Uncertain Multivariable Systems from a State Space Perspective

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Abstract

This paper introduces some new extensions of μ analysis for LTI systems with structured uncertainty to time varying and nonlinear systems.

1. Introduction

This paper will review the μ-based methods for analyzing the performance and robustness properties of uncertain linear feedback systems and then introduce some powerful extensions of this theory to time-varying/nonlinear systems. Although many ideas and results are presented here, the new material is reasonably straightforward. This paper in no way attempts to give a complete historical background, but only reviews those results which are specifically used in this paper.

The general framework for the linear case is illustrated in the diagram in Figure 1a. Any linear interconnection of inputs, outputs, commands, perturbations, and a controller can be rearranged to match this diagram. For the purpose of analysis the controller may be thought of as just another system component and the diagram reduces to that in Figure 1b. G will be taken to be a linear, time-invariant, lumped system and be represented by a rational transfer function. The interconnection structure G can be partitioned so that the transfer function from v to e can be expressed as the linear fractional transformation

$$e = F_u(G,\Delta)\ v = \left[G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}\ G_{12}\right]\ v.$$

The external input v is an additive signal entering the system and is typically used to model disturbances, commands, and noise. Additional uncertainty from sources such as unmodeled dynamics and parameter variations is modeled as the perturbation Δ to the nominal interconnection structure G. It is conventional to absorb any scalings, weights, or coloring filters into G so that v, e, and Δ are normalized to norm 1.

This paper will use an H_∞ performance objective, which results from several different assumptions of practical and theoretical interest. In terms of Figure 1b, the performance objective is $\|G_{22}\|_\infty \leq 1$. The ideal "norm" that captures all the features of both time and frequency domain performance objectives has not yet emerged. For now, weighted H_∞ in the frequency domain seems to be a reasonable compromise. It directly handles familiar cases of bounded power and energy, as well as sinusoidal steady state response, and can approximate other important cases. The real payoff for using H_∞, though, is that it allows for the direct treatment of robust performance using μ.

Section 2 considers robust stability and performance in the frequency domain using μ. For simple unstructured perturbations, robust stability leads naturally to a $\|\bullet\|_\infty$ norm test, but now on G_{11}. The $\|\bullet\|_\infty$ norm thus provides a single norm which handles both nominal performance and robust stability. Unfortunately, norm bounds are inadequate in dealing with more realistic models of plant uncertainty with structure and more complicated mathematical objects involving the structured singular value, μ, are required. This leads to a robust stability test of the form $\|\mu(G_{22})\|_\infty \leq 1$. Obviously, it would be desirable to treat performance with both noise and perturbations occurring simultaneously. This also leads to tests using μ, but now involving the entire transfer function G. Thus μ emerges as an essential analysis tool in dealing with *robust performance* as well as with structured perturbations.

The mathematical properties and computation of μ are briefly reviewed in Section 3 for the case of complex perturbations. Here μ is viewed as a natural generalization of both spectral radius and spectral norm, and this viewpoint leads to useful characterizations of μ in terms of these more familiar quantities. One consequence is that estimates for μ can be obtained by scaling of ordinary singular values.

The extension of the μ-based methods to time-varying and nonlinear systems is outlined in Sections 4-6. Section 4 bridges the gap between the frequency domain methods of Sections 1-3 and the Lyapunov theory of Section 5 by giving a state-space interpretation of the frequency domain μ-tests. Section 5 uses a Lyapunov function approach to study the robust stability of nonlinear and time-varying systems with structured uncertainty. Section 6 summarizes the theory. Section 7 describes the application of the theory to a problem involving nonlinear control input saturation.

One cautionary note for the reader is that stability is never carefully defined for the uncertain systems described here, and there is lots of jumping around between the frequency and time domain. We hope the reader can keep track.

2. Robust Stability and Performance

In figure 1b, Δ is a member of a set of the form

$$\mathbf{\Delta} = \{\text{diag } (\delta_1 I, \delta_2 I, \ldots, \delta_m I, \Delta_1, \Delta_2, \ldots, \Delta_n) \Big| \delta_i \in \mathbf{C}, \Delta_j \in \mathbf{C}^{k_j \times k_j}\} \tag{2.1}$$

or, more specifically, its bounded subset:

$$\mathbf{B\Delta} = \left\{\Delta \in \mathbf{\Delta} \,\Big|\, \bar{\sigma}(\Delta) < 1\right\}. \tag{2.2}$$

Nonsquare perturbations can easily be handled in what follows by augmenting the interconnection structure with rows or columns of zeros.

Norm-bounded perturbations often arise when trying to capture the effect of unmodeled dynamics and are themselves dynamic systems. This would typically lead to norm-bounded frequency-varying perturbations, but for analysis, it is sufficient to instead consider constant complex matrix perturbations. The focus here is on complex perturbations because the theory is far more developed for complex perturbations than for real perturbations. Real perturbations typically arise from uncertain coefficients in differential equation models of physical systems. Fortunately, it has always been possible in the applications that have been considered to use only complex perturbations by covering the effect of real variations with complex ones. In principle, this can always be done, but in practice it may require some ingenuity. This is clearly an area requiring more work and this is being pursued vigorously in other research efforts.

The positive real-valued function μ may be defined axiomatically as satisfying the property

$$\begin{aligned} &\det (I - M\Delta) \neq 0 \text{ for } \forall\, \Delta \in \mathbf{\Delta}, \bar{\sigma}(\Delta) < \gamma \\ &\text{iff } \gamma\mu(M) \leq 1. \end{aligned} \tag{2.3}$$

Note that μ is a function of M that depends on the structure of the Δ's in $\mathbf{\Delta}$. This dependency is typically not represented explicitly except in cases where confusion would otherwise arise. If $\mu(M) \neq 0$, (i.e. $\exists \Delta \in \mathbf{\Delta}$ such that $\det(I - M\Delta) = 0$) then

$$\frac{1}{\mu(M)} = \min_{\Delta \in \mathbf{\Delta}} \left\{\bar{\sigma}(\Delta) \,\Big|\, \det(I - M\Delta) = 0\right\}. \tag{2.4}$$

Unfortunately, (2.4) is not typically very useful in computing μ since the implied optimization problem is cumbersome and can have multiple local maxima which are not global. Computation of μ is a complicated problem and some results will be reviewed in the next section. For now, assume μ is the function defined above.

With these definitions, robust stability of the system in Figure 1b is characterized by the following theorem.

Theorem RSS (Robust Stability, Structured)

Stable for all $\Delta \in \mathbf{B\Delta}$
iff $\|\mu(G_{11})\|_\infty \leq 1$

$$\text{where } \|\mu(G)\|_\infty \stackrel{\text{def}}{=} \sup_\omega \mu\left[G(j\omega)\right]. \tag{2.5}$$

This theorem provides a test for robust stability with structured uncertainty and $\|G_{22}\|_\infty \leq 1$ is a test for nominal performance. Obviously, it would be desirable to treat performance with both noise and perturbations occurring simultaneously (Doyle, 1982-1985, Doyle, Wall, and Stein, 1982). The following theorem addresses exactly this problem.

Theorem RP:

$$F_u(G, \Delta) \text{ stable and } \|F_u(G, \Delta)\|_\infty < 1 \;\; \forall \Delta \in \mathbf{B\Delta}$$

$$\text{iff } \|\mu(G)\|_\infty \leq 1$$

where μ is taken w.r.t. the structure

$$\widetilde{\mathbf{\Delta}} = \{\text{diag } (\Delta, \; \Delta_{n+1}) \,\Big|\, \Delta \in \mathbf{\Delta}\}.$$

This theorem is the real payoff for using μ. It gives necessary and sufficient conditions for robust performance in the presence of structured uncertainty. It's made possible by the equivalence of performance and robust stability when using $\|\bullet\|_\infty$. The block Δ_{n+1} may be thought of loosely as a "performance block" used to turn the performance condition into a robust stability condition and finally into a test using μ. Note that μ is computed for the full G and is taken with respect to an augmented structure.

3. μ For Complex Perturbations

In the previous section, it was shown that robust performance and stability with structured uncertainty reduces to computing μ for constant matrices $G(j\omega)$ and then taking sup over all ω. This section reviews some of the properties of μ for complex perturbations. Recall that $\mathbf{\Delta}$ in (2.1) is a subalgebra of matrices satisfying

$$\{\lambda I | \lambda \in \mathbf{C}\} \quad \subset \quad \mathbf{\Delta} \quad \subset \quad \mathbf{C}^{N \times N}. \tag{3.1}$$

Indeed, in the special cases where $\mathbf{\Delta}$ is equal to one of its possible extreme sets in (3.1), μ is exactly either the usual spectral radius or maximum singular value:

$$\begin{array}{lll} \mathbf{\Delta} = \{\lambda I | \lambda \in \mathrm{C}\} & \Rightarrow & \mu(M) = \rho(M) \\ \mathbf{\Delta} = \mathrm{C}^{N \times N} & \Rightarrow & \mu(M) = \bar{\sigma}(M) \end{array} \tag{3.2}$$

Thus for any set $\mathbf{\Delta}$ it easy to see that

$$\rho(M) \quad \leq \quad \mu(M) \quad \leq \quad \bar{\sigma}(M). \tag{3.3}$$

These bounds may be improved by using simple properties of $\mathbf{\Delta}$. Suppose that $\mathbf{\Delta}$, $\mathcal{U}$, and $\mathcal{D}$ are the sets

$$\begin{aligned} \mathbf{\Delta} &= \{\mathrm{diag}\,(\delta_1 I, \delta_2 I, \ldots, \delta_m I, \Delta_1, \Delta_2, \ldots, \Delta_n)\} \\ \mathcal{U} &= \{\mathrm{diag}\,(u_1 I, u_2 I, \ldots, u_m I, U_1, U_2, \ldots, U_n)\} \\ \mathcal{D} &= \{\mathrm{diag}\,(D_1, D_2, \ldots, D_m, d_1 I, d_2 I, \ldots, d_n I)\} \end{aligned} \tag{3.4}$$

where the $\mathcal{U} \subset \mathbf{\Delta}$ are unitary ($U^*U = I, \forall U \in \mathcal{U}$) and the $\mathcal{D}$ are invertible. The restriction $D = D^* > 0$ may be made without loss of generality. From the definition for μ it is easy to see that for any $\Delta \in \mathbf{\Delta}$

$$\begin{array}{llll} U \in \mathcal{U} & \Rightarrow \bar{\sigma}(U\Delta) = \bar{\sigma}(\Delta) & \Rightarrow \mu(MU) = \mu(M) \\ D \in \mathcal{D} & \Rightarrow D^{-1}\Delta D = \Delta & \Rightarrow \mu\left(DMD^{-1}\right) = \mu(M) \end{array}$$

so the bounds in (3.3) can be improved to

$$\sup_{U \in \mathcal{U}} \rho(MU) \leq \mu(M) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}\left(DMD^{-1}\right). \tag{3.5}$$

The key theorems about μ show when these inequalities are actually equalities:

$$\sup_{U \in \mathcal{U}} \rho(MU) = \mu(M) \tag{3.6}$$

holds for all M and $\mathbf{\Delta}$ and

$$\mu(M) = \inf_{D \in \mathcal{D}} \bar{\sigma}\left(DMD^{-1}\right) \tag{3.7}$$

if $m = 0$ and $n \leq 3$ (three or fewer nonrepeated blocks) independent of block size. There are other conditions under which this upper bound is an equality but they are more cumbersome to state.

It is desirable in practice to use both upper and lower bounds for μ, since the existing bounds nicely complement each other. The lower bounds in terms of $\rho(MU)$ have the desirable property of always achieving μ independent of the number of blocks. Unfortunately, $\rho(MU)$ can have multiple local maxima which are not global. The upper bound $\bar{\sigma}\left(DMD^{-1}\right)$ in (3.5) has only global minima, because $\bar{\sigma}\left(e^D M e^{-D}\right)$ is convex in D, but may not give μ except in special cases. By having an upper bound it is much easier to recognize when a local maxima is not global and restart the algorithm with another initial guess. Extensive computational experience has yet to reveal a μ problem with nonrepeated blocks ($m = 0$) where the bounds obtained had a ratio smaller than about .85, independent of matrix size and number of blocks. In fact, less than .99 is unusual. This is encouraging but additional theoretical work is needed to guarantee the quality of the upper bound in general. We believe that lower ratios are possible for $m \neq 0$, which is particularly relevant for the results in the remaining sections.

4. State-space theory using μ

This section presents a "state-space" version of the theory that was outlined in earlier sections. The key idea here is that the Laplace transform variable s, when appropriately transformed to the unit disk can itself be interpreted as an additional block of repeated scalars in an augmented structure, eliminating the sup over $j\omega$. The resulting state-space approach is mathematically equivalent to the original frequency-domain one and yields an obvious restatement of Theorems RSS and RP, but the results are nevertheless interesting as they bridge the gap between the frequency domain analysis of the earlier sections and the Lyapunov theory of the next section.

Suppose that

$$G(s) \; = \; D + C(sI - A)^{-1}B \; = \; F_u\left(G_f, \; \frac{1}{s}I_p\right) \tag{4.1}$$

where $G_f = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\mathbf{\Delta}$ is from (2.1) or (3.6). Note that the state equations for the robust stability problem can be written as

$$\dot{x} = F_l(G_f, \Delta)x$$

$$\text{where} \quad F_l(G_f, \Delta) = A + B\Delta(I - D\Delta)^{-1}C. \tag{4.2}$$

The following standard lemma is the matrix version of the isomorphism of the right half plane with the unit disk. Let

$$Q = \begin{bmatrix} I_p & 2I_p \\ I_p & I_p \end{bmatrix} \quad \text{and} \quad (M)_H \stackrel{\text{def}}{=} \frac{1}{2}(M + M^*).$$

Lemma 4.1 $\forall\, M \in \mathrm{C}^{n \times n}$ with $I - M$ nonsingular:

a) $(M)_H \leq 0$ iff $\bar{\sigma}\left(F_l(Q, M)\right) \leq 1$

b) $\bar{\sigma}(M) \leq 1$ iff $(F_l(Q, M))_H \geq 0$

c) $F_l(Q, M) = F_u(Q, M)$

d) $F_l(Q, XMX^{-1}) = XF_l(Q, M)X^{-1}$

Since we plan on replacing s in the rhp by z with $|z| \leq 1$ using an LFT involving Q, we need formulas for the composition of LFT's. Let $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$, $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, and

$$T = \begin{bmatrix} F_l(R, P_{11}) & R_{12}(I - P_{11}R_{22})^{-1}P_{12} \\ P_{21}(I - R_{22}P_{11})^{-1}R_{21} & F_u(P, R_{22}) \end{bmatrix}.$$

Lemma 4.2

a) $F_l(T, \Delta) = F_l(R, F_l(P, \Delta))$

b) $F_u(T, \Delta) = F_u(P, F_u(R, \Delta))$.

The combination of Lemma 4.1b and Lemma 4.2 plus a few lines of algebra yield the following theorem on state-space robust stability

Theorem SRS $\sup_\omega \mu_{\mathbf{\Delta}}(G(j\omega)) \leq 1$ iff $\mu_{\widetilde{\mathbf{\Delta}}}(T) \leq 1$

where $\widetilde{\mathbf{\Delta}} = \{\text{diag}(zI, \Delta) \mid z \in \mathbf{C},\ \Delta \in \mathbf{\Delta}\}$ and

$$T = \begin{bmatrix} F_l(Q, A) & 2(I - A)^{-1}B \\ C(I - A)^{-1} & F_u(G_f, I) \end{bmatrix}. \quad (4.3)$$

Note that this theorem cannot be used directly to compute $\sup_\omega \mu_{\mathbf{\Delta}}(G(j\omega))$ but only tell whether or not it's less than or equal to 1. For other values we need to define

$$\tilde{\mu}_{\mathbf{\Delta}}(G_f) \overset{\text{def}}{=} \inf_\alpha \left\{ \alpha \;\middle|\; \mu_{\widetilde{\mathbf{\Delta}}} \begin{bmatrix} T_{11} & \frac{1}{\alpha}T_{12} \\ T_{21} & \frac{1}{\alpha}T_{22} \end{bmatrix} \leq 1 \right\} \quad (4.4)$$

where T and $\widetilde{\mathbf{\Delta}}$ are from Theorem 4.3. Then Theorem SRS can be restated as

$$\textbf{Theorem SRS}' : \sup_\omega \mu_{\mathbf{\Delta}}(G(j\omega)) = \tilde{\mu}_{\mathbf{\Delta}}(G_f). \quad (4.5)$$

Note that the left hand side of (4.5) involves a search over ω, while the right hand side involves a search over the α in (4.4). Thus we haven't totally eliminated the need to search. Furthermore, this search over α implies that if the upper bound for $\mu_{\widetilde{\mathbf{\Delta}}}$ is used in (4.4) to obtain an upper bound for $\tilde{\mu}_{\mathbf{\Delta}}(G_f)$, the latter bound may be arbitrarily conservative. This could happen even if the upper bound for $\mu_{\widetilde{\mathbf{\Delta}}}$ is close.

5. A Lyapunov Approach Using μ

In this section we will use a Lyapunov function approach to study robustness of nonlinear and time-varying systems. For structured nonlinearities and uncertainties this reduces to conditions that are the same as the upper bound for μ from Section 3. The advantage here is that the perturbations can be nonlinear and/or time-varying.

We'll call $Y = Y^* > 0$ a Lyapunov matrix (LM) for the system $\dot{x} = Ax$ if $d/dt(x^*Yx) < 0$ for all state trajectories. Recall that Lyapunov stability is characterized by the following standard result:

Lemma 5.1 $Y = Y^* > 0$ is an LM for $\dot{x} = XAX^{-1}x$ iff $(YXAX^{-1})_H < 0$ iff $(X^*YXA)_H < 0$.

Note that the degrees of freedom in X and Y are redundant. Thus we will typically take $Y = I$ and focus on the choice of X. As in the previous section, consider the uncertain differential equation

$$\dot{x} = F_l(G_f, \Delta)x \qquad \Delta \in \mathbf{B\Delta} \quad (5.1)$$

with $\mathbf{\Delta}$ as before. In the next theorem we'll use Lemma 1 with the LM $Y = I$ and replace A by $F_l(G_f, \Delta)$. A combination of Lemma 4.1, Lemma 5.1, and a little algebra yields

Theorem 5.2 $(XF_l(G_f, \Delta)X^{-1})_H < 0 \quad \forall \Delta \in \mathbf{B\Delta}$

$$\text{iff} \quad \mu_{\widehat{\mathbf{\Delta}}} \left(\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} T \begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix} \right) < 1 \quad (5.2)$$

where $\widehat{\mathbf{\Delta}} = \left\{\text{diag}(\Delta_1, \Delta_2) \mid \Delta_1 \in \mathbf{C}^{n\times n}, \Delta_2 \in \mathbf{\Delta}\right\}$ and T is from (4.3).

Since the choice of X exhausts the degrees of freedom in selecting an LM we can restate Theorem 5.2 as

Corollary 5.3 $\exists$ an LM for (5.1) iff (5.2) holds for some X.

As noted before we may take the LM to be the identity matrix without loss of generality. Naturally, we can inf over all X in (5.2), and combine this with the upper bound from (3.5). Note that this yields exactly the upper bound as for $\mu_{\widetilde{\mathbf{\Delta}}}$ in Theorem SSRS. The advantage in (5.2) is that Δ can be an arbitrary time-varying function in addition to its usual interpretation in the earlier sections. This lets us deal with nonlinear systems as well since Δ can depend in a nonlinear manner on any system quantities. The potential disadvantage is that (5.2) is only a *sufficient* condition for robust stability. If (5.2) is not satisfied, then the system may still be stable, but no fixed LM can be found to prove stability.

The approach in this section is similar in spirit to other Lyapunov function methods, particularly those of Toda and Patel and Yedavelli (see these proceedings for recent work and references). The methods here seem to be substantially less conservative, as has been verified on their published examples. More recent work by Yedavelli probably will reduce this gap somewhat. We haven't included these examples because of space and because they would be an unfair comparison of current work by us with earlier results by others.

6. Summary

Suppose that T is from (4.3), $\mathbf{\Delta}$ has structure such as (3.4), $\widetilde{\mathbf{\Delta}} = \{\text{diag}(zI, \Delta) \mid z \in \mathbf{C},\ \Delta \in \mathbf{\Delta}\}$, and $\widehat{\mathbf{\Delta}} =$

$\left\{\text{diag}(\Delta_1, \Delta_2) \mid \Delta_1 \in \mathbf{C}^{n\times n}, \Delta_2 \in \mathbf{\Delta}\right\}$. Suppose that $\mathcal{D}$, $\tilde{\mathcal{D}}$, $\hat{\mathcal{D}}$ are the sets of commuting D matrices for $\mathbf{\Delta}$, $\widetilde{\mathbf{\Delta}}$, $\widehat{\mathbf{\Delta}}$, respectively, as in (3.4).

Necessary and sufficient conditions for robust stability of the system in figure 1b (and equation (5.1)) for Δ constant or frequency varying are

$$\mu_{\tilde{\mathbf{\Delta}}}(T) \leq 1 \qquad (6.1a)$$

$$\Leftrightarrow \sup_{\omega} \mu_{\mathbf{\Delta}}\left(G(j\omega)\right) \leq 1 \qquad (6.1b)$$

$$\Leftrightarrow \sup_{\Delta \in \mathbf{B\Delta}} \rho\left(F_l\left(G_f,\, \Delta\right)\right) \leq 1 \qquad (6.2)$$

where $G(s) = F_u\left(G_f,\, \frac{1}{s}I_p\right)$. The test in (6.1) is a frequency domain test while (6.2) is a state-space test.

If Δ is allowed to be a structured, but arbitrarily time-varying complex matrix, then a necessary and sufficient condition for the existence of an LM, hence a ***sufficient*** condition for robust stability from (5.2) is:

$$\inf_X \mu_{\hat{\mathbf{\Delta}}}\left(\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} T \begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix}\right) < 1 \qquad (6.3)$$

Note however that a necessary condition is still $\mu_{\tilde{\mathbf{\Delta}}} \leq 1$ which shares the same upper bound with (6.3):

$$\begin{aligned} \mu_{\tilde{\mathbf{\Delta}}}(T) &\leq \inf_X \mu_{\hat{\mathbf{\Delta}}}\left(\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} T \begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix}\right) \\ &\leq \inf_{D \in \tilde{\mathcal{D}}} \bar{\sigma}\left(DTD^{-1}\right). \end{aligned} \qquad (6.4)$$

The results apply immediately to some nonlinear systems as well because Δ can be taken as a structured cone bounded nonlinearity. Furthermore, many other nonlinear systems fit in this framework since the nonlinearities can often be treated as gains which vary as functions of time and other system quantities.

To combine time and frequency varying perturbations and complete the connection between the two domains note that

$$\inf_{D \in \tilde{\mathcal{D}}} \bar{\sigma}\left(DTD^{-1}\right) \leq 1 \qquad (6.5a)$$

$$\Rightarrow \inf_{D \in \mathcal{D}} \sup_{\omega} \bar{\sigma}\left(DG(j\omega)D^{-1}\right) \leq 1 \qquad (6.5b)$$

and

$$\sup_{\omega} \mu_{\mathbf{\Delta}}(G(j\omega)) \leq \sup_{s=j\omega} \inf_{D \in \mathcal{D}} \bar{\sigma}\left(DG(j\omega)D^{-1}\right) \qquad (6.6)$$

$$\leq \inf_{D \in \mathcal{D}} \sup_{s=j\omega} \bar{\sigma}\left(DG(j\omega)D^{-1}\right). \qquad (6.7)$$

Because the D in the rhs of (6.5) is constant, it commutes with both time and frequency varying Δ. Thus the rhs of (6.5) is a sufficient condition for robust stability for Δ consisting of frequency varying and time varying as well as conic-bounded nonlinear blocks, in any combinations.

In summary, (6.1) is a necessary and sufficient condition for robust stability (and robust performance using Theorem RP) for frequency varying Δ, and (6.5a) is a sufficient condition for robust stability when Δ is time-varying and nonlinear as well. For the latter case there is currently neither a comparable necessary condition nor an interpretation for robust performance. Roughly speaking, all other conditions are between (6.1) and (6.5a). Additional conditions involving a mixture of frequency varying and constant D could be found between (6.6) and (6.7). Constant D are required for nonlinear or time varying Δ.

7. Example

In this section, we will use the results in this paper to analyze the nonlinear modified antiwindup (MAW) scheme used on the second MIMO example in [1]. Since [1] is in this proceedings right next to this paper, we'll save space and simply refer directly to the description of the example there.

The key feature of MAW is the variable gain α which multiplies both feedback channels around V^{-1} in Figure 8 in [1]. It is intended to keep c from saturating and V^{-1} from winding up (see Figures 7c, 7e, and 9c in [1]). In fact, this implementation of MAW cannot a priori guarantee no saturation but alternative implementations which are harder to describe can. For simplicity, we'll assume that the nonlinearity α as implemented does prevent saturation.

What must first be proven is that the MAW scheme is stable for large signals that push the system out of its linear range. Refering to Figure 8 in [1], the α loop must be extracted so the system looks like Figure 1b with $\Delta = \delta I$. We will treat α as a time-varying gain with $0 \leq \alpha \leq 10$. This requires the use of an additional LFT to map δ from the unit disc in $\mathbf{C}$ to a disc covering α. By forming T as in (4.3) and verifying that (6.5a) holds we obtain a sufficient condition for stability for arbitrarily time varying α, as desired. For this problem, $\widetilde{\mathbf{\Delta}} = \{\text{diag}(zI_6, \delta I_2)\}$.

We also would like to verify that MAW is not only stable but also provides a graceful degradation in robustness. The linear system has poor robustness to anything but δI input uncertainty, as can be easily verified. The assumption in [1] was that the linear controller was "ideal" when there was no saturation. This implies that the input uncertainty for this system must be negligible or be of this special form. To make this interesting we'll evaluate the robustness of this system to output uncertainty for which the unsaturated system is reasonably robust.

Referring to Figure 8 in [1], suppose that a full block of uncertainty $w\Delta$ with $\bar{\sigma}(\Delta) < 1$ is connected from d to

e. The smallest w, say w_0, that produces instability is a measure of multivariable stability margins at the output. For no saturation, the test for stability is a special case of (3.5) and (6.1) because Δ is unstructured. Since $e/d = S = (I+L)^{-1}$, the necessary and sufficient condition for stability is $w\|S\|_\infty \le 1$. Thus $w_0 = 1/\|S\|_\infty$ and for this example $w_0 = .72$. Since S is diagonal, w_0 does not happen to depend on the structure of Δ.

With saturation and MAW, $w_0 = .69$. To compute this, add the $w\Delta$ block to the above stability test for MAW, and find the largest w that still passes the test in (6.5a). This gives a lower bound for w_0. To get an upper bound for w_0 find the smallest w for which the test in (6.1a) fails, using a lower bound for μ. For this example, both upper and lower bounds were equal to 2 significant digits. For this problem, where T is 10×10, a test of (6.5a) and (6.1a) together takes approximately 5 minutes on a SUN 3/50 Workstation using very experimental software (lots of insecticide). This can vary greatly in both directions depending on the system, the block structure, and the desired accuracy of the answer. The existing software slows down dramatically on problems that have repeated block structures, such as this one. This is probably an artifice of the methods being used and not inherent in the problem.

The application of the results in this paper to the example in [1] shows that, for this example, MAW preserves not only stability, but also the robustness to uncertainty at the output. If the system were LTI, this would also have a robust performance interpretation as $1/w_0$ is the worst-case "sensitivity" $\|S\|_\infty$. Because the system is nonlinear, such an interpretation is not entirely correct, but it is consistent with the simulation results.

While this application is fairly exciting, it must be emphasized that they apply only to this specific example. We have not proven that MAW works in general. In fact, seemingly innocuous changes to R in the example here can foil MAW as implemented. The authors of [1] mention the limitations of their work, but perhaps do not emphasize sufficiently that this example was carefully selected so that CAW would fail and MAW would not.

It is also important to remember that while we were able to show that this nonlinear system is robustly stable, there are many other examples which are also stable, but would not be treated in any obvious way by these methods. It is obvious that the class of all uncertain nonlinear systems is simply too large to be treated systematically by any methodology. It is encouraging that the powerful μ analysis methods for uncertain linear systems do extend to some nonlinear and time varying systems. It remains to be seen how large a class this is. We expect that with some ingenuity, the methods in this paper can be applied to many practical problems as well as serving as a starting point for research in a robust nonlinear control theory. We be jammin'.

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References

[1] J.C. Doyle, R.S. Smith, and D.F. Enns, "Control of Plants with Input Saturation Nonlinearities," 1987 ACC, this proceedings, somewhere.

You're probably wondering where all the references are. You immediately looked here before even finding out what the paper was about. Sorry, but we didn't get to it. Plus there wasn't space to put even half of the references we'd need. Just so you won't get too upset, we'll include some psuedo-references:

[2] M.K. Fan and A. Tits, hot software.

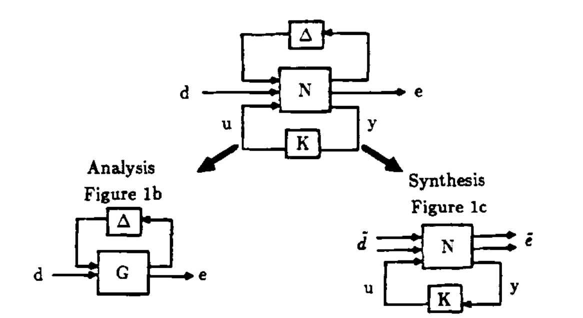
[3] M.G. Safonov, lots of nonlinear stability theory.

[4] A. Tannenbaum, he's our friend and he likes attention.

[5] R.K. Yedevalli, some Lyapunov methods similar to Section 5.

[6] G. Zames, because *everything* is really just small gain.

Figure 1a General Framework



$F_u(G,\Delta) =$

$G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}G_{12}$

$e = F_u(G,\Delta)d$

$F_l(N,K) =$

$N_{11} + N_{12}K(I - N_{22}K)^{-1}N_{21}$

$\tilde{e} = F_l(N,K)\tilde{d}$