma is proven.

Lemma 3: With the above definitions

$$J = \sup_{w_0 \in \mathcal{BS},\, Q \in \mathcal{Q}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\,\{S_{w_0 w_0} \Phi(j\omega)\} d\omega$$

where

$$\Phi(s) = \gamma^2 G_0^* (\gamma^2 I - G_1 G_1^*)^{-1} G_0 - (Q - N_3)^* (Q - N_3) \quad (16)$$

and $w_1 = W w_0$ with

$$W = N_1^{-1} N_2 + Q,\; Q \in \mathcal{Q}.$$

Note that depending on the assumptions on the signals w_0 and w_1, $\Phi(j\omega)$ need not be positive semidefinite for all ω, hence white noise is not, in general, the worst signal for w_0.

C. Performance Analysis with White and Causal Signals

In this case, w_0 is assumed to be white, i.e., $S_{w_0 w_0} = I$ and w_1 is assumed to depend causally on w_0, so $W \in \mathcal{H}_2$. We shall only present a frequency domain solution in this paper and a time domain solution will be given in the companion paper [7] together with synthesis results.

Since in this case $S_{w_0 w_0} = I$ and $\mathcal{Q} = \mathcal{H}_2$, from Lemma 3 we have

$$J = \sup_{Q \in \mathcal{H}_2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\,\Phi(j\omega) d\omega.$$

Theorem 4: Let $\gamma > \|G_1\|_\infty$ then

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\,\{\gamma^2 G_0^* (\gamma^2 I - G_1 G_1^*)^{-1} G_0 - N_3^* N_3\} d\omega$$
$$= \|G_0\|_2^2 + \|N_2\|_2^2 = \text{Trace}(B_0' X B_0)$$

with

$$W = N_1^{-1} N_2 = \left[\begin{array}{c|c} A + B_1 R^{-1}(D_1'C + B_1'X) & B_0 \\ \hline R^{-1}(D_1'C + B_1'X) & 0 \end{array}\right] \in \mathcal{RH}_2. \quad (17)$$

Proof: Since in this case

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\,\{\gamma^2 G_0^* (\gamma^2 I - G_1 G_1^*)^{-1} G_0\} d\omega - \inf_{Q \in \mathcal{H}_2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\,\{(Q - N_3)^* (Q - N_3)\} d\omega.$$

It is clear that the worst signal satisfies $Q = 0$ by orthogonal projection, so $w_1 = N_1^{-1} N_2 w_0$ and

$$\Phi(s) = \gamma^2 G_0^* (\gamma^2 I - G_1 G_1^*)^{-1} G_0 - N_3^* N_3.$$

Hence

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace} \left\{G_0^* \left(I + G_1 (\gamma^2 I - G_1^* G_1)^{-1} G_1^*\right) G_0 - N_3^* N_3\right\} d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace} \{G_0^* G_0 + G_0^* G_1 N_1^{-1} N_1^{-*} G_1^* G_0 - N_3^* N_3\} d\omega$$

and since N_2 and N_3 are orthogonal and $N_2 + N_3 = (N_1^*)^{-1} G_1^* G_0$ we obtain $J = \|G_0\|_2^2 + \|N_2\|_2^2$. This can be evaluated from the state-space realization of $\begin{bmatrix} G_0 \\ N_2 \end{bmatrix}$ on noting that the Riccati equation for X in (11) can be written as

$$XA + A'X + \begin{bmatrix} C \\ R^{-1/2}(D_1'C + B_1'X) \end{bmatrix}' \begin{bmatrix} C \\ R^{-1/2}(D_1'C + B_1'X) \end{bmatrix} = 0$$

i.e., X is the observability gramian of $\begin{bmatrix} G_0 \\ N_2 \end{bmatrix}$. Thus we have $J = \text{Trace}(B_0' X B_0)$. □

This gives the solution to Problem II and by Lemma 2 we can derive a solution to Problem I if we can find a suitable value of γ_o as follows.

Corollary 1: Let W be defined in (17), then there exists γ_o such that $\|W\|_2 = 1$ if

$$\lim_{\gamma \to \|G_1\|_\infty} \|W\|_2 > 1.$$

Further such a γ_o gives

$$\sup_{w \in \mathcal{BW}} \|z\|_{\mathcal{P}}^2 = \|G_0\|_2^2 + \|N_2\|_2^2 + \gamma_o^2 = J(\gamma_o) + \gamma_o^2.$$

Proof: First, it will be shown that $\|w_1\|_{\mathcal{P}}$ must be monotonically increasing as γ decreases towards $\|G_1\|_\infty$. Let $\|G_1\|_\infty < \gamma_o < \gamma_1$, and the corresponding signal norms in the optimal solutions to Problem II be respectively, $\|z^o\|_{\mathcal{P}}$, $\|w_1^o\|_{\mathcal{P}}$, $\|z^1\|_{\mathcal{P}}$ and $\|w_1^1\|_{\mathcal{P}}$. Then

$$\|z^o\|_{\mathcal{P}}^2 - \gamma_o^2 \|w_1^o\|_{\mathcal{P}}^2 \geq \|z^1\|_{\mathcal{P}}^2 - \gamma_o^2 \|w_1^1\|_{\mathcal{P}}^2$$
$$\|z^1\|_{\mathcal{P}}^2 - \gamma_1^2 \|w_1^1\|_{\mathcal{P}}^2 \geq \|z^o\|_{\mathcal{P}}^2 - \gamma_1^2 \|w_1^o\|_{\mathcal{P}}^2$$

which implies that $\gamma_o^2(\|w_1^o\|_{\mathcal{P}}^2 - \|w_1^1\|_{\mathcal{P}}^2) \leq (\|z^o\|_{\mathcal{P}}^2 - \|z^1\|_{\mathcal{P}}^2) \leq \gamma_1^2(\|w_1^o\|_{\mathcal{P}}^2 - \|w_1^1\|_{\mathcal{P}}^2)$, and hence $\|w_1^o\|_{\mathcal{P}}^2 \geq \|w_1^1\|_{\mathcal{P}}^2$. Further, it is clear that as $\gamma \to \infty$ that $\|W\|_2 \to 0$. Hence there will exist a γ_o giving $\|W\|_2 = 1$ if $\lim_{\gamma \to \|G_1\|_\infty} \|W\|_2 > 1$. The evaluation of the norm is then directly from Lemma 2. □

The conditions for the existence of γ_o are quite intricate as can be seen from examining the state-space realization for W. As $\gamma \to \|G_1\|_\infty$ one typically has a pole of W tending towards the imaginary axis and hence $\|W\|_2$ will tend to ∞ unless this pole is not minimal, by for example a suitable choice of B_0. This will also be the case if the stable poles of G_0 are canceled by the stable zeros of G_1^* in forming $(N_1^*)^{-1} G_1^* G_0 = N_2 + N_3$, and hence giving $W = N_1^{-1} N_2 = 0$.

Hence computing the power norm of z involves iterations on γ, as in the pure $\mathcal{H}_\infty$ case. We will now illustrate the above process though a simple example. Let

$$G = \left[\begin{array}{c|cc} -1 & 2 & 1 \\ \hline 1 & 0 & 0 \end{array}\right].$$

Then $G_1 = \frac{1}{s+1}$ and $\|G_1\|_\infty = 1$. It is clear that for $\gamma > 1$, the Riccati equation for X has a stabilizing solution

$$X = \gamma^2 - \gamma\sqrt{\gamma^2 - 1}$$

which gives

$$W(s) = \frac{2(1 - \sqrt{1 - \gamma^{-2}})}{s + \sqrt{1 - \gamma^{-2}}}$$

and

$$\|W(s)\|_2 = \frac{1}{\sqrt{1-\gamma^{-2}}} - 1.$$

Since $\|W(s)\|_2 \to \infty$ as $\gamma \to 1$ and $\|W(s)\|_2 \to 0$ as $\gamma \to \infty$, there is γ_0 such that $\|W(s)\|_2 = 1$. Indeed, $\gamma_0 = \frac{2}{\sqrt{3}}$ is the solution, which gives $W(s) = \frac{1}{s+1/2}$.

In general, however, neither X nor $\|W\|_2$ can be obtained explicitly in terms of γ, hence iteration on γ has to be done.

D. Performance Analysis with Nonwhite and Causal Signals

This is the case where $w_0 \in \mathcal{S}$ is not assumed to be white and $w_1 \in \mathcal{P}$ is assumed to depend causally on w_0, i.e., $w_1 = Ww_0$ for $W \in \mathcal{H}_2$. This is what we think of as the "real" problem, and it has a much better physical motivation than any other problem mentioned above. The difference between this case and the white and causal case considered in previous research is significant. The fact that white noise will not be the worst case signal can be inferred from (16), where we see that $\Phi(j\omega)$ need not be positive semi-definite for all ω. This is shown by an example.

We will construct an example where the white/causal problem can be solved analytically and then construct a nonwhite spectrum, $S_{w_0w_0}$, and causal W such that an increased norm of z is achieved. Let

$$[G_0 \quad G_1] = \left[\tfrac{\beta}{(s+1)^2} \quad \tfrac{1}{s+1}\right] = \left[\begin{array}{cc|cc} -1 & 1 & 0 & 1 \\ 0 & -1 & \beta & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right].$$

Since $\|G_1\|_\infty = 1$ we need $\gamma > 1$ and it will be convenient to define $\alpha := \sqrt{1-\gamma^{-2}} > 0$. Straightforward algebra now gives

$$\begin{aligned} N_1 &= \frac{(s+\alpha)}{\sqrt{1-\alpha^2}\,(s+1)} \\ (N_1^*)^{-1}G_1^*G_0 &= \frac{\beta\sqrt{1-\alpha^2}}{(\alpha-s)(s+1)^2} = N_2 + N_3 \\ N_2 &= \frac{\beta\sqrt{1-\alpha^2}\,(s+2+\alpha)}{(1+\alpha)^2(1+s)^2} \\ N_3 &= \frac{\beta\sqrt{1-\alpha^2}}{(1+\alpha)^2(\alpha-s)} \\ W &= \frac{\beta(1-\alpha)(s+2+\alpha)}{(1+\alpha)(s+\alpha)(s+1)} \\ &= \frac{2\beta}{(1+\alpha)(s+\alpha)} - \frac{\beta}{(s+1)} \\ G_0 + G_1W &= \frac{2\beta}{(1+\alpha)(s+1)(s+\alpha)} \\ &= \frac{2\beta}{(1-\alpha^2)}\left\{\frac{1}{s+\alpha} - \frac{1}{s+1}\right\}. \end{aligned}$$

Now considering the white/causal optimal case when $Q = 0$ we obtain

$$\begin{aligned} \Phi(j\omega) &= \frac{\gamma^2 G_0G_0^*}{N_1^*N_1} - N_3^*N_3 \\ &= \frac{\beta^2}{\omega^2+\alpha^2}\left\{\frac{1}{(\omega^2+1)} - \frac{1-\alpha^2}{(1+\alpha)^4}\right\}. \end{aligned}$$

Hence $\Phi(j\omega) < 0$ for all ω sufficiently large and it is not optimal in Problem II to use a white w_0.

It remains to be shown that the norm can be increased by choice of a nonwhite signal. We will demonstrate this by choosing a value of $\alpha = 0.5$ and then making the spectrum of w_0 unity for the frequencies up to $\omega^2 = 23/4 =: \omega_o^2$, where Φ is positive and zero is outside this range. The filter, W, will be as above and β will be chosen to give unity norm for w_1

$$\begin{aligned} \|w_1\|_{\mathcal{P}}^2 = \|W\|_2^2 &= \frac{\beta^2}{2\pi}\int_{-\omega_o}^{\omega_o} \frac{(\omega^2+25/4)}{9(\omega^2+1/4)(\omega^2+1)}d\omega \\ &= \frac{\beta^2}{9\pi}\{16\arctan(2\omega_o) - 7\arctan(\omega_o)\} \\ &= 1 \text{ if } \beta = 1.44114\cdots. \end{aligned}$$

With this value of β we calculate the norm of $z = (G_0 + G_1W)w_0$ as

$$\begin{aligned} \|z\|_{\mathcal{P}}^2 &= \frac{1}{2\pi}\int_{-\omega_o}^{\omega_o}\frac{4\beta^2}{(1+\alpha)^2(\omega^2+1)(\omega^2+\alpha^2)}d\omega \\ &= \frac{64\beta^2}{27\pi}\{2\arctan(2\omega_o) - \arctan(\omega_o)\} = 2.43637\cdots. \end{aligned}$$

This gives a lower bound on the squared norm in the nonwhite/causal case, and it will now be compared with the white/causal case for which we need to calculate the value of γ_o or equivalently α_o, so that

$$\|W\|_2^2 = 1 = \beta^2\left\{\frac{1}{2} - \frac{4}{(1+\alpha_o)^2} + \frac{2}{(1+\alpha_o)^2\alpha_o}\right\}$$

which is satisfied if α_o satisfies the cubic

$$\alpha_o^3 + 2\alpha_o^2 + \left(1 + \frac{8}{(2\beta^{-2}-1)}\right)\alpha_o - \frac{4}{(2\beta^{-2}-1)} = 0.$$

This results in $\alpha_o = 0.50529\cdots$ and gives a maximum norm for z as

$$\begin{aligned} \sup \|z\|_{\mathcal{P}}^2 &= \|G_0 + G_1W\|_2^2 \\ &= \frac{4\beta^2}{(1-\alpha_o^2)^2}\left\{\frac{1}{2} - \frac{2}{(1+\alpha_o)} + \frac{1}{2\alpha_o}\right\} = 2.41005\cdots. \end{aligned}$$

This gives the maximum value of the squared norm in the white/causal case and it is slightly smaller than the suboptimal nonwhite/causal value given above, hence verifying that the optimal input signal for w_0 is not generally white in this case, in contrast to the case when a noncausal W is allowed as in the next subsection.

E. Performance Analysis with Nonwhite and Noncausal Signals

In this section, we shall consider the analysis problem where w_0 is not restricted to be white and w_1 is not restricted to depend causally on w_0. Then the filter W in $w_1 = Ww_0$ is not necessarily causal, so $W \in \mathcal{L}_2$. The following study will show that in this case the worst-case signal w_0 is actually white, but the worst-case signal w_1 is not, in general, a causal function of w_0.

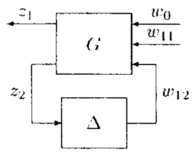

Fig. 5.

Theorem 5: If any $W \in \mathcal{L}_2$ is admissible then

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left\{\gamma^2 G_0^*(\gamma^2 I - G_1 G_1^*)^{-1} G_0\right\} d\omega = \|G_0\|_2^2 + \|N_2\|_2^2 + \|N_3\|_2^2$$

with the worst-case signal w_0 =white and

$$w_1 = (\gamma^2 I - G_1^* G_1)^{-1} G_1^* G_0 w_0 = N_1^{-1}(N_2 + N_3) w_0.$$

Proof: Since W can be any function in $\mathcal{L}_2$, the set $\mathcal{Q}$ equals $\mathcal{L}_2$, and for any given signal w_0, the worst-case signal w_1 must satisfy $\Gamma = 0$; that is, $Q = N_3$ and

$$W = (\gamma^2 I - G_1^* G_1)^{-1} G_1^* G_0. \tag{18}$$

So the worst-case signal w_1 is generated from passing w_0 through the noncausal linear system $(\gamma^2 I - G_1^* G_1)^{-1} G_1^* G_0$. Hence we have

$$\sup_{w \in \mathcal{W}}\left\{\|z\|_{\mathcal{P}}^2 - \gamma^2 \|w_1\|_{\mathcal{P}}^2\right\} = \sup_{w_0 \in \mathcal{BS}} \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\, S_{w_0 w_0} \times \left\{G_0^*(I - \gamma^{-2} G_1 G_1^*)^{-1} G_0\right\} d\omega.$$

Obviously, the worst-case signal w_0 is white, so the above is equal to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left\{G_0^*(I - \gamma^{-2} G_1 G_1^*)^{-1} G_0\right\} d\omega. \tag{19}$$

□

The results presented in Corollary 1 can also be applied here to compute $\sup_{w \in \mathcal{BW}} \|z\|_{\mathcal{P}}$ if desired.

IV. Robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance

In this section we will consider system performance analysis when the system model has structured norm-bounded perturbations as shown in Fig. 5 where G is partitioned according to the inputs and outputs as

$$G = \begin{bmatrix} G_{00} & G_{01} & G_{02} \\ G_{10} & G_{11} & G_{12} \end{bmatrix} =: [G_0 \quad G_1].$$

The uncertainty is structured such that $\Delta \in \boldsymbol{\Delta}$ where

$$\boldsymbol{\Delta} = \left\{\text{diag}\{\Delta_1, \Delta_2, \cdots \Delta_m\} : \Delta_i \in (\mathcal{H}_\infty)^{t_i \times t_i},\ \|\Delta_i\|_\infty \le 1\right\}.$$

We shall consider the evaluation of the system worst performance.

Problem IV: Given $\|\mu(G_1)\|_\infty < \gamma \le 1$, where $\mu(G_1)$ is the structured singular value of G_1 with respect to the structured uncertainty diag$\{\Delta_0, \Delta\}$ with $\Delta \in \boldsymbol{\Delta}$ and $\|\Delta_0\|_\infty \le 1$, compute

$$J_\Delta := \sup_{w_0 \in \mathcal{BS}, w_{11} \in \mathcal{P}, \Delta \in \boldsymbol{\Delta}} \left(\|z_1\|_{\mathcal{P}}^2 - \gamma^2 \|w_{11}\|_{\mathcal{P}}^2\right).$$

For information on the structured singular value (μ), the reader is referred to Doyle [4], Packard [11], and Fan and Tits [8]. Analysis of this mixed problem is more difficult than the pure $\mathcal{H}_\infty$ case, where the μ analysis theory applies directly. An upper bound for this problem can be obtained by combining the μ analysis and the mixed norm analysis results in the previous section. Define a set of scaling transfer matrices

$$\mathcal{D} = \left\{\text{diag}\{d_1(s) I_{t_1}, \cdots, d_m(s) I_{t_m}\} : d_i(s), d_i^{-1}(s) \in \mathcal{H}_\infty, \right\}.$$

Then $D \Delta D^{-1} = \Delta$ for all $\Delta \in \boldsymbol{\Delta}$ and $D \in \mathcal{D}$.

Let $D(s) \in \mathcal{D}$ and

$$z := \begin{bmatrix} z_1 \\ D(s) z_2 \end{bmatrix}, \quad w_1 := \begin{bmatrix} w_{11} \\ D(s) w_{12} \end{bmatrix}.$$

Then we have

$$z = \begin{bmatrix} G_{00} & G_{01} & G_{02} D^{-1} \\ D G_{10} & D G_{11} & D G_{12} D^{-1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} =: [\hat{G}_0 \quad \hat{G}_1] \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}.$$

Now consider the following mixed norm analysis problem

$$\hat{J} = \inf_{D \in \mathcal{D}} \sup_{w \in \mathcal{W}} \left\{\|z\|_{\mathcal{P}}^2 - \|w_1\|_{\mathcal{P}}^2\right\}. \tag{20}$$

Given D, the above maximization problem can be solved using the results obtained in the previous section.

Theorem 6: Suppose $\|\hat{G}_1\|_\infty < \gamma \le 1$ for some $D \in \mathcal{D}$, then

$$J_\Delta \le \hat{J}.$$

Proof: Note first that

$$\|z\|_{\mathcal{P}}^2 = \|z_1\|_{\mathcal{P}}^2 + \|D z_2\|_{\mathcal{P}}^2$$

and

$$\|w_1\|_{\mathcal{P}}^2 = \|w_{11}\|_{\mathcal{P}}^2 + \|D w_{12}\|_{\mathcal{P}}^2.$$

Then

$$\begin{aligned}\|z\|_{\mathcal{P}}^2 - \gamma^2\|w_1\|_{\mathcal{P}}^2 &= \|z_1\|_{\mathcal{P}}^2 - \gamma^2\|w_{11}\|_{\mathcal{P}}^2 + \|D z_2\|_{\mathcal{P}}^2 - \gamma^2\|D \Delta z_2\|_{\mathcal{P}}^2 \\ &= \|z_1\|_{\mathcal{P}}^2 - \gamma^2\|w_{11}\|_{\mathcal{P}}^2 + \|D z_2\|_{\mathcal{P}}^2 - \gamma^2\|\Delta D z_2\|_{\mathcal{P}}^2 \\ &\ge \|z_1\|_{\mathcal{P}}^2 - \gamma^2\|w_{11}\|_{\mathcal{P}}^2 + \|D z_2\|_{\mathcal{P}}^2 - \gamma^2\|D z_2\|_{\mathcal{P}}^2 \\ &\ge \|z_1\|_{\mathcal{P}}^2 - \gamma^2\|w_{11}\|_{\mathcal{P}}^2.\end{aligned}$$

Note that we have used $w_{12} = \Delta z_2$ in the first equation and $D\Delta = \Delta D$ in the second equation. □

To get the least conservative test possible, a search on D is required. If $w_0 = 0$, then the problem is exactly the μ analysis problem and the best D solves a convex optimization problem. Furthermore, if the number of uncertainty blocks less than two, the above criteria is necessary as well as sufficient for robust performance. The problem of selecting the best D scalings for the mixed problem is still open and not as simple as for the $\mathcal{H}_\infty$ case, where the problem can be reduced to constant matrices at each frequency.

V. Conclusions

In this paper, several system analysis problems based on a mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ criterion were introduced. The problems were divided into cases involving whether w_0 was white or not, and whether w_1 was causally related to w_0 or not. Solutions were given for the white causal case and for the nonwhite noncausal case. In the latter case we showed that white noise was in fact the worst-case signal. The most difficult case is the nonwhite causal case, and we presented an example showing that white noise is not the worst-case signal here. This problem remains unsolved. In addition, some applications to robust performance analysis with structured uncertainties were discussed.

Several issues in the paper need to be addressed further. For example, what is the best γ in the robust $\mathcal{H}_2$ performance bound that will give the least conservative bound for J_0? What are the best scaling matrices D? We believe that some μ-like computational algorithm can be developed to evaluate this robust performance. As far as the mixed norm analysis problem is concerned perhaps the most puzzling problem is what the worst-case signal w_0 for the nonwhite and causal case is. A better characterization of signals of bounded spectrum would also be helpful. In a related paper, we have successfully solved the synthesis problem for the white and causal case. The synthesis problem for the white and noncausal case has not been solved. These issues will be considered in our future research.

Appendix
Proofs of Induced Norms

We now prove the relationships given in Table I.

- $\mathcal{BL}_2 \to \mathcal{L}_2$: This is a standard result.
- $\mathcal{BS} \to \mathcal{S}$: If $u \in \mathcal{S}$, then

$$S_{zz}(j\omega) = G(j\omega)S_{uu}(j\omega)G^*(j\omega)$$

so

$$\|S_{zz}(j\omega)\|_\infty \leq \|G(j\omega)\|_\infty^2 \|S_{uu}(j\omega)\|_\infty.$$

Now suppose for some $w_0 \in \mathbb{R} \cup \{\infty\}$, we have

$$\bar{\sigma}[G(j\omega_0)] = \|G\|_\infty$$

and take a signal u such that $S_{uu}(j\omega_0) = I$ (for example a white signal). Then

$$\|S_{zz}(j\omega)\|_\infty = \|G\|_\infty^2.$$

- $\mathcal{BS} \to \mathcal{P}$: By definition, we have

$$\begin{aligned}\|z\|_\mathcal{P}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G(j\omega)S_{uu}(j\omega)G^*(j\omega)\}\, d\omega \\ &\leq \|S_{uu}(j\omega)\|_\infty \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G(j\omega)G^*(j\omega)\}\, d\omega \\ &= \|S_{uu}(j\omega\|)_\infty \|G\|_2^2.\end{aligned}$$

Now let u be white, i.e., $S_{uu} = I$. Then

$$\|z\|_\mathcal{P}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G(j\omega)G^*(j\omega)\}\, d\omega = \|G\|_2^2.$$

Note that if G is not strictly proper then the norm is unbounded.

- $\mathcal{BP} \to \mathcal{P}$: Since

$$\|z\|_\mathcal{P}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G(j\omega)S_{uu}(j\omega)G^*(j\omega)\}\, d\omega$$

we get immediately that

$$\|z\|_\mathcal{P} \leq \|G\|_\infty \|u\|_\mathcal{P}.$$

To show that $\|G\|_\infty$ is the least upper bound, first assume there exists some $\omega_0 < \infty$ such that

$$\bar{\sigma}[G(j\omega_0)] = \|G\|_\infty.$$

Let $G(j\omega_0)$ have a singular value decomposition

$$G(j\omega_0) = \bar{\sigma}u_1(j\omega_0)v_1^*(j\omega_0) + \sum_{i=2}^{r} \sigma_i u_i(j\omega_0)v_i^*(j\omega_0)$$

where r is the rank of $G(j\omega_0)$, and u_i and v_i are unit vectors. Write $v_1(j\omega_0)$ as

$$v_1(j\omega_0) = \begin{bmatrix} \alpha_1 e^{j\theta_1} \\ \alpha_2 e^{j\theta_2} \\ \vdots \\ \alpha_q e^{j\theta_q} \end{bmatrix}$$

where $\alpha_i \in \mathbb{R}$ is chosen so that $\theta_i \in (-\pi, 0]$ and q is the number of columns of G. Now let $\beta_i > 0$ be such that

$$\theta_i = \arg\left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0}\right)$$

and let the input u be generated from passing $\hat{u}$ through a filter

$$u(t) = \begin{bmatrix} \alpha_1 \frac{\beta_1 - s}{\beta_1 + s} \\ \alpha_2 \frac{\beta_2 - s}{\beta_2 + s} \\ \vdots \\ \alpha_q \frac{\beta_q - s}{\beta_q + s} \end{bmatrix} \hat{u}(t)$$

where

$$\hat{u}(t) = \sqrt{2}\sin(\omega_o t).$$

Then $R_{\hat{u}}(\tau) = \cos(\omega_o \tau)$, so

$$\|\hat{u}\|_\mathcal{P} = R_{\hat{u}}(0) = 1.$$

Also

$$S_{\hat{u}}(j\omega) = \pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)].$$

Then

$$S_{uu}(j\omega) = \begin{bmatrix} \alpha_1 \frac{\beta_1 - j\omega}{\beta_1 + j\omega} \\ \alpha_2 \frac{\beta_2 - j\omega}{\beta_2 + j\omega} \\ \vdots \\ \alpha_q \frac{\beta_q - j\omega}{\beta_q + j\omega} \end{bmatrix} S_{\hat{u}}(j\omega) \begin{bmatrix} \alpha_1 \frac{\beta_1 - j\omega}{\beta_1 + j\omega} \\ \alpha_2 \frac{\beta_2 - j\omega}{\beta_2 + j\omega} \\ \vdots \\ \alpha_q \frac{\beta_q - j\omega}{\beta_q + j\omega} \end{bmatrix}^*$$

and it is easy to show

$$\|u\|_\mathcal{P} = 1$$

so from (5)

$$\begin{aligned}\|z\|_{\mathcal{P}}^2 &= \frac{1}{2}\bar{\sigma}[G(j\omega_o)]^2 + \frac{1}{2}\bar{\sigma}[G(-j\omega_o)]^2 \\ &= \bar{\sigma}[G(j\omega_o)]^2 \\ &= \|G\|_\infty^2.\end{aligned}$$

Finally if $\omega_0 = \infty$, then the above procedure can give arbitrary close norm.

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REFERENCES

[1] B. D. O. Anderson, "An algebraic solution to the spectral factorization problem," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 410-414, 1967.

[2] D. S. Bernstein and W. M. Haddad, "LQG control with an $\mathcal{H}_\infty$ performance bound: A Riccati equation approach," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 293-305, 1989.

[3] S. Boyd, V. Balakrishnan, and P. Kabamba, "A bisection method for computing the $\mathcal{H}_\infty$ norm of a transfer matrix and related problems," *Math. Contr., Sig., Syst.*, vol. 2, no. 3, pp. 207-220, 1989.

[4] J. C. Doyle, "Analysis of feedback systems with structured uncertainties," in *IEE Proc.*, vol. 129, Part D, no. 6, 1982, pp. 242-250.

[5] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, no. 8, pp. 831-847, 1989.

[6] J. C. Doyle, J. Wall, and G. Stein, "Performance and robustness analysis for structured uncertainty," in *Proc. 21st IEEE Conf. Dec. Contr.*, 1982, pp. 629-636.

[7] J. C. Doyle, K. Zhou, K. Glover, and B. Bodenheimer, "Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives II: Optimal control," *IEEE Trans. Automat. Contr.*, vol. 39, no. 8, 1994.

[8] M. K. H. Fan and A. L. Tits, "Characterization and efficient computation of the structured singular value," *IEEE Trans. Automat. Contr.*, vol. AC-31, no. 8, pp. 734-743, 1986.

[9] W. A. Gardner, *Statistical Spectral Analysis: A Nonprobabilistic Theory*, Englewood Cliffs, NJ: Prentice-Hall, 1988.

[10] P. P. Khargonekar and M. A. Rotea, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control: A convex optimization approach," *IEEE Trans. Automat. Contr.*, vol. 36, no. 7, pp. 824-837, July 1991.

[11] A. K. Packard, "What's new with μ: structured uncertainty in multivariable control," Ph.D. dissertation, University of California, Berkeley, 1988.

[12] J. C. Willems, "Least-squares stationary optimal control and the algebraic Riccati equation," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 621-634, 1971.

[13] K. Zhou, J. C. Doyle, K. Glover, and B. Bodenheimer, "Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control," Amer. Contr. Conf., 1990.

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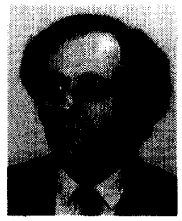

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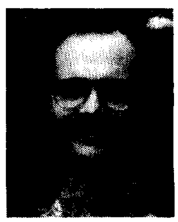

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