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[3] Carl Hering, *Conversion Tables*, New York, 1904.

[4] E. Bennett, "A Digest of the Relations between the Electrical Units and the Laws Underlying the Units," *Univ. of Wisconsin Bull.*, 1917.

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[7] A. E. Kennelly, "Magnetic Circuit Units," *Trans. Am. Inst. El. Engrs.*, Jan., 1930, **49**, No. 2, Apr., 1930, pp. 486–510. Discussion by Gokhale, p. 503, Tables II and III.

ON THERMODYNAMIC EQUILIBRIUM IN A STATIC EINSTEIN UNIVERSE

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§ 1. *Introduction.*—It has now become evident that the transformation of matter into radiation taking place throughout the universe and the red-shift observed in the light from the extra-galactic nebulae appear to imply a non-static quality in the universe which can be treated with some success with the help of a non-static cosmological line element.[1] If the quantity which gives the dependence of this non-static line element on the time is set equal to a constant, it is found that the line element then becomes the same as Einstein's original line element for a static universe. Hence the Einstein static universe may be regarded as a special case of the more general non-static universe, and we must continue to be interested in the properties of the Einstein universe not only because it is a limiting case of the more general model for the universe, but also because it represents a situation which might arise in the course of the evolution of the actual universe. The present article will deal with the thermodynamics of the Einstein universe, and in particular will treat the conditions for thermodynamic equilibrium between matter and radiation in such a universe assuming the possibility of their transformation into each other.

Treatments of the general problem of the equilibrium between matter and radiation have already been given for the case of a perfect monatomic gas interacting with black body radiation both in the absence and presence of gravitational fields. In the absence of any appreciable gravitational field, it was shown by the work of Stern[2] and myself[3] that the number of monatomic molecules of mass m present in unit volume at equilibrium at temperature T would be given by a formula of the form

$$N = bT^{3/2}\, e^{-\frac{mc^2}{kT}} \tag{1}$$

where b is a constant whose value cannot be determined solely from the first and second laws of thermodynamics, c is the velocity of light and k is Boltzmann's constant. On account of the large effect of the exponent $-mc^2/kT$ the equilibrium concentration of matter given by this formula would be exceedingly low, even for masses as small as that of the electron and for temperatures as high as the 40,000,000° assumed in the interior of the stars, unless indeed the constant b could be shown to have an enormous value.

Also in the presence of the gravitational field of a spherical distribution of fluid, I have recently been able to show[4] that the equilibrium concentration of monatomic gas would again be given by a formula of the same form (1) as for flat space-time. But in the presence of the gravitational field of the Einstein static universe I originally found[5] a slightly different formula, which however still had a similar large exponential dependence on $-mc^2/kT$.

Since that time, however, the development of the non-static line element for the universe has made it easier to understand the process by which the Einstein universe could be regarded as changed from one static state to another and thus clarified the problem of determining a static state which would correspond to thermodynamic equilibrium. And the present article will show that a correct application of the principles of relativistic thermodynamics actually leads to exactly rather than merely approximately the same expression for the equilibrium concentration of gas in the Einstein static universe as formula (1) for the concentration in the absence of a gravitational field and for the concentration in a spherical mass of gravitating fluid.

§ 2. *The Einstein Line Element as a Special Case of the Non-Static Line Element.*—The line element for the non-static universe can be written in the form[6]

$$ds^2 = -\frac{e^{g(t)}}{[1 + r^2/4R^2]^2}(dx^2 + dy^2 + dz^2) + dt^2 \qquad (2)$$

where R is a constant, r is an abbreviation for $\sqrt{x^2 + y^2 + z^2}$ and the dependence of the line element on the time t is determined by the form of the function $g(t)$. This line element corresponds to a universe in which particles stationary with respect to x, y, z will remain stationary and corresponds to a distribution of matter and radiation of uniform proper density and pressure which, however, will in general be changing with the time if $g(t)$ is actually changing with the time.

If, however, $g(t)$ is given a constant value in the above expression, the line element reduces to that for the static Einstein universe, corresponding to a constant uniform density and pressure, and indeed can then easily be thrown into one of the more familiar forms for the Einstein line element with the quantity $Re^{g/2}$ appearing as the radius of the universe. Thus for example if we take $g(t)$ as a constant g, and make the simple substitution

$$x = Xe^{-g/2}, \quad y = Ye^{-g/2}, \quad z = Ze^{-g/2} \tag{3}$$

we can write the line element in the form

$$ds^2 = -\frac{1}{\left[1 + \frac{X^2 + Y^2 + Z^2}{4(Re^{g/2})^2}\right]^2}(dX^2 + dY^2 + dZ^2) + dt^2 \tag{4}$$

which is one of the well-known expressions[7] for the Einstein line element with $Re^{g/2}$ as the radius.

Moreover, since the radius of the universe is the only adjustable parameter which occurs in the Einstein line element, it is now evident that we can regard a change in the Einstein universe from one static state to another as produced by a change in g from one given constant value to another. This makes it possible to apply to the change in state results which have been obtained from the study of non-static universes in which g is changing with the time.

§ 3. *Application of Relativistic Mechanics to Changes in State of the Einstein Static Universe.*—As shown in a previous article,[8] values for the energy-momentum tensor corresponding to the non-static line element (2) can be calculated from the principles of general relativity, and treating the material in the universe as a perfect fluid can be written in the form

$$8\pi T_1^1 = 8\pi T_2^2 = 8\pi T_3^3 = -8\pi p_0 = \frac{1}{R^2}e^{-g} + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \tag{5}$$

$$8\pi T_4^4 = 8\pi \rho_{00} = \frac{3}{R^2}e^{-g} + \frac{3}{4}\dot{g}^2 - \Lambda \tag{6}$$

$$8\pi T_\rho^\sigma = 0 \quad (\rho \neq \sigma) \tag{7}$$

where p_0 is the proper pressure and ρ_{00} the proper macroscopic density of the fluid, Λ is the cosmological constant and the dots indicate differentiation with respect to the time.

If we take g as a constant parameter in these equations, putting $\ddot{g} = \dot{g} = 0$, we obtain those relations between pressure, density and the metric which are necessary for a static Einstein universe.

On the other hand, if we let g vary with the time we can study the process through which the universe could change from one state to another, by substituting the values for the energy-momentum tensor given by equations (5), (6) and (7) into the well-known equation of relativistic mechanics

$$\frac{\partial \mathfrak{T}_\rho^\sigma}{\partial x_\sigma} - \frac{1}{2}\mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\rho} = 0. \tag{8}$$

Doing so we merely obtain identities for the cases $\rho = 1, 2, 3$, but substituting for the case $\rho = 4$ we easily obtain the important result

$$\frac{d}{dt}\left(\rho_{00}e^{\frac{3\mu}{2}}\right) + p_0 \frac{d}{dt}\left(e^{\frac{3\mu}{2}}\right) = 0, \tag{9}$$

where for simplicity we have set as an abbreviation

$$e^{\mu} = \frac{e^{g(t)}}{\left[1 + \frac{r^2}{4R^2}\right]^2}. \tag{10}$$

In accordance with the form of the line element, however, it is evident that $e^{\frac{3\mu}{2}}\,dx\,dy\,dz$ is the proper volume associated with the coördinate range $dx\,dy\,dz$. Hence the first term in equation (9) can be interpreted as the rate of increase in the proper energy in unit coördinate range and the second as the rate of expenditure of work on the surroundings. Furthermore, we remember that particles which are stationary with respect to x, y, z will remain stationary, so that the energy content of a given coördinate range will not be changing by the passage of matter through the boundaries of the coördinate range. Thus equation (9) has the simple physical interpretation that changes in the proper energy in any given coördinate range are due solely to the work done on the surroundings, and we can regard a change in the Einstein universe from one static state to another as the result of an *adiabatic* change in proper volume.

§ 4. *Application of the Relativistic Second Law of Thermodynamics to a Change in State of the Einstein Universe.* We must now consider the application of the second law of thermodynamics to changes in the state of an Einstein universe. In accordance with the principles of relativistic thermodynamics which I have previously developed,[9] the relativity analogue of the usual second law of thermodynamics can be expressed in the form

$$\frac{\partial}{\partial x_\rho}\left(\phi_0 \sqrt{-g}\,\frac{dx_\rho}{ds}\right)dx_1\,dx_2\,dx_3\,dx_4 \geq \frac{dQ_0}{T_0}, \tag{11}$$

where ϕ_0 is the proper density of entropy as measured by a local observer, dx_ρ/ds the macroscopic velocity of matter and T_0 the proper temperature, all at the point of interest, and dQ_0 is the heat measured in proper coördinates flowing into the infinitesimal region and in the time denoted by $dx_1dx_2dx_3dx_4$.

In applying this expression to the case of a change in state of the Einstein universe, we note that the macroscopic velocities dx_ρ/ds will be zero for the cases $\rho = 1, 2, 3$ since matter is stationary in the coördinate system x, y, z which is being used, and will be unity for the case $\rho = 4$ owing to the form of the line element. We also note that dQ_0 will be zero, owing to the adiabatic character of the change pointed out in the last section. Hence the general principle reduces for our case to

$$\frac{d}{dt}\left(\phi_0 \sqrt{-g}\right) = \frac{d}{dt}\left(\phi_0 e^{\frac{3\mu}{2}}\right) \geqq 0 \tag{12}$$

and since $\phi_0 e^{\frac{3\mu}{2}}$ is evidently the entropy associated with unit coördinate range, the result has the simple physical interpretation that the only possible changes in the state of an Einstein universe must be such as not to decrease the entropy content associated with each range of coördinates.

§ 5. *Conditions for Thermodynamic Equilibrium in a Static Einstein Universe.*—In accordance with the result of the foregoing section, the condition of thermodynamic equilibrium will evidently require that the quantity $\phi_0 e^{\frac{3\mu}{2}}$, which cannot decrease with the time, shall be a maximum at each point x, y, z. And noting equation (10) which defines μ, we can express this somewhat more conveniently by the requirement that $\phi_0 e^{\frac{3g}{2}}$ shall be a maximum.

Since we are interested, however, in the condition of thermodynamic equilibrium in a static Einstein universe, it is evident that the maximum value of this quantity must be achieved as a result of changes which preserve a static Einstein universe. In other words, when we vary the metrical quantity g and the pressure and density of matter and radiation in the universe in order to make $\phi_0 e^{\frac{3g}{2}}$ a maximum, we must preserve the truth of the equations of relativistic mechanics which relate the pressure and density to the metric in the way necessary to give a static Einstein universe, namely, as will be seen from § 3, the truth of equations (5) and (6) with g taken as independent of the time.

Hence we may now write as the desired conditions for thermodynamic equilibrium

$$\phi_0 e^{\frac{3g}{2}} \text{ to be a maximum} \tag{13}$$

under the subsidiary conditions obtained from (5) and (6)

$$p_0 + \frac{1}{8\pi R^2} e^{-g} - \frac{\Lambda}{8\pi} = 0 \tag{14}$$

and

$$\rho_{00} - \frac{3}{8\pi R^2} e^{-g} + \frac{\Lambda}{8\pi} = 0 \tag{15}$$

§ 6. *Equilibrium for a Monatomic Gas Interacting with Black Body*

Radiation.—We may now apply these conditions for thermodynamic equilibrium to the case of an Einstein universe filled with a monatomic gas interacting with black body radiation. Under these circumstances the proper entropy density ϕ_0, pressure p_0 and energy density ρ_{00} will evidently be functions of the proper temperature T_0 and number of atoms of gas N_0 per unit proper volume.

Hence employing the usual methods for treating a conditional maximum with the introduction of undetermined multipliers, we shall obtain from (13), (14) and (15) the following five equations

$$
\begin{aligned}
e^{\frac{3g}{2}}\frac{\partial \phi_0}{\partial N_0} + \mu_1 \frac{\partial p_0}{\partial N_0} + \mu_2 \frac{\partial \rho_{00}}{\partial N_0} &= 0 \\
e^{\frac{3g}{2}}\frac{\partial \phi_0}{\partial T_0} + \mu_1 \frac{\partial p_0}{\partial T_0} + \mu_2 \frac{\partial \rho_{00}}{\partial T_0} &= 0 \\
\phi_0 \frac{\partial}{\partial g} e^{\frac{3g}{2}} + \frac{\mu_1}{8\pi R^2}\frac{\partial}{\partial g} e^{-g} - \frac{3\mu_2}{8\pi R^2}\frac{\partial}{\partial g} e^{-g} &= 0 \\
p_0 + \frac{1}{8\pi R^2} e^{-g} - \frac{\Lambda}{8\pi} &= 0 \\
\rho_{00} - \frac{3}{8\pi R^2} e^{-g} + \frac{\Lambda}{8\pi} &= 0
\end{aligned}
\tag{16}
$$

for evaluating the undetermined multipliers μ_1 and μ_2 and the three unknown variables N_0, T_0 and g.

To treat these equations, we shall need explicit expressions for the quantities ϕ_0, p_0, and ρ_{00} in terms of the independent variables, N_0 the number of atoms of gas per unit volume and T_0 the proper temperature. For the proper density of entropy we can evidently write from the known expressions for the entropy of a monatomic gas and the entropy of radiation

$$\phi_0 = \frac{3}{2} N_0 k \log T_0 - N_0 k \log N_0 + N_0 k \log b e^{\frac{5}{2}} + \frac{4}{3} a T_0^3, \tag{17}$$

where k is Boltzmann's constant, a is the Stefan-Boltzmann constant, and b is a constant of the right magnitude to assure the same starting point for the entropy of the gas and the radiation, the factor $e^{\frac{5}{2}}$ being introduced in the interests of simplicity of form in the final formula. Furthermore for the pressure p_0 and proper macroscopic density ρ_{00}, we evidently have the expressions

$$p_0 = N_0 k T_0 + \frac{1}{3} a T_0^4 \tag{18}$$

and

$$\rho_{00} = N_0 m c^2 + \frac{3}{2} N_0 k T_0 + a T_0^4, \tag{19}$$

where m is the mass of one atom and c is the velocity of light.

Introducing these equations into (17) and performing the indicated differentiations we then obtain the following five algebraic equations for the five unknown quantities N_0, T_0, g, μ_1 and μ_2.

$$e^{\frac{3g}{2}}\left(k \log \frac{bT_0^{\frac{3}{2}}}{N_0} + \frac{3}{2}k\right) + \mu_1 kT_0 + \mu_2\left(mc^2 + \frac{3}{2}kT_0\right) = 0$$

$$e^{\frac{3g}{2}}\left(\frac{3}{2}\frac{N_0k}{T_0} + 4aT_0^2\right) + \mu_1\left(N_0k + \frac{4}{3}aT_0^3\right) + \mu_2\left(\frac{3}{2}N_0k + 4aT_0^3\right) = 0$$

$$\frac{3}{2}e^{\frac{3g}{2}}\left(N_0k \log \frac{bT_0^{\frac{3}{2}}}{N_0} + \frac{5}{2}N_0k + \frac{4}{3}aT_0^3\right) - \frac{\mu_1}{8\pi R^2}e^{-g} + \frac{3\mu_2}{8\pi R^2}e^{-g} = 0 \qquad (20)$$

$$N_0kT_0 + \frac{a}{3}T_0^4 + \frac{1}{8\pi R^2}e^{-g} - \frac{\Lambda}{8\pi} = 0$$

$$N_0mc^2 + \frac{3}{2}N_0kT_0 + aT_0^4 - \frac{3}{8\pi R^2}e^{-g} + \frac{\Lambda}{8\pi} = 0.$$

These equations are rather complicated but by perfectly straightforward, although somewhat lengthy, algebraic manipulation they can be shown to be equivalent to the simpler set

$$N_0 = bT_0^{\frac{3}{2}} e^{-\frac{mc^2}{kT_0}}$$

$$\mu_1 = 0$$

$$\mu_2 = -\frac{e^{\frac{3g}{2}}}{T_0} \qquad (21)$$

$$N_0kT_0 + \frac{a}{3}T_0^4 + \frac{1}{8\pi R^2}e^{-g} - \frac{\Lambda}{8\pi} = 0;$$

$$N_0mc^2 + \frac{3}{2}N_0kT_0 + aT_0^4 - \frac{3}{8\pi R^2}e^{-g} + \frac{\Lambda}{8\pi} = 0,$$

five equations for the five unknown quantities N_0, T_0, g, μ_1 and μ_2 in terms of the constants a, b, k, m, Λ and R.

§ 7. *Conclusion.*—The first of these five equations gives the relation between equilibrium concentration N_0 of our monatomic gas and proper temperature T_0 which was the matter of chief interest for the present article. The relation has the same form as for the equilibrium concentration in flat space-time and in the gravitational field of a sphere of perfect fluid. We can again conclude that the equilibrium concentration of matter would be exceedingly low even for masses as small as that of the electron and temperatures as high as 40,000,000° unless the constant b could be shown to have an enormous value.

[1] For references to the work of Friedman, LeMaitre, Robertson, Tolman, Eddington and de Sitter in this field, see Tolman, *Proc. Nat. Acad. Sci.*, **16**, 582 (1930).

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[3] Tolman, *Proc. Nat. Acad. Sci.*, **12**, 670 (1926).

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[5] Tolman, *Proc. Nat. Acad. Sci.*, **14**, 353 (1928).

[6] Tolman, *Ibid.*, **16**, 320 (1930).

[7] Einstein, *Berl. Ber.*, 1918, p. 448.

[8] Tolman, *Proc. Nat. Acad. Sci.*, **16**, 409 (1930), Equations (2).

[9] Tolman, *Ibid.*, **14**, 701 (1928); *Phys. Rev.*, **35**, 896 (1930).

ANSWER TO PROF. STÖRMER'S REMARK

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Referring to my "Note on the Nature of Cosmic Rays,"[1] Prof. Störmer draws attention to the fact[2] that he had treated the problems of the motion of electrons in the magnetic field of the earth many years ago. He gives the complete list of his publications on the subject and, indeed, I must confess that I was not aware of his work on the particular phase of the problem to which my note is devoted.

But if I am guilty of having overlooked Prof. Störmer's priority, I may claim extenuating circumstances on several counts, and I believe that my note was not quite superfluous. *In the first place,* Prof. Störmer's papers appeared in magazines which are not readily accessible to the physicist. Even now, after the bibliography has been given by him, I have no access to that of his publications (Geneva, 1907) which contains the data, answering the questions put in my note, or the formulas from which these data could be derived. As all my colleagues in Southern California, and many in other places, are in the same position, it was well to restate the problem and its solution.

In the second place, Prof. Störmer's work dates from pre-relativistic days and is, therefore, based on classical mechanics while my note takes into account relativity. For the high velocities in question, one expects, at first sight, greatly different results. That Prof. Störmer's result is of interest also in the relativistic case is surprising and requires an explanation. The analysis can be based on the Hamilton-Jacobi partial differential equation. In the case of a magnetic dipole, acting upon one