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Approximate Behaviors

John C. Doyle * doyle@hot.caltech.edu
Fernando Paganini † paganini@mit.edu
Raffaello D'Andrea* raff@hot.caltech.edu
Sven Khatri* skhatri@hot.caltech.edu

Abstract

The motivation for this paper is to contribute to a unified approach to modeling, realization, approximation and analysis for systems with a rich class of uncertainty structures. The specific focus is on what is the appropriate framework to model components with uncertainty, and what is the appropriate notion of approximation for such components. Components and systems are conceptualized in terms of their behaviors, which can be specified by parametrized equations. More questions are posed than are answered.

1 Introduction

In building hierarchical system models from components, it is convenient to have basic building blocks, with models of resistors, capacitors, masses, springs, ducts, valves, etc., and an abstract data type for their representation. As a simple example consider the equations $d/dt(cv) = i$ for an ideal capacitor with capacitance c. We need not belabor how inappropriate an a priori choice of inputs and outputs would be, but the form of our behavioral model could also involve some unfortunate a priori assumptions. Suppose we write the component as $c\dot{v} - i = 0$ and proceed to interconnect it with other components and begin some circuit analysis. At some point we must choose a (possibly uncertain) value for c. Unfortunately, we will be unable to investigate nonparametric uncertainty in c such as nonlinearities, time-variations or parasitics in the capacitor model, because we used the apparently harmless identity $d/dt(cv) = cd/dt(v)$. We must redo the component model without making this assumption, just as we would need to redo the model if we had chosen an inappropriate input-output partition.

While this example is trivial and would only be a minor inconvenience, it illustrates that once we begin considering uncertainty or nonlinearities in our components, this issue of an appropriate representation is not obvious. We would prefer a component model that made no unnecessary a priori assumptions, whether they be an input-output partitioning or an assumption about the way the component's uncertainty description can be refined. During analysis we may want to think of c as, in turn, a real parameter, an uncertain real parameter, a time-varying parameter, a static nonlinear function, a dynamic element with parasitics, a nonlinear function depending on external variables, and so on. At what point in the modeling process should these assumptions be fixed? What are reasonable ways to represent uncertainty in components and what is an appropriate notion of approximation at the component level?

A number of recent developments are relevant to answering these questions:

1. The emergence of the behavioral setting for system theory [19] as the natural paradigm for modeling. In this theory, a component is represented by a *relation* between variables which all play a symmetric role, with no a priori input/output (I/O) distinction. This provides a more natural modeling technique, since such I/O partitions are usually not available at the component modeling stage (for more discussion see [19, 9]). Moreover, behaviors are conceptually the adequate setting for uncertainty modeling, as discussed below and in [16].
2. The development of a multi-dimensional system theory to understand parameter-dependent and uncertain systems [11, 13, 1]. In this point of view, uncertainty is represented by a linear-fractional dependence on a set of *indeterminates*, which can be manipulated in algebraic form and must only be given system-theoretic interpretation at the analysis stage. Correspondingly, a multidimensional theory of minimality and model reduction [1, 3] has emerged, where many aspects of standard state-space realization theory are extended to this more general situation.
3. The theory of the *gap metric* [10, 17], which has identified the natural topology in which system approximation should be performed in regard to questions of stability. Interestingly, this establishes a connection between the previous areas, since it points to the system *graph* as the natural object for approximation; in other words the correct metric is obtained by deleting the I/O partition!

These advances suggest the possibility of a unified approach to modeling, realization, approximation and analysis for systems involving a rich class of uncertainty structures. A system would be conceptualized in terms of its behavior, which can be specified by parametrized equations. Such parameters would be abstract indeterminates for questions of minimality and realization, but could be given a concrete dynamical interpretation for the problems of approximation (in the gap metric sense) and analysis. In this way one would provide a generic "data type" for system modeling, for which a series of reduction, approximation and analysis routines could be developed at a very general level. The development of such approach involves formulating the appropriate theoretical questions, and investigating to what degree they can be turned into tractable computational tasks. While it fair to say that at this point we have more questions than answers, this paper is written with the objective of laying out such questions, in the hope of stimulating research in this area.

Background is given in Section 2. Section 3 contains a standard form for behavioral modeling in terms of parametrized equations, adapted from [15]. Section 4 studies equivalence and minimality of these representations, and Section 5 introduces an approximation theory inspired in the gap.

2 Preliminaries

2.1 Behaviors

The behavioral approach to system theory has been motivated by the realization that the "signal-flow" idea is not a primitive concept in dynamical modeling: components do not have intrinsic inputs and outputs, except when they are deliberately built to have them (for example, a computer). A more natural modeling philosophy is to let all variables be a priori on an equal footing. These and other ideas have led Willems [19] to propose an alternative axiomatic foundation for system theory, from which we extract the following

Definition 1 *A dynamical system* Σ *is a triple* $\Sigma = (\mathbb{T}, \mathbb{V}, \mathcal{B})$, *with* $\mathbb{T} \subset \mathbb{R}$ *the time axis,* $\mathbb{V}$ *the signal space, and* $\mathcal{B} \subset \mathbb{V}^{\mathbb{T}}$ *the*

*Caltech, M/S 116-81,Pasadena, CA 91125
†MIT, M/S 32-410, Cambridge, MA 02139

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behavior.

In Definition 1, one identifies a set of variables $v(t)$ evolving over time, and a system is defined by restricting the *universum* $\mathbb{V}^{\mathbb{T}}$ of all signals to an allowed *behavior* $\mathcal{B}$. Note that the variables v play a priori a symmetric role; for additional motivation and details on this approach, see [19, 9]. An interesting class of systems is given by the general form

$$\mathcal{B} = \{v(t) : Rv = 0\}, \tag{1}$$

where R is a differential operator (for $\mathbb{T} = \mathbb{R}$) or a difference operator (for $\mathbb{T} = \mathbb{Z}$). These arise frequently in models derived from physical laws; for example the forced mass-spring system

$$m\ddot{x} + kx - F = 0 \tag{2}$$

corresponds to $R = [m\frac{d^2}{dt^2} + kI, -I]$, $v = \mathrm{col}(x, F)$. This is an example of a *linear* system, which means that the behavior $\mathcal{B}$ is a linear subspace of $\mathbb{V}^{\mathbb{T}}$. Such systems have received most of the attention in the literature on behaviors, leading to new versions of linear system theory (see [19]) and optimal control (see e.g. [20, 6]).

2.2 The Gap Metric

A separate stream of research in recent years (see [10, 17]) has studied the weakest topology and/or metric in which systems should be approximated if stability properties must be preserved. While these papers are written in the I/O setting, they remarkably end up pointing to the system behavior as the adequate object for these problems, as we now briefly discuss.

The *gap* between two subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ of a Hilbert space $\mathcal{H}$ is defined as $d(\mathcal{S}_1, \mathcal{S}_2) = \|\mathbf{P}_{\mathcal{S}_1} - \mathbf{P}_{\mathcal{S}_2}\|$ where $\mathbf{P}_{\mathcal{S}_i}$ is the orthogonal projection onto $\mathcal{S}_i$. This defines a metric, taking values between 0 and 1, that can also be rewritten as

$$d(\mathcal{S}_1, \mathcal{S}_2) = \max(\vec{d}(\mathcal{S}_1, \mathcal{S}_2), (\vec{d}(\mathcal{S}_2, \mathcal{S}_1)) \tag{3}$$

where the *directed gap* $\vec{d}$ is defined as

$$\vec{d}(\mathcal{S}_1, \mathcal{S}_2) = \|(I - \mathbf{P}_{\mathcal{S}_1})\mathbf{P}_{\mathcal{S}_2}\| = \sup_{x \in \mathcal{S}_1, \|x\|=1} \left(\inf_{y \in \mathcal{S}_2} \|x - y\| \right). \tag{4}$$

This latter expression has a natural interpretation in terms of an "approximation game", where subspace $\mathcal{S}_1$ must choose a signal in its unit ball which is the furthest away from subspace $\mathcal{S}_2$.

The application of this concept to system theory is as follows: given two I/O systems P_1 and P_2, the $\mathcal{H}_2$ ($\mathcal{L}_2$) gap between them is defined as the subspace gap between their $\mathcal{H}_2$ ($\mathcal{L}_2$) *graphs*. We focus here on the $\mathcal{H}_2$ gap of [10], although other versions are available [17]. The graph of an I/O operator G in $\mathcal{H}_2$ is defined as $\{(y, u) \in \mathcal{H}_2 : y = Pu\}$, which is nothing more than the intersection with $\mathcal{H}_2$ of the system *behavior* $\mathcal{B} = \{v = (y, u) : y - Pu = 0\}$.

The $\mathcal{H}_2$ gap can be computed by introducing normalized coprime factorizations for the systems, with equivalent complexity to a 2 block $\mathcal{H}_\infty$ synthesis problem (see [10]). How does the gap relate to stability issues? Consider the standard plant-controller interconnection, described by the equations

$$\begin{bmatrix} I & -P \\ -C & I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = 0 \tag{5}$$

and assume that the interconnection is stable. The *generalized stability margin* is defined as

$$b_{P,C} = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \; C] \right\|_\infty^{-1}. \tag{6}$$

Note that $b_{P,C}$ can also be interpreted in terms of the behavior. If $\mathcal{B}_P$ and $\mathcal{B}_C$ are the behaviors of P and C in (5), then it can be shown that

$$b_{P,C} = \inf_{v \in \mathcal{B}_P \cap \mathcal{H}_2, \|v\|=1} \; \inf_{\bar{v} \in \mathcal{B}_C \cap \mathcal{H}_2} \|v - \bar{v}\| \tag{7}$$

which says that $\mathcal{B}_P$ is now trying to find a signal of unit norm which is *closest* to the behavior of $\mathcal{B}_C$. For this inf-inf problem there is symmetry, $b_{P,C} = b_{C,P}$. The main property of the gap theory [10] is that any plant at a distance less than β from the nominal will be stabilized by any controller stabilizing the nominal plant with a stability margin of β. Formally:

Proposition 1 *Let the interconnection of P_1 and C be stable, with margin $b_{P_1,C}$. The interconnection of P_2 and C is stable for all P_2 satisfying $d(P_1, P_2) \leq \beta$ if, and only if, $\beta < b_{P_1,C}$.*

Heuristically (for a proof see [10]), if $d(P_1, P_2) < b_{P_1,C}$ then (4) says that the furthest a unit norm signal in $\mathcal{B}_{P_2}$ can be from $\mathcal{B}_{P_1}$ is not enough to reach $\mathcal{B}_C$, which is further away due to (7). This implies that $\mathcal{B}_{P_2}$ and $\mathcal{B}_C$ intersect trivially over $\mathcal{H}_2$, a necessary condition for stability.

2.3 Uncertainty

In many situations one does not have such strong information as in (1) available: the simplest generalization is when the equations depend on some unknown parameters $\theta \in \Theta$, i.e. we have an operator R_θ in (1). While this could be considered a family of systems, it can also be thought as one system with behavior

$$\mathcal{B} = \bigcup_{\theta \in \Theta} \{v(t) : R_\theta v = 0\}, \tag{8}$$

which fits more naturally with the modeling of less structured "uncertainty" sources. These arise, and are often prevalent, in complex systems where detailed models are either unavailable or must be sacrificed for coarser, simpler descriptions, which are *relations* in signal space, for example the static "conic sector" relation

$$\mathcal{B} = \{v(t) : [\, v_1(t) \;\; v_2(t) \,] \Pi \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \leq 0 \;\; \forall\, t\} \tag{9}$$

with Π an indefinite quadratic form. Such relations do not fit naturally in a I/O point of view since they are not functions; standard robust control has handled this situation by defining a *family* of functions which "covers" the relation, in some, often not very precise sense. For example the family $f(\Delta) = a + b\Delta$, with a, b constants and $\Delta(t)$ a memoryless contractive gain, can be chosen to cover (9), in the sense that

$$\mathcal{B} = \bigcup_{\Delta} \{(f(\Delta)v_2, v_2)\}. \tag{10}$$

Note that in this case the "covering" only implies that the behavior is the union of the graphs of the I/O maps, but does not mean that the "true" system (which could be nonlinear) is a special instance of this family. In fact, the idea that the true system should be a specific element of the parametrization is an artifice of the I/O point of view, which has made the notion of "uncertainty" unnecessarily mysterious. Stability and performance questions asked for a set of I/O systems in the worst-case are more naturally viewed as questions about the behavior $\mathcal{B}$, which is *parametrized* in this way.

This viewpoint simplifies the theory conceptually, and it could be argued that such parametrizations should be avoided, focusing on "intrinsic" descriptions of the behavior such as (9). In fact, a useful generalization of (9) is the Integral Quadratic Constraint (IQC)

$$\langle \Pi v, v \rangle = \int v(\omega)\Pi(\omega)v(\omega)d\omega \leq 0 \tag{11}$$

which has been promoted by Megretski [12] as a general modeling tool for uncertainty, showing that it leads to strong theoretical properties. Note that such IQCs are "behavioral" in nature and need not be given an I/O interpretation. Many such IQCs could be superimposed to model a particular component, and in addition they can be used for signal characterization (see [14]).

In real world problems, however, it is common to encounter a combination of very structured uncertainty sources such as real parameters (8), together with the less structured IQCs. Somewhere in the middle lie unmodeled LTI dynamics, which correspond to LTI systems which have been undermodeled by a low-order approximation. A general modeling tool should provide a standard "data type" in which to represent these different descriptions of $\mathcal{B}$; subsequent theory and computational algorithms should provide a way to interconnect, simplify, model reduce, analyze and possibly design systems in this standard form.

3 A Data Type for Modeling

In this section we present a proposed parametrization of behaviors, which has the potential of covering the various types of uncertainty described in Section 2. The following exposition is based on previous work [8, 14, 15], with a slight change in emphasis.

Definition 2 *A dynamical system Σ is parametrized as an implicit uncertain system if the behavior is written in the form*

$$\mathcal{B} = \bigcup_{\Delta} \{v : G(\Delta)v = 0\} \tag{12}$$

where $G(\Delta)$ is a family of bounded equation operators.

In other words, the behavior is described by a parametrized set of equations, which directly extends the representation of parametric uncertainty in (8). The parameter Δ, however, need not vary on a finite dimensional space.

For example, we can use an LTI system as a parameter ("LTI uncertainty") and describe systems which are essentially LTI, but where an accurate model of the form $R(\frac{d}{dt})v = 0$ would be of too high an order. By approximating R by R_0 in a frequency band of interest, we can write the implicit uncertain description

$$[R_0(\frac{d}{dt}) + \Delta W(\frac{d}{dt})]v = 0 \tag{13}$$

where W is a frequency weighing function and Δ an LTI operator with $\|\Delta\|_\infty \leq 1$.

By allowing time-variation in the operator Δ, Definition 2 extends to descriptions based on quadratic constraints. We have already given an example of a memoryless operator to capture static constraints (9). An operator with memory can be used to parametrize a general IQC of the form (11), as shown in [14]: by factoring $\Pi(\omega) = P(\omega)^* P(\omega) - Q(\omega)^* Q(\omega)$, we have

$$\{v \in l_2 : \int v(\omega)' \Pi(\omega) v(\omega) d\omega \leq 0 \} = \bigcup_{\|\Delta\| \leq 1} \{v \in l_2 : (P - \Delta Q)v = 0\}.$$

Note that the use of an LTV operator as a parameter is no indication of time variation, or for that matter linearity, in the underlying system. If a nonlinear, time invariant system is described by an IQC as in (11), we would still parametrize this constraint in terms of an LTV operator, "covering" the behavior in the weak sense described in Section 2.3.

To recapitulate, by considering the set of solutions to a parametrized family of equations, we are able to represent "nominal" equations of the form (1), parametric and LTI uncertainty and the above types of quadratic constraints. In addition to representing separately the various types of uncertainties, these can be put together in the following standard way. Take for example the intersection of the constraints $(P_1 - \Delta_1 Q_1)v = 0$, $(P_2 - \Delta_2 Q_2)v = 0$. In fact, these can be written jointly in the form

$$\left(\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} - \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \right) v = 0, \tag{14}$$

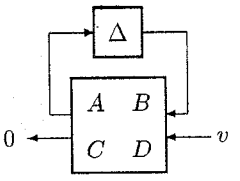


Figure 1: Implicit LFT system

which is an implicit description in terms of spatially *structured* parameter Δ. More generally, these parameters could have the block diagonal form

$$\Delta = \mathbf{diag}\,[\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}] \tag{15}$$

where the blocks $\delta_i I_{r_i}$, Δ_{L+j} are real parameters or dynamic operators. The δI blocks allow repetition in one of these operators. In each case, there is a restricted class Δ of allowed parameters.

Note that the previous superimposition maintains the form $(P - \Delta Q)v = 0$, and the complexity is transferred to the structure of Δ. More generally, one could use a linear fractional parametrization of the form

$$(\Delta \star M)v = 0, \tag{16}$$

where

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{17}$$

and the LFT operator is defined as

$$\Delta \star \mathbf{M} := D + C\Delta\,(I - A\Delta)^{-1}\,B \tag{18}$$

and corresponds to the feedback diagram of Figure 1. This would allow rational dependence of the equation coefficients on the parameter.

The difficulty with this parametrization is that one has to ensure the well posedness of such LFT (i.e. the existence of the operator inverse). A way to study this well posedness simultaneously with the parametrized behavior is to introduce *latent* variables z (as opposed to the "manifest" variables v, see [19]) as indicated in Figure 1, and write the equations

$$\begin{bmatrix} I - \Delta A & -\Delta B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} = 0. \tag{19}$$

This reduces the problem to the case of affine dependence. Correspondingly, from now on we will only study the general affine parametrization of (19). Note that any finite-dimensional dynamics present in the operators A, B, C, or D can be represented as a state space realization in the differentiation operator $\frac{d}{dt}$ (or the delay operator). With this modification, Δ is augmented to include a block $\frac{d}{dt} I$, and without loss of generality matrices A, B, C and D are constant.

We formalize this as follows: the allowable class of uncertainty, Θ, is taken to be an L+F tuple of operators, δ_1 through δ_N and Δ_{L+1} through Δ_{L+F}. Each δ_i and Δ_j is restricted to be a certain type of operator: LTV, LTI, a real parameter, or the operator $\frac{d}{dt}$. For example, a particular choice for Θ is the following:

$$\Theta := \{\theta = (\delta_1, \cdots, \delta_L) : \|\delta_i\| \leq 1\}; \tag{20}$$

in this example, Θ consists of arbitrary (equivalently LTV) contractive operators from $\mathcal{L}_2$ to $\mathcal{L}_2$. Another choice could be the δ_i being real parameters such that $|\delta_i| \leq 1$. Associated with Θ is an L-tuple of integers $\mathbf{r}$, which establishes the multiplicity of each δ_i, and is used to define the uncertainty:

$$\mathbf{r} := (r_1, \cdots, r_L), \tag{21}$$

$$\Delta(\mathbf{r}, \theta) := \mathbf{diag}[\delta_1 I_{r_1}, \cdots, \delta_L I_{r_L}, \Delta_{L+1}, \cdots, \Delta_{L+F}]. \tag{22}$$

Let matrix M (referred to as the *representation matrix*) with the following structure be given

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{23}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{r\times n}$, and $D \in \mathbb{R}^{r\times q}$, and let $\mathbf{r}$ be given. For a fixed θ, and hence $\Delta(\mathbf{r},\theta)$, denote $\mathcal{B}_\theta$ as the following set of time trajectories:

$$\mathcal{B}_\theta := \{v \in \mathbb{V} : \exists\, z \in \mathbb{Z} \text{ s.t. equation (19) is satisfied}\}. \tag{24}$$

The behavior $\mathcal{B}$ of system Σ is then defined as follows:

$$\mathcal{B} := \bigcup_{\theta\in\Theta} \mathcal{B}_\theta. \tag{25}$$

The triple $(M, \mathbf{r}, \Theta)$ is referred to as a *representation* for system Σ.

4 Equivalent Representations and Minimality

4.1 Weak and Strong Equivalence

Let the uncertainty class Θ be given. $(M, \mathbf{r}, \Theta)$ and $(\tilde{M}, \tilde{\mathbf{r}}, \Theta)$ are said to be *weakly equivalent* representations if they parametrize the same behavior:

$$\mathcal{B} = \tilde{\mathcal{B}}. \tag{26}$$

$(M, \mathbf{r}, \Theta)$ and $(\tilde{M}, \tilde{\mathbf{r}}, \Theta)$ are said to be *strongly equivalent* if

$$\mathcal{B}_\theta = \tilde{\mathcal{B}}_\theta \;\; \forall\theta \in \Theta. \tag{27}$$

It is clear that strong equivalence implies weak equivalence; the converse, however, is typically not true. To illustrate the difference between these two notions of equivalence, let Θ consist of the uncertainty structure defined in equation (20) and consider the three following system equations:

$$v_1 = \delta_1 v_2; \tag{28}$$
$$v_1 = -\delta_1 v_2; \tag{29}$$
$$v_1 = \delta_1\delta_2 v_2. \tag{30}$$

All of the above systems capture the same behavior; it thus follows that all representations for these systems will be weakly equivalent. For different values of δ_i, however, the behaviors of the above systems are all different, and are thus not strongly equivalent.

A class of strongly equivalent representations for a given $(M, \mathbf{r}, \Theta)$ is of the form $(\tilde{M}, \mathbf{r}, \Theta)$, where $\tilde{M}$ is defined as

$$\tilde{M} := \begin{bmatrix} T^{-1}(A+LC)T & T^{-1}(B+LD) \\ PCT & PD \end{bmatrix}, \tag{31}$$

where T is an invertible matrix such that $T\Delta(\mathbf{r},\theta) = \Delta(\mathbf{r},\theta)T$ for all $\theta \in \Theta$, L is any matrix of compatible dimension, and P is any invertible matrix of compatible dimension. The above does not parametrize all strongly (and thus weakly) equivalent representations; for example, there might exist a representation with fewer latent variables (n) or fewer equations (r), or both. For the special case where $\Theta = \{\frac{d}{dt}\}$, (C, A) is an observable pair, and M is full row rank, it can be shown that the above captures all equivalent representations [6], [18] (note that when Θ consists of a single operator, the notions of strong and weak equivalence are the same).

Let the uncertainty structure Θ be fixed, and let representation $(M, \mathbf{r}, \Theta)$ be given. If for all other weakly (strongly) equivalent representations $(\tilde{M}, \tilde{\mathbf{r}}, \Theta)$ the size of matrix $\tilde{M}$ is greater than or equal to the size of M, $(M, \mathbf{r}, \Theta)$ is said to be a weakly (strongly) *minimal* representation. Returning to the special case of $\Theta = \{\frac{d}{dt}\}$ (for which strong and weak equivalence are the same), it can be shown that a representation is minimal if and only if (C, A) is an observable pair and M is full row rank. Thus equation (31) parametrizes all equivalent minimal representations when $\Theta = \{\frac{d}{dt}\}$.

A natural question arises as to when a representation is minimal for a given Θ, and how to construct minimal representations. This appears to be a difficult problem; to date, only when special cases for Θ are considered and an I/O version of equivalence is adopted have computationally tractable methods for constructing minimal representations been obtained. This is discussed below.

4.2 I/O Equivalence

When the B and D matrices in representation matrix M are restricted to be of the form

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 & B_u \\ -I & D_u \end{bmatrix} \tag{32}$$

and the manifest variables are partitioned into $v = (y, u)$, the following I/O equations are obtained:

$$z = \Delta(\mathbf{r},\theta)(Az + B_u u), \tag{33}$$
$$y = Cz + D_u u. \tag{34}$$

Define

$$M_{io} := \begin{bmatrix} A & B_u \\ C & D_u \end{bmatrix}, \; \tilde{M}_{io} := \begin{bmatrix} \tilde{A} & \tilde{B}_u \\ \tilde{C} & \tilde{D}_u \end{bmatrix}, \tag{35}$$

which are obtained from the corresponding representation matrices M and $\tilde{M}$. Representations $(M, \mathbf{r}, \Theta)$ and $(\tilde{M}, \tilde{\mathbf{r}}, \Theta)$ are said to be *weakly I/O equivalent* if

$$\forall\theta \in \Theta, \exists\tilde{\theta} \in \Theta \text{ s.t. } \Delta(\mathbf{r},\theta) \star M_{io} = \Delta(\tilde{\mathbf{r}},\tilde{\theta}) \star \tilde{M}_{io},$$
$$\forall\tilde{\theta} \in \Theta, \exists\theta \in \Theta \text{ s.t. } \Delta(\mathbf{r},\theta) \star M_{io} = \Delta(\tilde{\mathbf{r}},\tilde{\theta}) \star \tilde{M}_{io}.$$

$(M, \mathbf{r}, \Theta)$ and $(\tilde{M}, \tilde{\mathbf{r}}, \Theta)$ are said to be *strongly I/O equivalent* if

$$\Delta(\mathbf{r},\theta) \star M_{io} = \Delta(\tilde{\mathbf{r}},\theta) \star \tilde{M}_{io} \;\; \forall\theta \in \Theta. \tag{36}$$

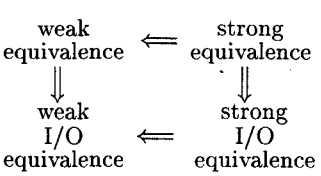
In general, I/O equivalence is a more restrictive notion of equivalence; the I/O equivalence of two representations does not imply that they parametrize the same behavior. For example, the following two sets of equations are I/O equivalent, but capture different behaviors:

$$\begin{array}{lll} \dot{z} = 0u & & \\ y = z & & y = 0u \end{array} \tag{37}$$

When the uncertainty class Θ is of the type described in equation (20) (LTV uncertainty) and the notion of strong I/O equivalence is adopted, a computationally tractable method for constructing minimal representations is presented in [1] and [7]. The conditions for equivalence and minimality are algebraic in nature, as in the 1-D case (i.e., when Θ is a single operator); the notions of controllability and observability can be introduced, and it can be shown that a representation is minimal if and only if it is both controllable and observable. Similarly, it can be shown that all minimal representations are related via similarity transformations. Other notions such as Kalman decomposition and balanced realizations can be introduced, as in the 1-D case.

4.3 Strong Equivalence and LTV Uncertainty

All the notions of equivalence are related as per the following diagram:



One can also ask the question of how equivalence is related for different types of uncertainty class Θ. It is clear that the following relation holds:

$$\begin{array}{c}\text{LTV}\\ \text{equivalence}\end{array} \Longrightarrow \begin{array}{c}\text{LTI}\\ \text{equivalence}\end{array} \Longrightarrow \begin{array}{c}\text{parametric}\\ \text{equivalence}\end{array}$$

Thus a strong case can be made for considering strongly equivalent representations for LTV uncertainty: if given a representation one substitutes for it a strongly equivalent representation for LTV Θ, it will also be equivalent irrespective of what kind of equivalence or what type of uncertainty is desired at some later stage. Thus if one does not know a-priori how the model will be used or whether the uncertainty is LTV, LTI, or parametric, the right equivalence class to consider is that induced by strongly equivalent representations and LTV uncertainty.

Another reason for considering strong equivalence and LTV Θ is that strong equivalence is much easier to check for than weak equivalence, and that working with LTV uncertainty typically leads to more tractable conditions than when working with LTI or parametric uncertainty; as of the writing of this paper, the only type of equivalence which has yielded computationally tractable methods for determining whether two realizations are equivalent and for constructing minimal representations is strong I/O equivalence for LTV Θ. Current research includes extending these techniques to the more general case of strong equivalence for LTV Θ. It is conjectured that for uncertainty other than LTV Θ or when the notion of weak equivalence is adopted, the question of equivalence and the characterization of minimality are NP-Hard in the number of elements in Θ, $L+F$.

4.4 LTV Uncertainty and Modeling

A natural question is the following: how conservative is the LTV assumption in terms of model reduction if we know a priori that the uncertainty is LTI or consists of real parameters? From a purely algebraic standpoint, LTV uncertainty is equivalent to treating the elements of Θ as non-commuting indeterminates [3]. It is a straight forward matter to construct representations which are minimal when the elements of Θ are assumed to be non-commuting but are not minimal when the elements commute (as is the case for LTI or parametric uncertainty): $G(\Delta) = \delta_1\delta_2 - \delta_2\delta_1$.

Recently several algorithms have been presented to construct low order LFT representations for uncertain state space models for systems with parametric uncertainty [5], [4] (the notion of equivalence used here is that of strong I/O equivalence). Both of these methods try to exploit the special structure of commuting uncertain parameters, where $\delta_i\delta_j = \delta_j\delta_i \forall i,j$. The claim made in this section is that *in general* the constructed realizations obtained with these methods are of order no lower than if the parameters are assumed not to commute: Non-minimal models are usually the artifice of the modeling methodology; for real systems it is extremely unlikely that parameters will enter in a manner where the property of commuting or non-commuting is an issue. Thus not only does working with commuting indeterminates not result in any substantial advantage when dealing with real systems, the added computational complexity is simply not warranted.

4.4.1 First Principles Example

We present an example where non-minimality is the result of the modeling methodology. In [5], a model of an electromagnetic suspension system is considered; an uncertain representation is constructed by applying their proposed algorithm to a state-space representation of the system equations with uncertainty in the elements of the matrices A, B, C, and D. The resulting representation is compared against one obtained by a direct representation algorithm, which typically results in artificially high uncertainty dimension, and it is shown that their proposed algorithm results in a lower-order representation. If the uncertain model were constructed from first principles, however, and not from the state-space data, the resulting uncertain model would be minimal; this is outlined below.

$$\ddot{x} = g - f/m, \tag{38}$$

$$e = Ri + L\dot{i}, \tag{39}$$

$$f = k(i/(x+x_0))^2. \tag{40}$$

Equations (38), (39), and (40) are the equations which describe the dynamics of the magnetic suspension system. Performing a taylor series expansion of f and eliminating the higher order terms yields

$$\ddot{x} = \overline{f}/m, \quad \overline{f} = \frac{2kI_0}{(x+x_0)^2}\left(\frac{e-Ri}{L} - \frac{I_0\dot{x}}{x+x_0}\right). \tag{41}$$

The suspension system is an interconnection of two subsystems, the mass and the electromagnet. Each of these models is represented by an implicit LFT system. The interconnection variables are the position of the ball and the magnetic force on the ball. The model of the mass has no uncertainty, and the uncertainty structure for the electromagnet is given by $\Delta = \mathbf{diag}[\delta_1, \delta_2 I_2, \delta_3 I_3, \delta_4, \delta_5]$ where $\delta_1 = k$, $\delta_2 = I_0$ (the nominal current), $\delta_3 = (x+x_0)^{-1}$, $\delta_4 = R$, and $\delta_5 = L^{-1}$. This yields

$$\overline{f} = 2\delta_1\delta_2\delta_3^2\left[\delta_5(e - \delta_4 i) - \delta_2\delta_3\dot{x}\right]. \tag{42}$$

When the two systems are interconnected and an input-output partition is chosen, the uncertainty structure will not change. In this case the input is naturally chosen as e and the output as x. This representation has the same uncertainty structure as that found in [5] and is in fact minimal.

For this example, if the uncertainty description is determined from the initial equations rather than the state-space terms, a minimal model is obtained.

4.4.2 Curve Fitting Example

In the following example, state space terms are curve fitted using parameters. In [4], an algorithm which assumes that the uncertainty consists of real parameters is proposed to generate low order realizations, and is applied to a phugoid motion model. Using this algorithm, a realization with an uncertainty block of $\Delta = \mathbf{diag}[\delta_1 I_6, \delta_2 I_6]$ was obtained; the original uncertainty structure was $\Delta = \mathbf{diag}[\delta_1 I_6, \delta_2 I_7]$. If the uncertainty elements are assumed not to commute, the algorithms in [1] and [7] result in uncertainty structure $\Delta = \mathbf{diag}[\delta_1 I_5, \delta_2 I_7]$, which is of the same total size; thus the commuting nature of the uncertain parameters does not affect the total order of the realization.

5 Approximations and Gaps

Having discussed issues of exact reducibility for systems in our standard form, the next question is approximate reducibility: can one define a notion of distance between parameterizations of the type (19), such that "close" systems inherit desirable properties? The answer will of course depend on the properties considered; following the model of the gap metric theory, we focus on the question of approximation from the point of view of stability.

The material reviewed in Section 2.2 suggests a path to follow. In fact, returning to the definitions for the gap metric, we see that the "approximation game" in (4) can still be posed when the subspaces $\mathcal{S}_i$ are replaced by more complicated behaviors. This leads us to the following

Definition 3 *The $\mathcal{H}_2$ gap between two systems $\Sigma_1 = (\mathbb{T}, \mathbb{V}, \mathcal{B}_1)$ and $\Sigma_2 = (\mathbb{T}, \mathbb{V}, \mathcal{B}_2)$ is defined as $d(\Sigma_1, \Sigma_2) = \max(\vec{d}(\Sigma_1, \Sigma_2), (\vec{d}(\Sigma_2, \Sigma_1))$, where*

$$\vec{d}(\Sigma_1, \Sigma_2) = \sup_{x \in \mathcal{B}_1 \cap \mathcal{H}_2, \|x\|=1} \left(\inf_{y \in \mathcal{B}_2 \cap \mathcal{H}_2} \|x - y\| \right). \tag{43}$$

Provided that the $\mathcal{B}_i$ are cones of signals (which holds for all systems which admit the parameterization (12)), it follows from Definition 3 that d is in fact a metric, with values between 0 and 1.

Analogously to (7), we could also define the stability margin of a stable interconnection between two systems Σ_1 and Σ_2 by

$$b_{\Sigma_1, \Sigma_2} = \inf_{x \in \mathcal{B}_1 \cap \mathcal{H}_2, \|x\|=1} \left(\inf_{y \in \mathcal{B}_2 \cap \mathcal{H}_2} \|x - y\| \right). \tag{44}$$

With regards to the computation of these quantities for systems Σ_i described by $(M_i, \mathbf{r}_i, \theta_i)$ as in Section 3, stability margin evaluation is in fact of the same complexity as robustness analysis: the question $b_{\Sigma_1, \Sigma_2} < \alpha$ is equivalent to testing for non-trivial solutions to the constraints

$$\theta_1 \in \Theta_1, x \in \mathcal{B}_{\theta_1} \cap \mathcal{H}_2, \tag{45}$$

$$\theta_2 \in \Theta_2, y \in \mathcal{B}_{\theta_2} \cap \mathcal{H}_2, \tag{46}$$

$$\|x - y\| \leq \alpha \|x\|, \tag{47}$$

which is equivalent to a robustness analysis question in implicit form (see [16]).

As for the gap evaluation, we can write

$$\vec{d}(\Sigma_1, \Sigma_2) = \sup_{\theta_1 \in \Theta_1} \sup_{\substack{x \in \mathcal{B}_{\theta_1} \cap \mathcal{H}_2 \\ \|x\|=1}} \inf_{\theta_2 \in \Theta_2} \inf_{y \in \mathcal{B}_{\theta_2} \cap \mathcal{H}_2} \|y - x\|, \tag{48}$$

which appears to be difficult; also, it is not clear how to exploit the parameterization in (48). By changing the order of the middle sup-inf (in general a conservative step), we obtain

$$\vec{d}(\Sigma_1, \Sigma_2) \leq \sup_{\theta_1 \in \Theta_1} \inf_{\theta_2 \in \Theta_2} \vec{d}(\mathcal{B}_{\theta_1} \cap \mathcal{H}_2, \mathcal{B}_{\theta_2} \cap \mathcal{H}_2), \tag{49}$$

where $\vec{d}(\mathcal{B}_{\theta_1} \cap \mathcal{H}_2, \mathcal{B}_{\theta_2} \cap \mathcal{H}_2)$ is the standard subspace gap. So the upper bound (49) can be interpreted as a sup-inf game over the uncertainty, superimposed to a gap-type calculation, still in principle very hard. With the objective of further simplifying the analysis, one can force $\theta_1 = \theta_2$ and provide a notion of approximation in correspondence with the notion of strong equivalence in Section 4 (Definition 3 corresponds to the notion of weak equivalence):

Definition 4 *The* strong *$\mathcal{H}_2$ gap between two representations $(M, \mathbf{r}, \theta)$ and $(\tilde{M}, \tilde{\mathbf{r}}, \theta)$ is defined by $d_s((M, \mathbf{r}, \theta), (\tilde{M}, \tilde{\mathbf{r}}, \theta)) := \max[\vec{d}_s((M, \mathbf{r}, \theta), (\tilde{M}, \tilde{\mathbf{r}}, \theta)), \vec{d}_s((\tilde{M}, \tilde{\mathbf{r}}, \theta), (M, \mathbf{r}, \theta))]$ where*

$$\vec{d}_s((M, \mathbf{r}, \theta), (\tilde{M}, \tilde{\mathbf{r}}, \theta)) := \sup_{\theta \in \Theta} \vec{d}(\mathcal{B}_\theta \cap \mathcal{H}_2, \tilde{\mathcal{B}}_\theta \cap \mathcal{H}_2). \tag{50}$$

Note that this second gap is a function of the parameterization, not just the two behaviors. This method of approximation is more difficult to interpret than Definition 3, but it provides an upper bound for it: it follows from (49) that if $(M, \mathbf{r}, \theta)$ and $(\tilde{M}, \tilde{\mathbf{r}}, \theta)$ parametrize Σ and $\tilde{\Sigma}$ respectively,

$$d_s((M, \mathbf{r}, \theta), (\tilde{M}, \tilde{\mathbf{r}}, \theta)) \geq d(\Sigma, \tilde{\Sigma}).$$

An illustrative example: let

$$\mathcal{B}_1 = \bigcup_{\theta \in [-1,1]} \{v = (y, u) : y - \theta u = 0\}, \tag{51}$$

$$\mathcal{B}_2 = \bigcup_{\theta \in [-1,1]} \{v = (y, u) : y + \theta u = 0\}. \tag{52}$$

It is clear that $\mathcal{B}_1 = \mathcal{B}_2$, so the gap of Definition 3 is 0. However, the strong gap between the representations is 1, corresponding to $\theta = 1$.

The potential payoff of this conservatism is a more tractable computation: we have now only one supremum added to the standard gap computation. This is still potentially hard, and its precise computational properties are an open question for research [2].

Assuming the quantities can be computed, is it true that gap perturbations smaller than the stability margin preserve stability? An additional difficulty here is that certain controllability restrictions in the sense of [19] are required to ensure that the $\mathcal{H}_2$ behaviors give enough information about the complete behavior. These are automatically satisfied for the I/O systems of Section 2.2, but the correct extension to the general uncertain case is an area of future research.

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