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A PLASMA CAUSED BY THE RELATIVISTIC MASS EFFECT*

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ABSTRACT

A plasma excited by two short pulses at the electron gyrofrequency which have a time separation τ, is considered in the single particle approach. It is shown that the relativistic mass effect can lead to a series of radiation maxima after the second pulse. In the case of a cold plasma in an inhomogeneous magnetic field these maxima arise at multiples of the time τ; in the case of a hot plasma in a homogeneous magnetic field at multiples of $\tau/|1 \pm D|$, where D is the strength of the second pulse relative to the first one. The shape of the radiation maxima is given by the square of the Fourier transform of the distribution of the inhomogeneities or the initial energies, respectively. The two effects have the tendency to cancel each other. If the plasma is excited by three pulses, the time separation of the second and third pulse being T, radiation maxima occur at times $t = K\tau + LT$, ($\pm K, L = 0,1,2\ldots$ but $t > 0$) after the third pulse in the case of a cold plasma with field inhomogeneities, and at $t = (K\tau + LT)/|1 \pm D \pm D_2|$ in the case of a hot plasma. If collisions are taken into account the dependence on T of the radiation maxima with $L = 0$ is determined by inelastic collisions only, while the other decay times are determined by all kinds of collisions.

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I. INTRODUCTION

Recently the observation of echoes radiated from a plasma was reported[1]. One type of experiment was to excite the plasma at the electron gyrofrequency by two short pulses with a time separation τ. The echo radiation was then observed at a time τ after the second pulse. This effect is related to the well known spin echo[2].

Any theory of echo-like phenomena has to be nonlinear. The aim of this paper is to study a special nonlinearity caused by the relativistic mass effect, which can lead to such radiation maxima after the second (third ...) pulse. There is a general relation to the spin echo[3], but the results obtained here show also essential differences with it.

We consider a magneto plasma, which is so dilute that the single particle approach is valid. For simplicity the plasma dimensions are assumed to be small compared to the wavelength of the cyclotron radiation. The radiation by the plasma at the cyclotron frequency essentially depends on the relative phase of the gyrating particles. The energy radiated per second into the solid angle $d\Theta$ is in the nonrelativistic ($v \ll c$) case[4]:

$$dI_c = d\Theta \frac{e^2 \overline{\omega}_c^2}{c^3\, 8\pi} (2 - \sin^{-2}\theta) \left| \sum_{\ell=1}^{N} v_\ell \exp\left[i(\omega_c^\ell t + \alpha_\ell)\right] \right|^2 \qquad (1)$$

where N is the number of particles considered, v_ℓ is the magnitude of the velocity of the ℓth particle perpendicular to the magnetic field $\underset{\sim}{H}$, ω_c^ℓ is its gyrofrequency and α_ℓ its phase at $t = 0$. $\overline{\omega}_c$ is the average gyrofrequency and θ is the angle between the direction of observation and $\underset{\sim}{H}$. If the dimensions of the plasma are not small compared to the wavelength, retardation effects have to be taken into account.

In the case where the phases of the particles are randomly distributed (incoherent radiation), the last factor in (1) reduces to

$\sum v_\ell^2 = \overline{v_\ell^2}\, N$. If all particles have the same phase (complete coherence) the value is $(\sum v_\ell)^2 = \overline{v_\ell^2}\, N^2$, i.e., the radiation is increased by about a factor N compared to the incoherent case. In order to measure the degree of coherence, we introduce the function

$$\Phi(t) = \left(\sum_{\ell=1}^{N} v_\ell\right)^{-2} \left| \sum_{\ell=1}^{N} v_\ell \exp\left[i(\omega_c^\ell t + \alpha_\ell)\right] \right|^2 \qquad (2)$$

which is unity in the case of complete coherence and $1/N$ if the phases are randomly distributed. If one neglects the statistical fluctuations in the phase distribution in order to substitute the sums in (2) by integrals, one obtains $\Phi = 0$ in the case of equally distributed phases.

In order to have an effect like the echo[1], the quantity Φ must depend on time and have a sharp maximum at the time the radiation peak is to occur. This means that the phase correlations between the particles have to be time dependent. In the approximation used here this requires the introduction of individual gyrofrequencies for the different particles. One way for this to occur is through inhomogeneities of the magnetic field. An additional possibility is the relativistic mass effect which causes the gyrofrequency to become energy dependent. This effect also provides the necessary nonlinearity in the equations.

The assumed validity of the single particle approach implicitly includes the assumption that the total energy contained in the radiation peaks emitted by the plasma is small compared to the total kinetic energy of the plasma.

II. THE MODEL

At first we consider the acceleration of the electrons by the pulses. For this we treat the nonrelativistic ($v \ll c$) motion of an electron in a homogeneous magnetic field under the influence of a plane electric wave. We neglect the magnetic field of the pulses as well as the spatial variation of their electric field. As we are interested in the gyration of the electrons, we assume the $\underset{\sim}{E}$-vector to be perpendicular to the static $\underset{\sim}{H}$, and we choose our coordinate system so that the z-axis is parallel to $\underset{\sim}{H}$ and the x-axis parallel to $\underset{\sim}{E}$. The equation of motion for an electron in this approximation is

$$\dot{\underset{\sim}{v}} = -\frac{e}{m}\left[\underset{\sim}{E}^{o}\sin(\omega t + \psi) + \frac{1}{c}\,\underset{\sim}{v} \times \underset{\sim}{H}\right] \tag{3}$$

ψ being the phase of the electric field at $t = 0$. Separation of the equations leads to

$$\ddot{v}_x = -\omega\frac{e}{m}E^{o}\cos(\omega t + \psi) - \omega_c^2 v_x \tag{4a}$$

$$\ddot{v}_y = -\omega_c\frac{e}{m}E^{o}\sin(\omega t + \psi) - \omega_c^2 v_y \tag{4b}$$

ω_c being the electron gyrofrequency. In the case of $E^{o} = 0$, i.e., before and after the pulses, the solution of (4) is

$$v_x = v_o\cos(\omega_c t + \alpha_o) \tag{5a}$$

$$v_y = v_o\sin(\omega_c t + \alpha_o) \tag{5b}$$

where v_o is the component of the velocity of the electron under consideration perpendicular to $\underset{\sim}{H}$, and α_o is its phase at $t = 0$.

When the frequency of the exciting field is exactly the gyrofrequency of the electron under consideration ($\omega = \omega_c$), the general solution of (4) is:

$$v_x = -(pt + b)\sin(\omega_c t + \psi) + a\cos(\omega_c t + \psi) + v_o\cos(\omega_c t + \alpha_o) \tag{6a}$$

$$v_y = (pt + b)\cos(\omega_c t + \psi) + d\sin(\omega_c t + \psi) + v_o \sin(\omega_c t + \chi_o) \tag{6b}$$

with

$$p = \frac{e}{2m} E^o \quad ; \qquad a = \frac{p}{\omega_c} \frac{tg^2\psi}{1 + tg^2\psi} \tag{7a,b}$$

$$b = \frac{p}{\omega_c} \frac{tg\,\psi}{1 + tg^2\psi} \quad ; \qquad d = -\frac{p}{\omega_c} \frac{1}{1 + tg^2\psi} \tag{7c,d}$$

The motion of the electron as given by (6) consists of two parts: that induced by the external electric field and that associated with the initial condition. As an initial condition we required for $t = 0$ that $\underset{\sim}{v}$ have the value given by (5). So, if we assume that for $t < 0$ there is no electric field, while at $t = 0$ a pulse starts, the motion of the electron is given by (5) for $t < 0$ and by (6) for $t \geq 0$. If we further assume that the pulse starts at $t = 0$ with the phase $\psi = 0$, i.e., with $E(0) = 0$, the solution (6) reduces to:

$$v_x = -pt\sin\omega_c t + v_o\cos(\omega_c t + \chi_o) \tag{8a}$$

$$v_y = pt\cos\omega_c t - \frac{p}{\omega_c}\sin\omega_c t + v_o\sin(\omega_c t + \chi_o) \tag{8b}$$

In the case $\omega \neq \omega_c$ the general solution of (4) is

$$v_x = a_1\cos(\omega t + \psi) + a_2\sin\omega_c t + a_3\cos\omega_c t + v_o\cos(\omega_c t + \alpha_o) \tag{9a}$$

$$v_y = a_4\sin(\omega t + \psi) + a_3\sin\omega_c t - a_2\cos\omega_c t + v_o\sin(\omega_c t + \alpha_o) \tag{9b}$$

with

$$a_1 = \frac{\omega}{\omega^2 - \omega_c^2} \frac{e}{m} E^0 \quad ; \quad a_2 = \frac{\omega_c}{\omega} a_1 \sin \psi \tag{10a,b}$$

$$a_3 = - a_1 \cos \psi \quad ; \quad a_4 = \frac{\omega_c}{\omega} a_1 \tag{10c,d}$$

If we again assume that $\psi = 0$ and write $\omega = \omega_c + \Delta\omega$, we obtain from (9)

$$v_x = a_1 \cos \omega_c t \left[-1 + \cos \Delta\omega t \right] - a_1 \sin \omega_c t \sin \Delta\omega t$$
$$+ v_o \cos(\omega_c t + \alpha_o) \tag{11a}$$

$$v_y = a_1 \sin \omega_c t \left[-1 + \frac{\omega_c}{\omega} \cos \Delta\omega t \right] + \frac{\omega_c}{\omega} a_1 \cos \omega_c t \sin \Delta\omega t$$
$$+ v_o \sin(\omega_o t + \alpha_o) \tag{11b}$$

If we further assume that the pulse acting on the electron is short and its frequency near the cyclotron frequency, i.e., that

$$\Delta \omega t_1 \ll 1 \quad \text{and} \quad \frac{\Delta\omega}{\omega_c} \ll 1 \tag{12a,b}$$

where t_1 is the duration of the pulse, we can expand the trigonometric functions with the argument $\Delta\omega t$. Keeping only terms linear in $\Delta\omega t$ and neglecting terms of the order $\Delta\omega/\omega_c$ compared to unity, the equations (11) reduce to the solution given by (8).

In the following we shall always make the assumptions (12), i.e., assume that the pulses are so short that the differences in the gyrofrequency of the different particles are not essential for the acceleration. By using equations (8) we neglect all nonlinear effects during the pulses.

We now consider an ensemble of electrons, i.e., an electron plasma (without interactions) excited at the gyrofrequency by two short pulses.

The quantity of interest is the radiation after the second pulse, which is characterized by $\Phi(t)$ given by (2).

In order to have differences in the gyrofrequencies after the first pulse, we consider the influence of an initial temperature and account for field inhomogeneities by attributing a different gyrofrequency to each electron. (This implies that the inhomogeneities are perpendicular to the field lines). We assume a distribution $h(\eta)$ over the different gyrofrequencies, where $\eta = \Delta_{inh}\omega_c$ is the deviation from the average gyrofrequency due to the inhomogeneities

At first we assume that all electrons have the same initial (transverse) energy with the corresponding (transverse) velocity v_o, but different, equally distributed phases. Then we have, in the two-dimensional v-space, the distribution given in Fig. 1a. As it follows from (2) that only phase differences are essential, it is convenient to consider this diagram in a velocity space system, rotating with an average gyrofrequency ω_c (defined by equations (17) and (24)). Then φ is the phase difference with respect to a specified particle.

The velocities v'_x and v'_y in this rotating system are obtained from the velocities in the nonrotating system by the transformation

$$v'_x = - v_x \sin \overline{\omega}_c t + v_y \cos \overline{\omega}_c t \tag{13a}$$

$$v'_y = - v_x \cos \overline{\omega}_c t - v_y \sin \overline{\omega}_c t \tag{13b}$$

If we apply this transformation to equation (8) we obtain:

$$v'_x = pt + v_o \sin \alpha_o - \frac{p}{\omega_c} \sin \omega_c t \cos \omega_c t \tag{14a}$$

$$v'_y = - v_o \cos \alpha_o + \frac{p}{\omega_c} \sin^2 \omega_c t \tag{14b}$$

If we assume v_o, $pt \gg p/\omega_c$ or $\sin \omega_c t_1 = 0$ (i.e., the pulse consisting of an integer number of cycles), we can neglect in (14) the terms with p/ω_c. We then have at the end of the first pulse the distribution given in Fig. 1b with

$$V = pt_1 = \frac{e}{2m} E^o t_1 \tag{15}$$

The electrons are now equally distributed on the small dashed circle. The velocities of the electrons are now between $V - v_o$ and $V + v_o$, while the phase differences are smaller or equal to 2 artg v_o/V. For the treatment which follows we further assume that

$$v_o \ll V \tag{16}$$

i.e. that the energy the electrons gained during the first pulse is large compared to the initial (thermal) energy. Then all particles in Fig. 1b have almost the same phase and we approximate the distribution on the dashed circle by a uniform distribution on its solidly drawn diameter or, as we later allow for different initial energies, by a distribution $g(v)$ on this line. This means, the only effect of the initial energy we keep is that the particles have different energies after the first pulse according to their phase at the onset of the pulse.

As time proceeds, phase differences arise between the particles according to the differences in their gyrofrequencies. In our model the gyrofrequency depends on the energy and on the local magnetic field, and is given by[4]

$$\omega_c^{\ell} = \frac{ecH}{E^{\ell}(v_{\ell})} \tag{17}$$

where E^{ℓ} is the relativistic energy. As we are interested in the phase differences, we ask for the differences in the gyrofrequency. With the assumption $v_\ell^2 << c^2$ we have in first order

$$\Delta \omega_c^\ell = A(1 - v_\ell^2/V^2) + \eta_\ell \tag{18}$$

with

$$A = \overline{\omega}_c V^2/2c^2 \tag{19}$$

when we attribute $\Delta\omega_c^\ell = 0$ to particles with $v_\ell = V$.

Correspondingly the relative phase at time τ after the first pulse is

$$\varphi_\ell(\tau) = A\tau(1 - v_\ell^2/V^2) + \eta_\ell\tau \tag{20}$$

Fig. 1c gives the distribution in the v'-space at the end of the second pulse, which follows the first pulse after a time τ. The electrons are now distributed in the hatched ring. The quantities v_ℓ^* and φ^*_ℓ of a particle are determined by the corresponding quantities v_ℓ and $\varphi_\ell(\tau)$ at the onset of the second pulse and we have

$$v_\ell^{*}/V^2 = v_\ell^2/V^2 + D^2 + 2(v_\ell/V)\, D \cos\left[\varphi(\tau) - \varphi_o\right] \tag{21}$$

$$\varphi_\ell^*(t=0) = \varphi_o + \beta_\ell \tag{22}$$

$$\sin \beta_\ell = (v_\ell/v_\ell^*) \sin\left[\varphi(\tau) - \varphi_o\right] \tag{23a}$$

$$\cos \beta_\ell = \frac{v_{\ell'}^2 V^2 + D^2 - v_\ell^2/V^2}{2D\, v_{\ell'}^* V} \tag{23b}$$

$$\Delta^* \omega_c^\ell = A(B - v_\ell^{*2}/V^2) + \eta_\ell \tag{24}$$

$$\varphi_\ell^*(t) = \varphi_\ell^*(t=0) + \Delta^* \omega_c^\ell t \tag{25}$$

φ_o gives the phase of the electric field of the second pulse relative to a particle with $\varphi(\tau) = 0$. In any actual experiment this is a statistical quantity. B is an arbitrary constant which defines the particle with respect to which $\Delta^* \omega_c$ is measured, D gives the strength of the second pulse relative to the first one ($\mu_2 t_2 = DV$), and t is now the time measured from the end of the second pulse.

Having determined the velocity and the relative phase of the particles for any time after the second pulse, we now can calculate $\Phi(t)$ according to (2) and study its time dependence. For this we introduce the distribution functions $g(v)$ and $h(\eta)$ and substitute the sum in (2) by an integral over v and η . Then we find

$$\Phi(t) = \left| \iint dv\, d\eta\, g(v)\, h(\eta)\, v^* \right|^{-2} \cdot$$
$$\cdot \left| V \cdot \iint dv\, d\eta\, g(v)\, h(\eta) \left\{ D \exp(if) + (v/V) \exp(if_1) \right\} \right|^2 \tag{26}$$

with

$$f = \varphi_o + \Delta^* \omega_c t \tag{27}$$

$$f_1 = \varphi(\tau) - \varphi_o + f \quad . \tag{28}$$

III. RESULTS

We now calculate $\Phi(t)$ from (26) using several approximations. The normalizing factor in (26) can be given approximately by

$$\left| \iint dv\, d\eta\, g(v)\, g(\eta)\, v^* \right|^2 \approx V^2(1 + D^2) \tag{29}$$

From equations (21), (24) and (27) we find

$$f = \varphi_o + \Lambda t \left\{ B - v^2/V^2 - D^2 - 2(v/V)\, D \cos \left[\varphi(\tau) - \varphi_o \right] \right\} + \eta t \tag{30}$$

By virtue of the relation

$$\exp(-iz\cos\Theta) = \sum_{\ell=-\infty}^{+\infty} J_\ell(z) \exp\left[i\ell(\Theta-\pi/2)\right] \tag{31}$$

J_ℓ being the Bessel function of order ℓ, we find

$$\exp(if) = \sum_{\ell=-\infty}^{+\infty} J_\ell(2At\,Dv/V) \exp\left\{ i\left[(t+\ell\tau)\left[\eta + A(1-v^2/V^2)\right]\right.\right.$$
$$\left.\left. - (\ell-1)\varphi_0 - \ell\pi/2 + At(B-D^2-1)\right]\right\} \tag{32}$$

Before discussing the general case, we consider two special cases:

a) An initially cold plasma in an inhomogeneous magnetic field.

In this limit

$$g(v) = \delta(v - V) \tag{33}$$

Then we have only to perform the integration over η. We see from (32) that at a time

$$t_\ell = -\ell\tau \tag{34}$$

the ℓ-th term in the sum (32) becomes independent of η. If we integrate for a time t_ℓ the first term in (26), the ℓ-th term of (32) gives the essential contribution. Or more precisely: If we perform the integral and consider it as a function of t, then the ℓ-th term of (32) gives a contribution which is the Fourier transform of $h(\eta)$ with its maximum at t_ℓ. If the width of this maximum is small compared to the separation from the next maximum which is due to the next term in (32), then the maximum at t_ℓ is essentially determined only by the ℓ-th term. In this case $\Phi(t)$ can easily be calculated. It shows maxima at times given by (34) (with the condition $t > 0$), i.e., we have a series of radiation maxima at times after the second pulse that are multiples of τ.[5]

The second term in (26) has essentially the same structure as the first. As we have, according to (28)

$$f_1 = \eta\tau + A\tau(1 - v^2/V^2) - \varphi_o + f \tag{35}$$

we see that at a time t where the ℓ-th term of (32) gives the main contribution to the integral, the $(\ell - 1)$-th term of the corresponding expansion of $\exp(if_1)$ contributes. These two contributions have a phase difference of $\pi/2$. So we find for the maxima

$$\Phi(t_\ell) = (1 + D^2)^{-1} \left\{ \left| DJ_\ell(2ADt_\ell) \right|^2 + \left| J_{\ell-1}(2ADt_\ell) \right|^2 \right\} \tag{36}$$

If the argument of the Bessel functions is small, we may use the approximation[6]

$$J_\ell(z) = \frac{z^\ell}{2^\ell \, \ell!} \quad ; \qquad \ell \geq 0 \quad \text{integer} \tag{37}$$

and we obtain

$$\Phi(t_\ell) = \frac{1}{1 + D^2} \left[\frac{D^2 (ADt_\ell)^{2|\ell|}}{(|\ell|!)^2} + \frac{(ADt_\ell)^{2|\ell-1|}}{(|\ell - 1|!)^2} \right] \tag{38}$$

In this approximation the amplitude of the radiation maxima grows with the pulse strength and grows with increasing τ.

If, on the other hand, the argument of the Bessel function is large, we may use the approximation[6]

$$J_\ell(z) = \sqrt{\frac{2}{\pi z}} \cos(z - (\ell/2 + 1/4)\,\pi) \tag{39}$$

Then we obtain

$$\Phi(t_\ell) = \frac{1}{\pi A t_\ell} \cdot \frac{1 + (D^2 - 1)\cos^2[2ADt_\ell - (\ell/2 + 1/4)\,\pi]}{1 + D^2} \tag{40}$$

Fig. 2 gives $\Phi(t_1)$ for the first radiation maximum ($\ell=-1$) as a function of τ, calculated from (36), (38) and (40) with $D = 1$. If one takes into account collisions, the decay for large τ becomes nearly exponential. The maximum is then determined by the pulse strength and by the collision frequency.

We observe that the normalizing factor (29) is proportional to V^2 and in the approximation (40) we have $\Phi(t\ ) \sim 1/A \sim 1/V^2$. This shows that in the region of validity of (40) the absolute intensity of the radiation maxima does not depend on A, i.e. on the strength of the pulse, in the case $D = 1$.

The shape of the radiation peaks is the square of the Fourier transform of $h(\eta)$. If $h(\eta)$ is a Gaussian with a width η_0, then the shape of the radiation peaks is also a Gaussian with a width $\Delta_0 t = 4/\eta_0$. In making our approximations we assumed that the width of the radiation peaks is small compared to the time separation of the different peaks. This assumption is equivalent to the assumption

$$\eta_0 \tau \gg 1 \tag{41}$$

where η_0 is a characteristic spread in the gyrofrequencies due to the inhomogeneities. If this condition is not fulfilled, or more precisely, if $\eta_0\tau$ is of the order π or less, the wings of the different peaks overlap and this means that one has really to employ the entire sum (32) in order to determine $\Phi(t)$. One sees that in this case the actual value of $\Phi(t)$ depends now in an essential way on φ_0 which determines the relative phase of the different terms. As in any actual experiment, φ_0 is a statistical quantity, one should find under this condition that the amplitudes of the radiation peaks are different, each time one performs the experiment without changing any of the other parameters.

For $\eta_o \tau \ll 1$ there arise no maxima in $\Phi(t)$.

b) A hot plasma in a homogeneous magnetic field. In this limit we have

$$h(\eta) = \delta(\eta) \tag{42}$$

and we deal only with the dependence of (32) on v . In this case we have the integration variable v also in the argument of the Bessel functions. We assume this argument to be large, so that the approximation (39) is valid. If we write the cosine in (39) as the sum of two exponential functions, we find for (32) (having performed in (26) the integration over η, i.e. dropped the η-dependence):

$$\exp(if) = \left(\frac{1}{4\pi\ ADtv/V}\right)^{1/2} \left\{ \sum_{\ell=-\infty}^{+\infty} \exp(iF_\ell) + \sum_{m=-\infty}^{+\infty} \exp(iF'_m) \right\} \tag{43}$$

with

$$F_\ell = -At(D + v/V)^2 + \ell A\tau(1 - v^2/V^2) - (\ell-1)\ \varphi_o + \pi/4 + ABt \tag{44a}$$

$$F'_m = -At(D - v/V)^2 + mA\tau(1 - v^2/V^2) - (m-1)\ \varphi_o - m\pi - \pi/4 + ABt \tag{44b}$$

We now write $v/V = 1 + u$ and make use of the assumption (16) by neglecting in (44) all terms which are quadratic in u . We then have

$$F_\ell = -2Au\left[t(D+1) + \ell\tau\right] - (\ell-1)\ \varphi_o + \pi/4 + At\left[B - (1+D)^2\right] \tag{45a}$$

$$F'_m = 2Au\left[t(D-1) - m\tau\right] - (m-1)\ \varphi_o - m\pi - \pi/4 + At\left[B - (1-D)^2\right] \tag{45b}$$

Using the same arguments as in the cold plasma case, we conclude that we now have two series of radiation maxima at times

$$t_\ell = -\ell\frac{\tau}{D+1} \quad ; \quad t_m = m\frac{\tau}{D-1} \tag{46a,b}$$

The essential difference from the cold plasma case is that the time for the occurrence of the radiation peaks depends on the relative pulse strength D.[7]

The term $\exp(if_1)$ in (26) can be treated in the same way. We find again that the $(\ell-1)$-th term of the expansion contributes to the integral at a time t where the ℓ-th term of (93) contributes. These two terms now have either the same phase, if they are determined by (45a), or have opposite sign, if they are determined by (45b). So we find for the maximum of $\Phi(t)$

$$\Phi(t_\ell) = \frac{1}{4\pi\, ADt_\ell} \; \frac{(1+D)^2}{1+D^2} \tag{47a}$$

$$\Phi(t_m) = \frac{1}{4\pi\, ADt_m} \; \frac{(1-D)^2}{1+D^2} \tag{47b}$$

In the case that

$$-\ell(D-1) = m(D+1) \tag{48}$$

the contribution from both series add to one radiation peak. The phase with which they add depends on the last term in (45a) and (45b), i.e., depends on A and D.

The shape of the radiation peaks is now essentially the square of the Fourier transform of $g(v)$. If we assume $g(v)$ to be a Maxwellian with its maximum at $v = V$, then the shape of the radiation peaks is also a Gaussian $\sim \exp[-(t-t_\ell)/(\Delta_o t)^2]$, where

$$\Delta_o t = \frac{1}{2A(D \pm 1)\; v_o/V} \tag{49}$$

v_o being the velocity of a particle with the energy kTe, i.e. $v_o/V = (kT_e/E_1)^{1/2}$ where T_e is the initial temperature and E_1 is the energy an electron gains by the first pulse if it is initially at rest. If we introduce the dimensionless quantity

$$\chi = \Lambda\tau \, v_o / V \tag{50}$$

we have

$$\Delta_o t = \frac{\tau}{2\chi(D \pm 1)} \, . \tag{49a}$$

According to (49a) the initial temperature of the plasma can, in principle, be determined by measuring the width of the radiation maxima.

The total energy in one of the radiation peaks is proportional to $\int \Phi(t) \, dt$. If the shape of the peak is a Gaussian with a width given by (49), we have

$$\int \Phi(t) \, dt = \Phi(t\ ) \, \Delta_o t \, \pi^{1/2} \tag{51}$$

The assumption that the width of the peaks is small compared to the separation between the different peaks is now equivalent to

$$\chi \gg 1 \tag{52}$$

and corresponds to the condition (41) in the cold plasma case. For $\chi \ll 1$ no radiation maxima arise.

Fig. 3 gives a numerical example for the case of a hot plasma in a homogeneous magnetic field, computed from equation (26) without further approximations. The initial energy distribution was assumed to be Maxwellian. The parameters for this example were chosen to $\Lambda\tau = 50$, $v_o/V = .1$, $\varphi_o = \pi$, and $D = 1$. Fig. 3 shows also how the results obtained from (26) are modified by collisions according to equation (77). A $\Phi(t)$ of 10^{-3} means that the radiation is $N \cdot 10^{-3}$ times the intensity which the plasma would radiate if the electrons had the same energy distribution but randomly distributed phases, i.e. for 10^8 electrons this would be a factor of 10^5.

We now consider the general case, i.e. the simultaneous influence of the field inhomogeneities and of the initial temperature. If the condition (41) is fulfilled, we retain for a time t_ℓ given by (34) just the ℓ-th term of (32) after integration over η. If we further assume the argument of the Bessel functions to be large, i.e. the approximation (39) to be valid, and $g(v)$ to be Maxwellian, the integration over v gives us the result (40) multiplied by a correction factor

$$\exp(-2\ell^2 D^2 \chi^2) \tag{53}$$

If we, on the other hand, assume (52) to be valid and integrate at first over v, we find correspondingly for times given by (46) the result (47) multiplied by a factor

$$\exp\left[-\frac{1}{2}\left(\frac{\psi_0 \tau \ell D}{D \pm 1}\right)^2\right] \tag{54}$$

if $h(\eta)$ is assumed to be Gaussian.

This result shows that the effect of the initial temperature and that of the inhomogeneities have the tendency to cancel each other. If both conditions (41) and (52) are fulfilled simultaneously, essentially no radiation maxima arise.

IV. EXCITATION BY THREE OR MORE PULSES

Next we consider a plasma excited by three short pulses. Let D_2 be the strength of the third pulse relative to the first and φ_1 the phase of the electric field of the third pulse in the rotating coordinate system, and

v_2, $\varphi_2(t)$, β_2, $\Delta_2\omega$, B_2 be quantities after the third pulse. Furthermore, let T be the time between the second and third pulse. Then we have, corresponding to equations (21-25):

$$v_2^2/V^2 = v^{*2}/V^2 + D_2^2 + 2D_2(v^*/V)\cos[\varphi^*(T) - \varphi_1] \tag{55}$$

$$\varphi_2(t=0) = \varphi_1 + \beta_2 \tag{56}$$

$$\sin\beta_2 = \frac{v^*}{v_2}\sin[\varphi^*(T) - \varphi_1] \tag{57a}$$

$$\cos\beta_2 = \frac{v_2^2/V^2 + D_2^2 - v^{*2}/V^2}{2D_2 v_2/V} \tag{57b}$$

$$\Delta_2\omega_c = A(B_2 - v_2^2/V^2) + \eta \tag{58}$$

$$\varphi_2(t) = \varphi_2(t=0) + \Delta_2\omega_c t \tag{59}$$

where t now is measured from the end of the third pulse. By analogy with (26) we find

$$\Phi(t) = \left|\iint dv\, d\eta\, g(v)\, h(\eta)\, v_2\right|^{-2} \cdot$$
$$\cdot \left| V\iint dv\, d\eta\, g(v)\, h(\eta)\, \Big\{D_2\exp(if_2) + D\exp(if_3) + (v/V)\exp(if_4)\Big\}\right|^2 \tag{60}$$

with

$$f_2 = \varphi_1 + \Delta_2\omega_c t \tag{61a}$$

$$f_3 = \varphi_o - \varphi_1 + \Delta^*\omega_c T + f_2 \tag{61b}$$

$$f_4 = -\varphi_o + \Delta\omega_c\tau + f_3 \tag{61c}$$

More explicitly we have

$$f_2 = \varphi_1 + t\eta - At\Big\{-B_2 + D_2^2 + D^2 + v^2/V^2 + 2(v/V)\,D\cos[\varphi(\tau) - \varphi_0]$$

$$+ 2D_2D\cos[\varphi_0 - \varphi_1 + \Delta^*\omega_cT] + 2D_2(v/V)\cos[\varphi(\tau) - \varphi_1 + \Delta^*\omega_cT]\Big\} \qquad (62)$$

where $\Delta^*\omega_cT$ is to be calculated from (24).

We now specialize to the case of an initially cold plasma in an inhomogeneous magnetic field, i.e. we make the approximation (33). Performing the v-integration in (60) means essentially dropping the v-dependence. As (62) contains three cosines, the expansion into Bessel functions now gives a threefold product of sums of the type (31). As $\Delta^*\omega_cT$ still contains a cosine we apply (31) once more and obtain finally:

$$\exp(if_2) = \sum_{k,\ell,m,n=-\infty}^{+\infty} J_k(2ADt)\, J_\ell(2ADD_2t)\, J_m(2AD_2t)\, J_n[2ADT(m+\ell)]\,\cdot$$

$$\cdot\exp\Big\{i[\eta(t + (k+m+n)\tau + (\ell+m)T) - (k+\ell+m+n)\pi/2$$

$$-(\ell+m-1)\varphi_1 + (\ell-k-n)\varphi_0 + (\ell+m)(B-1-D^2)AT$$

$$- At(D_2^2 + D^2 + 1 - B_2)]\Big\} \qquad (63)$$

With the assumption (41) we conclude from (63) that in the case of three exciting pulses we have radiation maxima at times

$$t_{KL} = -K\tau - LT \qquad (64)$$

K and L being integers with the restriction $t_{KL} > 0$. As we consider in (63) for a time t_{KL} only those terms which are independent of η, the fourfold sum reduces to a double sum over k and m. We perform the sum over l by means of the addition theorem[6]

$$J_\nu(x+y) = \sum_{\mu=-\infty}^{+\infty} J_\mu(x)\, J_{\nu-\mu}(y) \tag{65}$$

and obtain*

$$G_2 \sim \sum_m J_{K-m}(x_1)\, J_{L-m}(x_2)\, J_m(x_3)\, \exp(im\,\pi/2) \tag{66}$$

$$= \sum_m J_{K-m}(x_1)\, J_{L-m}(x_2)\, J_{-m}(x_3)\, \exp(-im\,\pi/2) \tag{66a}$$

with

$$x_1 = -2ADK\tau \quad ; \quad x_2 = 2ADD_2 t_{KL} \quad ; \quad x_3 = 2AD_2\, t_{KL} \tag{67a,b,c}$$

$$G_2 = \int d\eta\, h(\eta)\, \exp(if_2) \tag{67d}$$

It seems that the expression (66) can only be simplified by making further restrictive assumptions. If we assume $D = 1$, i.e. $x_2 = x_3$, we can use the formula[6]

$$J_\mu(z)\, J_\nu(z) = \frac{2(-1)^\nu}{\pi} \int_0^{\pi/2} J_{\mu-\nu}(2z\cos\psi)\, \cos(\mu+\nu)\psi\; d\psi \tag{68}$$

We apply (68) to the last two Bessel functions in (66a) so that $J_{k-m}(x_1)$ remains the only Bessel function depending on m. We now can perform the sum over m by virtue of (31). Thus we find:

$$G_2 \sim \frac{2}{\pi} \int_0^{\pi/2} J_L(2x_2\cos\psi)\, \exp\left\{ix_1\cos 2\psi - iK\,\pi/2\right\} \cdot$$

$$\cdot \cos(2K - L)\,\psi\; d\psi \tag{69}$$

Only in special cases have we found an analytic expression for the integral (69). If $K = 0$, from which it follows that $x_1 = 0$, we can apply (68) and find

$$G_2 \sim J_L(-2AD_2 LT)\;\; J_0(-2AD_2 LT) \tag{70}$$

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*In (66) there is only a phase factor omitted, which is common to all terms in the sum.

In the case that $x_1 \ll 1$ and $x_1 \ll x_2$, one can neglect the exponential function in (69) and obtain

$$G_2 \sim J_{-K}(2\Lambda D_2 t_{KL}) J_{L-K}(2\Lambda D_2 t_{KL}) \exp(iK\pi/2) \tag{71}$$

This approximation is not valid for $L = 0$ as we than have $x_1 \approx x_2$ and the variation of the exponential function in (69) with ψ must not be neglected in comparison with the variation of the Bessel function.

In the same way one finds for the other two terms in (60):

$$G_3 \sim \sum_m J_{K-m}(x_1) J_{L-1-m}(x_2) J_m(x_3) \exp(im\pi/2) \tag{72}$$

$$G_4 \sim \sum_m J_{K-1-m}(x_1) J_{L-1-m}(x_2) J_m(x_3) \exp(im\pi/2) \tag{73}$$

where G_3 and G_4 are defined as similar to G_2 (67d). These sums can be treated as (66).

If we consider the case of a hot plasma in a homogeneous magnetic field excited by three pulses, we obtain an expression similar to (63) where the arguments of the Bessel functions, except the second, depend on v. Making the large argument approximation (39) leads us to expect radiation maxima at times

$$t_{KL} = \frac{-K\tau - LT}{1 \pm D \pm D_2} \tag{74}$$

where K and L are again integers with the restriction $t_{KL} > 0$ and the signs of D and D_2 may occur in each combination.

The results obtained for the case of three exciting pulses may be generalized in a straightforward manner to the case of n+1 pulses. The relations (55-59) become recurrence formulae by substituting the index 2 by n, the index 1 and the star by the index (n-1), and T by T_n

For t being the time elapsed after the last pulse, one has

$$\Phi_n(t) \sim \left| \iint dv\, d\eta\, g(v)\, h(\eta) \left\{ D_n \exp\left[i(\varphi_{n-1} + \Delta_n \omega_c t) \right] + v_{n-1} \exp\left[i(\varphi_{n-1}(T_n) + \Delta_n \omega_c t) \right] \right\} \right|^2 \tag{75}$$

In the case of an initially cold plasma in an inhomogeneous magnetic field one concludes that radiation maxima arise at times

$$t = \sum_{i=1}^{n} L_i T_i \tag{76}$$

where L_i are integers with the requirement that $t > 0$.

V. THE INFLUENCE OF COLLISIONS

As collisions destroy phase correlations, it is obvious that they give rise to a much faster decay of the radiation maxima than that given by the previous formulae, which were derived in the approximation of a collisionless plasma. It was observed in the experiment of Hill and Kaplan[1] that the dependence of the radiation maxima on τ was determined by all phase-destroying collisions, while in the three-pulse case the dependence on T was determined by the inelastic collisions only. This is due to the fact that after the second pulse there is information stored not only in the phases but also in the energy distribution.

Let us consider at first the two-pulse case. If we assume that the phase of a particle after a collision is not related to its phase before the collision, it follows that only particles can contribute to the radiation maxima which did not undergo any collision. If we assume that the probability of undergoing a collision is the same for all particles (independent of their velocity) we have

$$\phi_c(t) = \phi(t) \exp(-2(t+\tau)\, \nu_1) \tag{77}$$

where $\nu(t)$ is the value obtained for a collisionless plasma and ν_1 is the collision frequency accounting for all kinds of collisions.

Under conditions as in the experiment of Hill and Kaplan[2] most of the collisions the electrons undergo are with neutrals. As the mass of an electron is very small compared to that of an atom, most collisions only change the phases of the electrons but not their energies. We call these collisions elastic.

We now consider the three-pulse case and study at first the influence of the elastic collisions during the time between the second and third pulse in the limit $T\nu_{el} \gg 1$. In this case the phases have been randomized at the onset of the third pulse and $\alpha = \Delta^* \omega_c T$ is now a statistical quantity which is to be integrated over , instead of being given by (24). Furthermore, the problem now has become three dimensional, as the electrons can, through collisions, acquire a velocity component parallel to the magnetic field. So we have

$$v_\perp^*(t = T) = v^*(t=0) \sin\theta \; ; \quad v_{||}^*(t=T) = v^*(t=0) \cos\theta \tag{78}$$

where θ is the angle between the electron velocity and the z-axis. We also have to integrate our final result over θ. Consequently we have to substitute in (55) and (57) v^* by $v_\perp^*(t=T)$ and (58) by

$$\Delta_2\omega_c = A(B_2 - v_2^2/V^2 - (v^{*2}(t=0)/V^2) \cos^2\theta) + \eta \tag{79}$$

v_2 now being again only the transverse part of the velocity. With these modifications we now find instead of (62):

$$f_2 = \varphi_1 + t\eta - At \Big\{ -B_2 + D_2^2 + D^2 + v^2/V^2 + 2(v/V)\, D \cos\left[\varphi(\tau) - \varphi_o\right]$$

$$+ 2DD_2 \sin\Theta \cos\left[\varphi_o - \varphi_1 + \alpha\right] + 2(v/V)\, D_2 \sin\Theta \cos\left[\varphi(\tau) - \varphi_1 + \alpha\right]\Big\} \quad (80)$$

If we now make the expansion into Bessel functions, we see that only terms with $L = 0$ give a contribution after integrating over α. So there are, in spite of $\nu_1 T \gg 1$, still radiation maxima after the third pulse, but their number is decreased by the limitation to $L = 0$.

In the case of a cold plasma in a homogeneous magnetic field we derive from (80) again equation (66) with $L = 0$ and x_2 and x_3 to be substituted by $x_2 \sin\Theta$ and $x_3 \sin\Theta$. Assuming again $D = 1$ and integrating over Θ we find

$$G_2' \sim \frac{2}{\pi} \int_0^{\pi/2} d\Theta \int_0^{\pi/2} d\psi \, J_o(2x_2 \cos\psi \sin\Theta) \exp\left\{ix_1 \cos 2\psi + iK\frac{\pi}{2}\right\} \cdot$$

$$\cdot \sin\Theta \cos 2K\psi \quad (81)$$

We observe that the value of (81) is not changed if we substitute $\sin\Theta$ by $\cos\Theta$. So we apply (68) in order to perform the integration over Θ. This yields

$$G_2' \sim \int_0^{\pi/2} J_{1/2}^2(x_2 \cos\psi) \exp\left\{ix_1 \cos 2\psi + iK\frac{\pi}{2}\right\} \cos 2K\psi \, d\psi \quad (82)$$

In order to account for the inelastic collisions during the time between the second and third pulse and all kinds of collisions during the time before the second and after the third pulse, the value (82) has to be multiplied by the factor

$$\exp(-(\tau+t)\,\nu_1 - T\,\nu_2) \quad (83)$$

where ν_2 is an effective collision frequency for inelastic collisions.

VII. DISCUSSION

In the previous sections it has been shown that the relativistic mass effect can give rise to radiation maxima in a plasma excited by a sequence of short pulses at the electron gyrofrequency. The essential point in the treatment was that the influence of the relativistic mass effect was neglected during the exciting pulses, while it was taken into account between the pulses. This approximation can only be made if (12) is fulfilled, i.e. if the pulses are short and if $\tau \gg t_1$ and further if $v \ll c$.

The occurrence of a series of radiation maxima is essentially different from the spin echo case[2] and is due to the fact that the gyrofrequency in our treatment is energy dependent[3]. A further essential difference is the new result that when the plasma is hot the time at which the radiation maxima occur depends on the relative pulse strength D.

A not very essential assumption in our treatment was that the dimensions of the plasma are small compared to the wavelength of the radiation at the gyrofrequency. If this is no longer true, one sees that the $\underset{\sim}{k}$-vector of the exciting pulses has to be perpendicular to the magnetic field. If $\underset{\sim}{k}$ and $\underset{\sim}{H}$ are parallel, particles excited at different phases can interchange their places by moving along the lines of force, giving rise thereby to statistical phase differences and spoiling the correlations which have been generated. A further consequence is that the radiated energy in this case is essentially radiated into the same direction as the exciting pulses.

It has already been pointed out[5] that the model of a cold plasma in an inhomogeneous magnetic field gives all the characteristics of the observations by Hill and Kaplan[1]. But a general discussion[3] shows that there are also other nonlinearities beside the relativistic mass effect which can give rise to radiation maxima. The relative importance of the different effects depend on the details of the experiment. It should be noted, however, that the results obtained in this paper are qualitatively correct also for other nonlinearities, by which the gyrofrequency of a particle becomes energy dependent, as e.g. the influence of spatial gradients of the magnetic or electric field[3,8].

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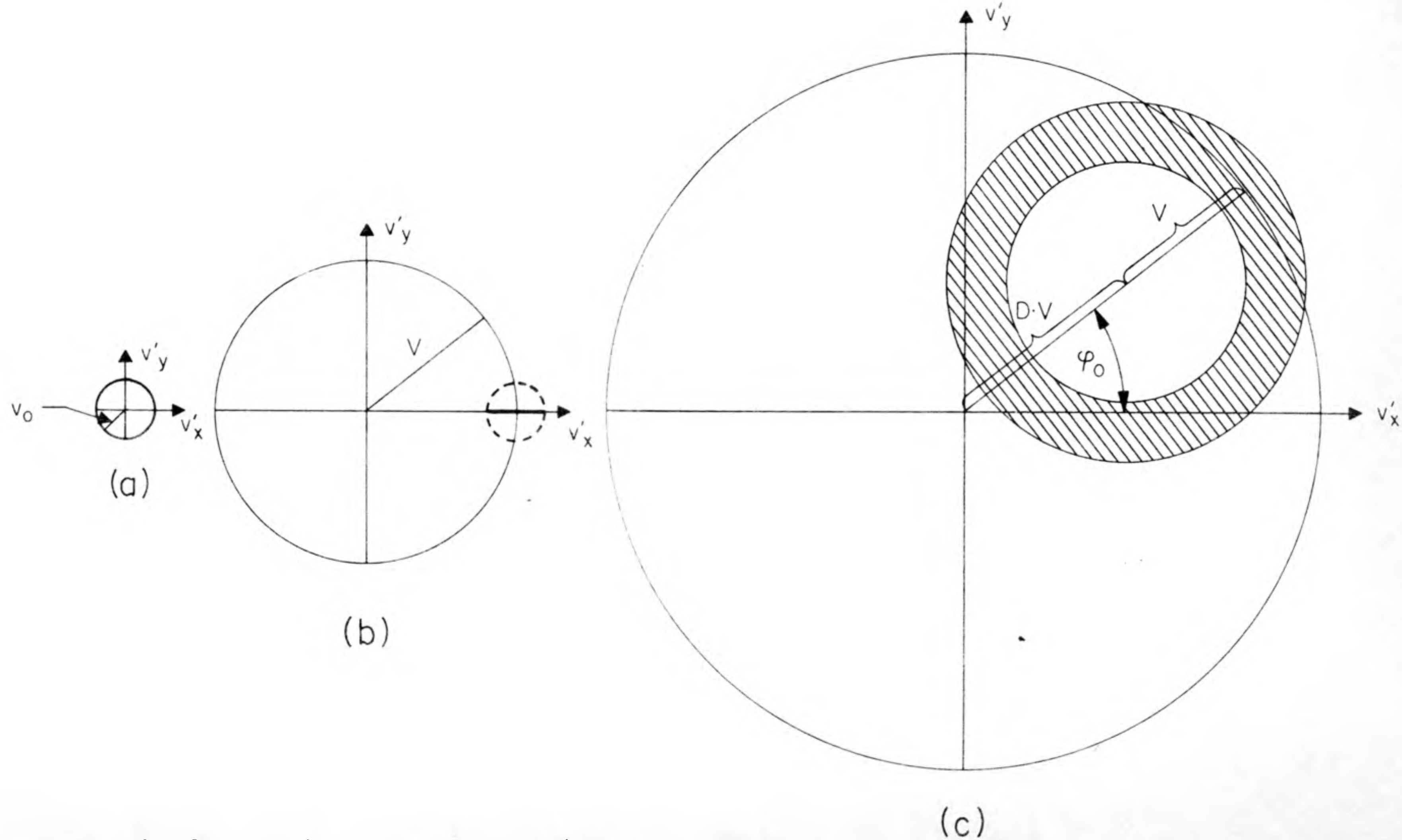

Fig. 1. Particle distribution in v'-space:
a) before the first pulse,
b) at the end of first pulse,
c) at the end of second pulse.

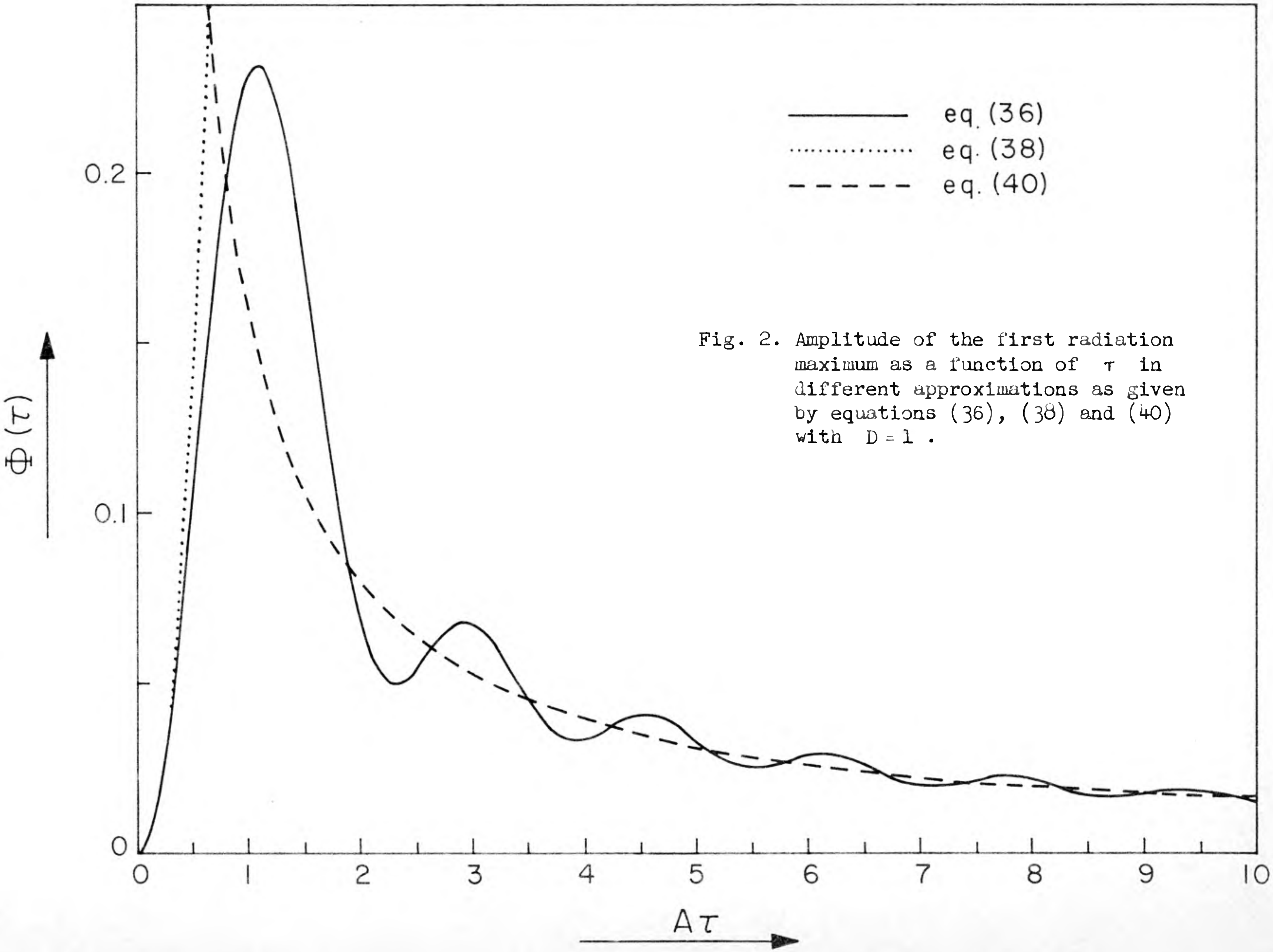

Fig. 2. Amplitude of the first radiation maximum as a function of τ in different approximations as given by equations (36), (38) and (40) with $D = 1$.

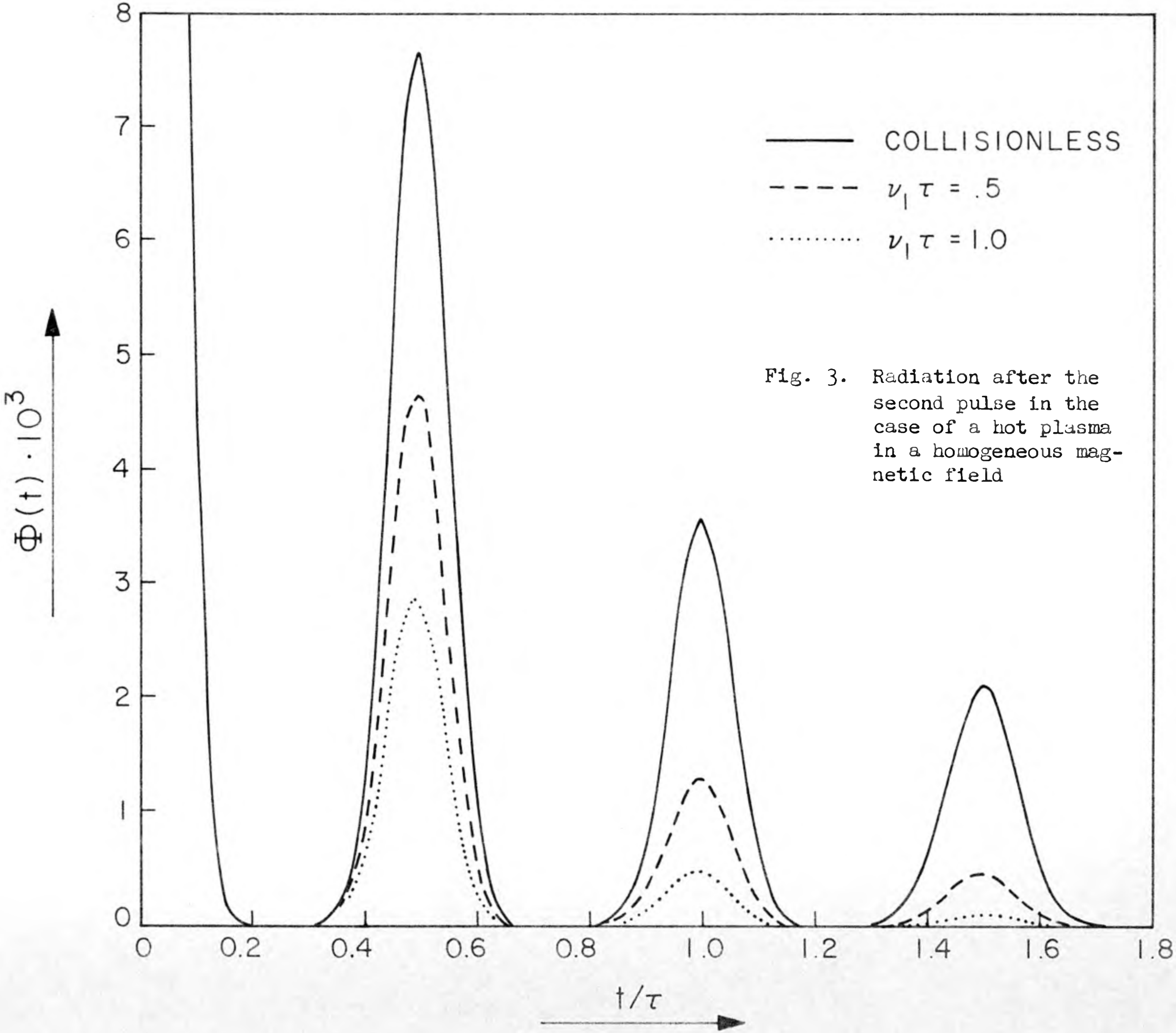

Fig. 3. Radiation after the second pulse in the case of a hot plasma in a homogeneous magnetic field